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Local duality for 2-dimensional local ring

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Abstract. We prove a local duality for some schemes associated to a 2-dimensional complete local ring whose residue field is an n-dimensional local field in the sense of Kato–Parshin. Our results generalize the Saito works in the case $n = 0$ and are applied to study the Bloch–Ogus complex for such rings in various cases.

Keywords. Hasse principle; purity; local duality; curves over higher local fields.

1. Introduction

Let A be a 2-dimensional complete local ring with finite residue field. The Bloch–Ogus complex associated to A has been studied by Saito in [14]. In this prospect, he calculated the homologies of this complex and obtained (for any integer $n \geq 1$) the following exact sequence:

$$\begin{aligned} 0 &\longrightarrow (\mathbb{Z}/n)^r \longrightarrow H^3(K, \mathbb{Z}/n(2)) \\ &\longrightarrow \bigoplus_{v\in P} H^2(k(v), \mathbb{Z}/n(1)) \longrightarrow \mathbb{Z}/n \longrightarrow 0, \end{aligned} \tag{1.1}$$

where P denotes the set of height one prime ideals of A, K is the fractional field of A, $k(v)$ is the residue field at $v \in P$, and $r = r(A)$ is an integer depending on the degeneracy of Spec$A$. This result is based upon the isomorphism (Lemma 5.4 of [14])

$$H^4(X, \mathbb{Z}/n(2)) \simeq \mathbb{Z}/n, \tag{1.2}$$

where $X = \mathrm{Spec}A \setminus \{x\}$; and x is the unique maximal ideal of A. A decade later, Matsumi [10] generalized the result by Saito to 3-dimensional complete regular local ring of positive characteristic. Indeed, he proved the exactness of the complex

$$\begin{aligned} 0 &\longrightarrow H^4(K, \mathbb{Z}/\ell(3)) \longrightarrow \bigoplus_{v\in(\mathrm{Spec}A)_2} H^3(k(v), \mathbb{Z}/\ell(2)) \\ &\longrightarrow \bigoplus_{v\in(\mathrm{Spec}A)_1} H^2(k(v), \mathbb{Z}/\ell(1)) \longrightarrow \mathbb{Z}/\ell \longrightarrow 0 \end{aligned} \tag{1.3}$$

for all ℓ prime to char(A), where $(\mathrm{Spec}A)_i$ indicates the set of all points in Spec$A$ of dimension i. Besides, if the ring A is not regular, then the map

$$H^4(K, \mathbb{Z}/\ell(3)) \xrightarrow{\Psi_K} \bigoplus_{v\in(\mathrm{Spec}A)_2} H^3(k(v), \mathbb{Z}/\ell(2))$$

is non-injective.

We proved in Theorem 3 of [5] that $\ker \Psi_K$ contains a sub-group of type $(\mathbb{Z}/\ell)^{r_1'}(A)$, where $r_1'(A)$ is calculated as the $\mathbb{Z}$-rank of the graph of the exceptional fiber of a resolution of Spec$A$. The main tools used in this direction are the isomorphism

$$H^6(X, \mathbb{Z}/\ell(3)) \simeq \mathbb{Z}/\ell \tag{1.4}$$

and the perfect pairing

$$H^i(X, \mathbb{Z}/\ell) \times H^{6-i}(X, \mathbb{Z}/\ell(3)) \longrightarrow H^6(X, \mathbb{Z}/\ell(3)) \simeq \mathbb{Z}/\ell \tag{1.5}$$

for all $i \geq 1$ (Section 0, D2 of [5]).

In order to generalize the previous results, we consider a 2-dimensional complete local ring A whose residue field k is a n-dimensional local field in the sense of Kato–Parshin. To be more precise, let $X = \mathrm{Spec}A \backslash \{x\}$, where x is the unique maximal ideal of A. Then the main result of this paper is the following:

Theorem (Theorem 3.1). *There exist an isomorphism*

$$H^{4+n}(X, \mathbb{Z}/\ell(2+n)) \simeq \mathbb{Z}/\ell \tag{1.6}$$

and a perfect pairing

$$H^1(X, \mathbb{Z}/\ell) \times H^{3+n}(X, \mathbb{Z}/\ell(2+n)) \longrightarrow H^{4+n}(X, \mathbb{Z}/\ell(2+n)) \simeq \mathbb{Z}/\ell \tag{1.7}$$

for all ℓ prime to char(A).

We apply this result to calculate the homologies of the Bloch–Ogus complex associated to A (§0 of [8]). Indeed, let $\pi_1^{c.s}(X)$ be the quotient group of $\pi_1^{ab}(X)$ which classifies abelian c.s coverings of X (see Definition 4.1 below). We then prove the following.

