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An optimal version of an inequality involving the third symmetric means

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Abstract. Let $(GA)_n^{[k]}(a)$, $A_n(a)$, $G_n(a)$ be the third symmetric mean of k degree, the arithmetic and geometric means of $a_1, \ldots, a_n$ ($a_i > 0, i = 1, \ldots, n$), respectively. By means of descending dimension method, we prove that the maximum of p is $\frac{k-1}{n-1}$ and the minimum of q is $\frac{n}{n-1}\left(\frac{k-1}{k}\right)^{\frac{k}{n}}$ so that the inequalities

$$(G_n(a))^{1-p}(A_n(a))^p \le (GA)_n^{[k]}(a) \le (1-q)G_n(a) + qA_n(a) \quad (2 \le k \le n-1)$$

hold.

Keywords. Third symmetric mean of k degree; optimal values; inequality; descending dimension method.

1. Introduction and main results

When n is an integer ≥