Theorem (Theorem 4.2). *Let A be a 2-dimensional complete normal local ring of positive characteristic whose residue field is a n-dimensional local field. Then the exact sequence*

$$\begin{aligned} 0 \longrightarrow \pi_1^{c.s}(X)/\ell \longrightarrow H^{3+n}(K, \mathbb{Z}/\ell(2+n)) \\ \longrightarrow \bigoplus_{v \in P} H^{2+n}(k(v), \mathbb{Z}/\ell(1+n)) \longrightarrow \mathbb{Z}/\ell \longrightarrow 0 \end{aligned} \tag{1.8}$$

holds.

To prove these results, we rely heavily on the Grothendieck duality theorem for strict local rings (§3) as well as the purity theorem of Fujiwara–Gabber, which we recall next.

In sentences just below Corollary 7.1.7 of [6], Fujiwara confirmed that the absolute cohomological purity in equicharacteristic is true. In other words, we get the following.

Theorem of Fujiwara–Gabber. *Let T be an equicharacteristic Notherian excellent regular scheme and Z be a regular closed subscheme of codimension c. Then for an arbitrary natural number ℓ prime to* char (T), *the following canonical isomorphism*

$$H_Z^i(T, \mathbb{Z}/\ell(j)) \simeq H^{i-2c}(Z, \mathbb{Z}/\ell(j-c)) \tag{1.9}$$

holds.

Finally, we complete the partial duality (1.7) in the case $n = 1$. So, we obtain the following.

Theorem (Theorem 5.1). *Let A be a 2-dimensional normal complete local ring whose residue field is one-dimensional local field. Then, for every ℓ prime to* char(A), *the isomorphism*

$$H^5(X, \mathbb{Z}/\ell(3)) \simeq \mathbb{Z}/\ell \tag{1.10}$$

and the perfect pairing

$$H^i(X, \mathbb{Z}/\ell) \times H^{5-i}(X, \mathbb{Z}/\ell(3)) \longrightarrow H^5(X, \mathbb{Z}/\ell(3)) \simeq \mathbb{Z}/\ell \tag{1.11}$$

hold for all $i \in \{0, \dots, 5\}$.

Our paper is organized as follows. Section 2 is devoted to some notations. Section 3 contains the main theorem of this work concerning the duality of the scheme $X = \mathrm{Spec} A \setminus \{x\}$, where A is a 2-dimensional complete local ring whose residue field is n-dimensional local field and x is the unique maximal ideal of A. In §4, we study the Bloch–Ogus complex associated to A. In §5, we investigate the particular case $n = 1$.

2. Notations

For an abelian group M and a positive integer n, we denote by M/n the cokernel of the map $M \xrightarrow{n} M$. For a scheme Z, and a sheaf $\mathcal{F}$ over the étale site of Z, $H^i(Z, \mathcal{F})$ denotes the i-th étale cohomology group. For a positive integer ℓ invertible on Z, $\mathbb{Z}/\ell(1)$ denotes the sheaf of ℓ-th root of unity and for an integer i, we denote $\mathbb{Z}/\ell(i) = (\mathbb{Z}/\ell(1))^{\otimes i}$.

A local field k is said to be *n-dimensional local* if there exists the following sequence of fields k_i $(1 \le i \le n)$ such that
(i) each k_i is a complete discrete valuation field having k_{i-1} as the residue field of the valuation ring O_{k_i} of k_i, and
(ii) k_0 is a finite field.

For such a field, and for ℓ prime to char(k), the well-known isomorphism (§3.2, Proposition 1 of [9])

$$H^{n+1}(k, \mathbb{Z}/\ell(n)) \simeq \mathbb{Z}/\ell \tag{2.1}$$

and for each $i \in \{0, \dots, n+1\}$ a perfect duality

$$H^i(k, \mathbb{Z}/\ell(j)) \times H^{n+1-i}(k, \mathbb{Z}/\ell(n-j)) \longrightarrow H^{n+1}(k, \mathbb{Z}/\ell(n)) \simeq \mathbb{Z}/\ell \tag{2.2}$$

hold.

For a field L, $K_i(L)$ is the i-th Milnor group. It coincides with the i-th Quillen group for $i \le 2$. For ℓ prime to char L, there is a Galois symbol

$$h^i_{\ell,L}\ K_i L/\ell \longrightarrow H^i(L,\ \mathbb{Z}/\ell(i)) \tag{2.3}$$

which is an isomorphism for $i = 0, 1, 2$ ($i = 2$ is Merkur'jev–Suslin).

3. Local duality

We start this section by a description of the Grothendieck local duality. Let B denote a d-dimensional normal complete local ring with maximal ideal x'. By Cohen structure theorem (§31.1 of [12]), B is a quotient of a regular local ring. Hence Spec B admits a dualizing complex. Now, assume in the first step that the residue field of B is separably closed (B is strictly local). Then, for $X' = \text{Spec } B\backslash\{x'\}$ and for any ℓ prime to char(A), there is a Poincaré duality theory (Exposé I, Remarque 4.7.17 of [18]). Namely, there is a trace isomorphism

$$H^{2d-1}(X', \mathbb{Z}/\ell(d)) \xrightarrow{\sim} \mathbb{Z}/\ell \tag{3.1}$$

and a perfect pairing

$$H^i(X', \mathbb{Z}/\ell) \times H^{2d-1-i}(X', \mathbb{Z}/\ell(d) \longrightarrow H^{2d-1}(X', \mathbb{Z}/\ell(d)) \simeq \mathbb{Z}/\ell \tag{3.2}$$

for all $i \in \{0, \ldots, 2\text{d} - 1\}$.

Assume at this point that the residue field k of B is arbitrary. Let k_s be a separable closure of k. The strict henselization B^{sh} of B (with respect to the separably closed extension k_s of k) at the unique maximal ideal x of B is a strictly local ring. It coincides with the integral closure of B in the maximal unramified extension L^{ur} of the fraction field L of B. Let x' be the maximal ideal of B^{sh} and let $X' = \text{Spec}B^{\text{sh}}\backslash\{x'\}$. So, the Galois group of X' over X is Gal(L^{ur}/L) which is isomorphic to Gal(k_s/k). Then for any integer $j \geq 0$, we get the Hochschield–Serre spectral sequence (Remark 2.21 of [11])

$$E_2^{p,q} = H^p(k, H^q(X', \mathbb{Z}/\ell(j)) \Longrightarrow H^{p+q}(X, \mathbb{Z}/\ell(j)). \tag{3.3}$$

Let A denote a 2-dimensional normal complete local ring whose residue field is an n-dimensional local field. Let x be the unique maximal ideal of A. Then by normality A admits at most one singularity at x in such a way that the scheme $X = \text{Spec}A\backslash\{x\}$ becomes a regular scheme.

In what follows, we put K, the fractional field of A, k, the residue field of K, and P, the set of height one prime ideals of A.

For each $v \in P$ we denote by K_v the completion of K at v and by $k(v)$ the residue field of K_v.

Let $X = \text{Spec}A\backslash\{x\}$ as above. Generalizing (1.2), (1.4) and (1.5), we get the following.

Theorem 3.1. *For all ℓ prime to* char(A), *the isomorphism*

$$H^{4+n}(X, \mathbb{Z}/\ell(2+n)) \simeq \mathbb{Z}/\ell \tag{3.4}$$

and the perfect pairing

$$H^1(X, \mathbb{Z}/\ell) \times H^{3+n}(X, \mathbb{Z}/\ell(2+n)) \longrightarrow H^{4+n}(X, \mathbb{Z}/\ell(2+n)) \simeq \mathbb{Z}/\ell \tag{3.5}$$

occur. Furthermore, this duality is compatible with duality (2.2) *in the sense that the commutative diagram*

$$\begin{array}{ccccccc} H^1(X, \mathbb{Z}/\ell) & \times & H^{3+n}(X, \mathbb{Z}/\ell(2+n) & \longrightarrow & H^{4+n}(X, \mathbb{Z}/\ell(2+n)) & \xrightarrow{\sim} & \mathbb{Z}/\ell \\ \downarrow i^* & & \uparrow i_* & & \uparrow i_* & & \| \\ H^1(k(v), \mathbb{Z}/\ell) & \times & H^{n+1}(k(v), \mathbb{Z}/\ell(n+1)) & \longrightarrow & H^{n+2}(k(v), \mathbb{Z}/\ell(n+1)) & \xrightarrow{\sim} & \mathbb{Z}/\ell \end{array} \tag{3.6}$$

holds, where i^* is the map on H^i induced from the map $v \longrightarrow X$ and i_* is the Gysin map.

Proof. The proof is slightly different from the proof of Theorem 1 in [4]. Let k_s be a separable closure of k. We consider the strict henselization A^{sh} of A (with respect to the separably closed extension k_s of k) at the unique maximal ideal x of A. Then, we denote x' the unique maximal ideal of A^{sh}, $X' = \mathrm{Spec} A^{\mathrm{sh}} \backslash \{x'\}$ and we use the spectral sequence (3.3). As k is an n-dimensional local field, we have $H^{n+2}(k, M) = 0$ for any torsion module M and as X' is of cohomological dimension $2d - 1$ (the last paragraph of Introduction of [17]), we obtain

$$\begin{aligned} H^{4+n}(X, \mathbb{Z}/\ell(2+n)) &\backsimeq H^{n+1}(k, H^3(X', \mathbb{Z}/\ell(2+n)) \\ &\backsimeq H^{n+1}(k, \mathbb{Z}/\ell(n)) \quad \text{by (3.1)} \\ &\backsimeq \mathbb{Z}/\ell \quad \text{by (2.1)}. \end{aligned}$$

We now prove the duality (3.5). The filtration of the group $H^{3+n}(X, \mathbb{Z}/\ell(2+n))$ is

$$H^{3+n}(X, \mathbb{Z}/\ell(2+n)) = E_n^{3+n} \supseteq E_{n+1}^{3+n} \supseteq 0$$

which leads to the exact sequence

$$0 \to E_\infty^{n+1,2} \longrightarrow H^{3+n}(X, \mathbb{Z}/\ell(2+n)) \longrightarrow E_\infty^{n,3} \longrightarrow 0.$$

Since $E_2^{p,q} = 0$ for all $p \geq n+2$ or $q \geq 4$, we see that

$$E_2^{n,3} = E_3^{n,3} = \cdots = E_\infty^{n,3}.$$

The same argument yields

$$E_3^{n+1,2} = E_4^{n+1,2} = \cdots = E_\infty^{n+1,2}$$

and $E_3^{n+1,2} = Co\ker d_2^{n-1,3}$ where $d_2^{n-1,3}$ is the map

$$H^{n-1}(k, H^3(X', \mathbb{Z}/\ell(2+n)) \longrightarrow H^{n+1}(k, H^2(X', \mathbb{Z}/\ell(2+n)).$$

Hence, we obtain the exact sequence

$$\begin{aligned} 0 \to Co\ker d_2^{n-1,3} &\longrightarrow H^{3+n}(X, \mathbb{Z}/\ell(2+n)) \\ &\longrightarrow H^n(k, H^3(X', \mathbb{Z}/\ell(2+n)) \longrightarrow 0. \end{aligned} \tag{3.7}$$

Combining duality (2.2) for k and duality (3.2), we deduce that the group $H^0(k, H^1(X', \mathbb{Z}/\ell))$ is dual to the group $H^{n+1}(k, H^2(X', \mathbb{Z}/\ell(2+n))$ and the group $H^2(k, H^0(X', \mathbb{Z}/\ell))$ is dual to the group $H^{n-1}(k, H^3(X', \mathbb{Z}/\ell(2+n))$. On the other hand, we have the commutative diagram

$$\begin{array}{ccccccc} H^{n-1}(k, H^3(X', \mathbb{Z}/\ell(2+n)) & \times & H^2(k, H^0(X', \mathbb{Z}/\ell)) & \longrightarrow & H^2(k, \mathbb{Z}/\ell(1)) & \xrightarrow{\backsim} & \mathbb{Z}/\ell \\ \downarrow & & \uparrow & & \| & & \| \\ H^{n+1}(k, H^2(X', \mathbb{Z}/\ell(2+n)) & \times & H^0(k, H^1(X', \mathbb{Z}/\ell)) & \longrightarrow & H^2(k, \mathbb{Z}/\ell(1)) & \xrightarrow{\backsim} & \mathbb{Z}/\ell \end{array} \tag{3.8}$$

given by the cup products and the spectral sequence (3.3), using the same argument as (diagram 46 of [1]). We infer that $Co\ker d_2^{n-1,3}$ is the dual of $\ker' d_2^{0,1}$ where $'d_2^{0,1}$ is the boundary map for the spectral sequence ((3.4), $j = 0$)

$$'E_2^{p,q} = H^p(k, H^q(X', \mathbb{Z}/\ell) \Longrightarrow H^{p+q}(X, \mathbb{Z}/\ell). \tag{3.9}$$

Similarly, the group $H^n(k, H^3(X', \mathbb{Z}/\ell(2+n))$ is dual to the group $H^1(k, H^0(X', \mathbb{Z}/\ell))$. The required duality is deduced from the following commutative diagram:

$$\begin{array}{ccccccc} 0 \to Co\ker d_2^{n-1,3} & \longrightarrow & H^{3+n}(X, \mathbb{Z}/\ell(2+n)) & \longrightarrow & H^n(k, H^3(X', \mathbb{Z}/\ell(2+n)) & \longrightarrow 0 \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr & \\ 0 \to (\ker' d_2^{0,1})^\vee & \longrightarrow & (H^1(X, \mathbb{Z}/\ell))^\vee & \longrightarrow & (H^1(k, H^0(X', \mathbb{Z}/\ell)))^\vee & \longrightarrow 0 \end{array}$$

where the upper exact sequence is (3.7) and the bottom exact sequence is the dual of the well-known exact sequence

$$0 \to' E_2^{1,0} \longrightarrow H^1(X, \mathbb{Z}/\ell) \longrightarrow \ker' d_2^{0,1} \longrightarrow 0$$

deduced from the spectral sequence (3.9) and where $(M)^\vee$ denotes the dual $\mathrm{Hom}(M, \mathbb{Z}/\ell)$ for any $\mathbb{Z}/\ell$-module M.

Finally, to obtain the last part of the theorem, we remark that the commutativity of the diagram (3.6) is obtained via a same argument (projection formula (VI 6.5 of [11]) and compatibility of traces (VI 11.1 of [11])) as [1] to establish the commutative diagram in the proof of assertion (ii) at page 791 of [1]. □

COROLLARY 3.2

With the same notations as above, the following commutative diagram

$$\begin{array}{ccc} H^{n+1}(k(v), \mathbb{Z}/\ell(n+1)) & \xrightarrow{i_*} & H^{3+n}(X, \mathbb{Z}/\ell(2+n) \\ \downarrow & & \downarrow \\ (H^1(k(v), \mathbb{Z}/\ell))^\vee & \xrightarrow{(i^*)^\wedge} & (H^1(X, \mathbb{Z}/\ell))^\vee \end{array} \tag{3.10}$$

holds.

Proof. This is a consequence of diagram (3.6). □

The duality (3.5) will be completed (§5) to a general pairing by replacing $H^1(X, \mathbb{Z}/\ell)$ by $H^i(X, \mathbb{Z}/\ell)$; for $0 \le i \le 5$ in the case $n = 1$.

4. The Bloch–Ogus complex

In this section, we investigate the study of the Bloch–Ogus complex associated to the ring A considered previously. So, let A be a 2-dimensional normal complete local ring whose residue field is an n-dimensional local field. Next, we define a group which appears in the homologies of the associated Bloch–Ogus complex of A.

DEFINITION 4.1

Let Z be a Noetherian scheme. A finite etale covering $f: W \to Z$ is called a c.s covering if for any closed point z of Z, $z \times_Z W$ is isomorphic to a finite scheme-theoretic sum of copies of z. We denote $\pi_1^{c.s}(Z)$ the quotient group of $\pi_1^{ab}(Z)$ which classifies abelian c.s coverings of Z.

As above, let $X = \mathrm{Spec} A \setminus \{x\}$. The group $\pi_1^{c.s}(X)/\ell$ is the dual of the kernel of the map

$$H^1(X, \mathbb{Z}/\ell) \longrightarrow \prod_{v \in P} H^1(k(v), \mathbb{Z}/\ell) \tag{4.1}$$

(definition and sentence just below section 2 of [14]). Now, we are able to calculate the homologies of the Bloch–Ogus complex associated to the ring A.

Theorem 4.2. *For all ℓ prime to the characteristic of A, the following sequence is exact.*

$$\begin{aligned} 0 \longrightarrow \pi_1^{c.s}(X)/\ell \longrightarrow H^{n+3}(K, \mathbb{Z}/\ell(n+2)) \\ \longrightarrow \bigoplus_{v \in P} H^{n+2}(k(v), \mathbb{Z}/\ell(n+1)) \longrightarrow \mathbb{Z}/\ell \longrightarrow 0. \end{aligned} \tag{4.2}$$

Proof. Consider the localization sequence on $X = \mathrm{Spec} A \setminus \{x\}$,

$$\begin{aligned} \ldots \to H^i(X, \mathbb{Z}/\ell(n+2)) \longrightarrow H^i(K, \mathbb{Z}/\ell(n+2)) \\ \longrightarrow \bigoplus_{v \in P} H_v^{i+1}(X, \mathbb{Z}/\ell(n+2)) \to \ldots. \end{aligned}$$

Firstly, for any $v \in P$, we have the isomorphisms

$$H_v^i(X, \mathbb{Z}/\ell(2+n)) \simeq H_v^i(\mathrm{Spec} A_v, \mathbb{Z}/\ell(2+n))$$

by excision. Secondly, we can apply the purity theorem of Fujiwara–Gabber (from Introduction) for $Z = v$, $T = \mathrm{Spec} A_v$ and we find the isomorphisms

$$H_v^{3+n}(\mathrm{Spec} A_v, \mathbb{Z}/\ell(2+n)) \simeq H^{1+n}(k(v), \mathbb{Z}/\ell(1+n))$$

and

$$H_v^{4+n}(\mathrm{Spec} A_v, \mathbb{Z}/\ell(2+n)) \simeq H^{2+n}(k(v), \mathbb{Z}/\ell(1+n))$$

which lead to the isomorphisms

$$H_v^{3+n}(X, \mathbb{Z}/\ell(2+n)) \simeq H^{1+n}(k(v), \mathbb{Z}/\ell(1+n))$$

and

$$H_v^{4+n}(X, \mathbb{Z}/\ell(2+n)) \simeq H^{2+n}(k(v), \mathbb{Z}/\ell(1+n)).$$

Hence we derive the exact sequence

$$\begin{aligned} \bigoplus_{v \in P} H^{1+n}(k(v), \mathbb{Z}/\ell(1+n)) \xrightarrow{g} H^{3+n}(X, \mathbb{Z}/\ell(2+n)) \\ \longrightarrow H^{3+n}(K, \mathbb{Z}/\ell(2+n)) \\ \longrightarrow \bigoplus_{v \in P} H^{2+n}(k(v), \mathbb{Z}/\ell(1+n)) \\ \longrightarrow H^{4+n}(X, \mathbb{Z}/\ell(2+n)) \longrightarrow 0. \end{aligned}$$

The last zero on the right is a consequence of the vanishing of the group $H^{4+n}(K, \mathbb{Z}/\ell(2+n))$. Indeed, A is finite over $O_L[[T]]$ for some complete discrete valuation field L having the same residue field with A (§31 of [12]). By Serre (ch. I, Proposition 14 of [16]), $cd_\ell(A) \leq cd_\ell(O_L[[T]])$ and by Gabber [7], $cd_\ell(O_L[[T]]) = n + 3$ using the fact that $cd_\ell(k) = n + 1$.

Now, by the right square of the diagram (3.6), the Gysin map

$$\bigoplus_{v \in P} H^{2+n}(k(v), \mathbb{Z}/\ell(1+n)) \longrightarrow H^{4+n}(X, \mathbb{Z}/\ell(2+n))$$

can be replaced by the map $\oplus_{v \in P} H^{2+n}(k(v), \mathbb{Z}/\ell(1+n)) \longrightarrow \mathbb{Z}/\ell$ after composing with the trace isomorphism $H^{4+n}(X, \mathbb{Z}/\ell(2+n)) \simeq \mathbb{Z}/\ell$ (3.4). So, we obtain the exact sequence

$$0 \longrightarrow Co\ker g \longrightarrow H^{3+n}(K, \mathbb{Z}/\ell(2+n))$$
$$\longrightarrow \bigoplus_{v \in P} H^{2+n}(k(v), \mathbb{Z}/\ell(1+n)) \longrightarrow \mathbb{Z}/\ell \longrightarrow 0.$$

Finally, in view of the commutative diagram (3.10), we deduce that $Co\ker g$ equals to the group $\pi_1^{c.s}(X)/\ell$ taking in account (4.1). □

Remark 4.3. Theorem 4.2 can be seen as a generalization of the regular equal characteristic case. In fact, in this case, the injectivity of the first arrow in (4.2) follows in a straightforward manner from a general result of Panin [13] on the cohomology of the Cousin complex, together with the vanishing of $H^r(X, -) = H^r(k, -)$ for $i \geq \dim k + 2$.

Remark 4.4.

(1) The case $n = 1$ leads to the following exact sequence

$$0 \longrightarrow H^4(K, \mathbb{Z}/\ell(3)) \longrightarrow \bigoplus_{v \in P} H^3(k(v), \mathbb{Z}/\ell(2)) \longrightarrow \mathbb{Z}/\ell \longrightarrow 0$$

which is considered in [4].

(2) The case $n = 0$, implies the following exact sequence.

$$0 \longrightarrow \pi_1^{c.s}(X)/\ell \longrightarrow H^3(K, \mathbb{Z}/\ell(2))$$
$$\longrightarrow \bigoplus_{v \in P} H^2(k(v), \mathbb{Z}/\ell(1)) \longrightarrow \mathbb{Z}/\ell \longrightarrow 0 \qquad (4.3)$$

already obtained by Saito [14]. Recently, Yoshida [19] provided an alternative approach which includes the equal characteristic case.

Analogy with curves

The kernel of the first arrow can be calculated in some cases. In the case $n = 0$, Saito proceeded as follows: Choose a resolution of Spec$A$, by which we mean a proper birational morphism

$$f\colon X \longrightarrow \mathrm{Spec} A$$

which satisfies the following conditions: $\mathcal{X}$ is a two-dimensional regular scheme, $Y := (f^{-1}(x))_{\mathrm{red}}$ is a geometrically connected curve over A/x such that any irreducible component of Y is regular and it has only ordinary double points as singularity.

In [14], Saito announced the following two basic facts:

(1) The specialization map (Proposition 2.2 of [14])

$$\pi_1^{ab}(X) \longrightarrow \pi_1^{ab}(Y) \tag{4.4}$$

induces an isomorphism

$$\pi_1^{c.s}(X) \simeq \pi_1^{c.s}(Y). \tag{4.5}$$

(2) There exists an isomorphism (Proposition 2.2 of [14])

$$\pi_1^{c.s}(Y) \simeq (\hat{\mathbb{Z}})^r, \tag{4.6}$$

where r is the rank of A (Definition 2.4 of [14]).

To prove these facts, Saito refers to his previous work [15], and particularly Proposition 2.2 and Theorem 2.4, where the results are obtained for curves over usually local fields. In fact, this can be explained as follows: As pointed out (the end of section 0 in [5]), there is a basic analogy. The scheme $X = \mathrm{Spec} A \backslash \{x\}$ is not a variety, but a family of curves depending on a local uniformizer π and degenerating on Y. We pass from x the henselian line $\mathrm{Spec} W(k_0)$, k_0 is the residue field of A and $W(k_0)$ is the ring of Witt vectors. Then, we let $K = \mathrm{Frac}(W(k_0))$ and X can be considered as an analogue of a variety over the one-dimensional local field. This analogy is confirmed by the following isomorphisms and dualities:

- The ring A is two-dimensional with finite residue field (Saito case): The first part of Lemma 5.4 of [14] and proof of Proposition 1.2, sequence (1.3) and what follows in [15].
- The ring A is three-dimensional with finite residue field: Here X becomes a family of surfaces, see duality (D2), section 0 in [5] and duality (b), section II, 2' in [2].
- The ring A is two-dimensional with n-dimensional local field as residue field. Dualities (1.6) and (1.7) of this paper and dualities in Theorem 2.1 of [3].

To conclude this discussion, we deduce a way to calculate explicitly the group $\pi_1^{c.s}(X)$ for the ring considered in this section is to begin by studying the arithmetic of class field theory for curves over two-dimensional local ring.

We summarize, the previous comparison in the following diagram:

Let A be a two-dimensional local ring with n-dimensional local field as residue field

	$n = 0$	$n = 1$
outils	{regular local ring} Hasse principle Saito, Panin	{monodromy filtration} {two-dim local field}
	$\Downarrow$	$\Downarrow$
curves varieties	{curves over local field} $\pi_1^{c.s}(X)$ explicit Bloch, Saito, Yoshida	{curves over two-dim local field} $\pi_1^{c.s}(X)$ explicit in preparation
	$\Downarrow$	$\Downarrow$
schemes	{scheme $\mathrm{Spec} A \backslash \{x\}$} $\pi_1^{c.s}(X)$ explicit Kato, Saito	{scheme $\mathrm{Spec} A \backslash \{x\}$} ?

5. The case $N = 1$

Let A denote a 2-dimensional complete normal local ring of positive characteristic whose residue field is a one-dimensional local field. The aim of this section is to complete the duality (3.5) for $i \geq 1$. We prove the following.

Theorem 5.1. *For all ℓ prime to* $\mathrm{char}(A)$, *the isomorphism*

$$H^5(X, \mathbb{Z}/\ell(3)) \simeq \mathbb{Z}/\ell \tag{5.1}$$

and the perfect pairing

$$H^i(X, \mathbb{Z}/\ell) \times H^{5-i}(X, \mathbb{Z}/\ell(3)) \longrightarrow H^5(X, \mathbb{Z}/\ell(3)) \simeq \mathbb{Z}/\ell \tag{5.2}$$

hold for all $i \in \{0, \dots, 5\}$.

Proof. The first isomorphism is given by (3.4). Next, we proceed to the second part of the theorem. As in the proof of Theorem 3.1, we consider the strict henselization A^{sh} of A (with respect to the separably closed extension k_s of k) at x. If x' is the maximal ideal of A^{sh}, we recall that we denote $X' = \mathrm{Spec}A^{\mathrm{sh}} \backslash \{x'\}$ and we consider the spectral sequence (3.3). The filtration of the group $H^i(X, \mathbb{Z}/\ell(3))$ is

$$H^i(X, \mathbb{Z}/\ell(3)) = E_0^i \supseteq E_1^i \supseteq E_2^i \supseteq 0,$$

where the quotients are given by

$$E_0^i/E_1^i \simeq E_\infty^{0,i} \simeq \ker d_2^{0,i}$$

$$E_1^i/E_2^i \simeq E_\infty^{1,i-1} \simeq E_2^{1,i-1}, \text{ and}$$

$$E_2^i \simeq E_\infty^{2,i-2} \simeq Co\ker d_2^{0,i-1}.$$

The same computation is true for the group $H^{5-i}(X, \mathbb{Z}/\ell)$ and the filtration

$$H^{5-i}(X, \mathbb{Z}/\ell) =' E_0^{5-i} \supseteq' E_1^{5-i} \supseteq' E_2^{5-i} \supseteq 0$$

by considering the spectral sequence ((3.4), $j = 0$).

Now, combining duality (2.2) and duality (3.2) we observe that the group $E_2^{0,j}$ is dual to the group ${}'E_2^{2,3-j}$ and the group $E_2^{1,j}$ is dual to the group ${}'E_2^{1,3-j}$ for all $0 \leq j \leq 3$. On the other hand, we have the commutative diagram

$$\begin{array}{ccccccc} H^0(k, H^i(X', \mathbb{Z}/\ell(3)) & \times & H^2(k, H^{3-i}(X', \mathbb{Z}/\ell)) & \longrightarrow & H^2(k, \mathbb{Z}/\ell(1)) & \xrightarrow{\sim} & \mathbb{Z}/\ell \\ \downarrow & & \uparrow & & \| & & \| \\ H^2(k, H^{i-1}(X', \mathbb{Z}/\ell(3)) & \times & H^0(k, H^{4-i}(X', \mathbb{Z}/\ell)) & \longrightarrow & H^2(k, \mathbb{Z}/\ell(1)) & \xrightarrow{\sim} & \mathbb{Z}/\ell \end{array}$$

given by the cup products and the spectral sequence (3.4), using the same argument as (diagram 46 of [1]). We infer that $Co\ker d_2^{0,i-1}$ is the dual of $\ker' d_2^{0,5-i}$ and $\ker d_2^{0,i}$ is the dual of $Co\ker' d_2^{0,2d-i}$, where ${}'d_2^{p,q}$ is the boundary map for the spectral sequence ((3.4), $j = 0$)

$${}'E_2^{p,q} = H^p(k, H^q(X', \mathbb{Z}/\ell)) \Longrightarrow H^{p+q}(X, \mathbb{Z}/\ell).$$

This is illustrated by the following diagram:

$$
\begin{array}{ccccc}
H^i X, \mathbb{Z}/\ell(3)) = E_0^i & & & & 0 \\
| & \Big\} \ker d_2^{0,i} & \longleftrightarrow & Co\ker' d_2^{0,24-i} \Big\{ & | \\
E_1^i & & & & {}'E_2^{5-i} \\
| & \Big\} E_2^{1,i-1} & \longleftrightarrow & {}'E_2^{1,24-i} \Big\{ & | \\
E_2^i & & & & {}'E_1^{5-i} \\
| & \Big\} Co\ker d_2^{0,i-1} & \longleftrightarrow & \ker' d_2^{0,5-i} \Big\{ & | \\
0 & & & & {}'E_0^{5-i} = H^{5-i}(X, \mathbb{Z}/\ell)
\end{array}, \tag{5.3}
$$

where each pair of groups which are combined by $\longleftrightarrow$ consists of a group and its dual group.

We begin by calculating the dual group of E_1^i, using the following commutative diagram

$$
\begin{array}{ccccccc}
0 \to & E_2^i & \to & E_1^i & \to & E_1^i/E_2^i & \to 0 \\
& \downarrow \wr & & \downarrow & & \downarrow \wr & \\
0 \to & ({}'E_0^{5-i}/{}'E_1^{5-i})^\vee & \to & ({}'E_0^{5-i}/{}'E_2^{5-i})^\vee & \to & ({}'E_1^{5-i}/{}'E_2^{5-i})^\vee & \to 0
\end{array},
$$

where $(M)^\vee$ denotes the dual $\mathrm{Hom}(M, \mathbb{Z}/\ell)$ for any $\mathbb{Z}/\ell$-module M and where the left and right vertical isomorphisms are explained by the previous diagram. This yields that

$$
E_1^i \simeq ({}'E_0^{5-i}/{}'E_2^{5-i})^\vee. \tag{5.4}
$$

Finally, the required duality between $H^i(X, \mathbb{Z}/\ell(d+1)) = E_0^i$ and ${}'E_0^{5-i} = H^{5-i}(X, \mathbb{Z}/\ell)$ follows from the following commutative diagram

$$
\begin{array}{ccccccc}
0 \to & E_1^i & \to & E_0^i & \to & E_0^i/E_1^i & \to 0 \\
& \downarrow \wr & & \downarrow \wr & & \downarrow \wr & \\
0 \to & ({}'E_0^{5-i}/{}'E_2^{5-i})^\vee & \to & ({}'E_0^{5-i})^\vee & \to & ({}'E_2^{5-i})^\vee & \to 0
\end{array},
$$

where the right vertical isomorphism is given by (5.3) and the left vertical isomorphism is the isomorphism (5.4). □

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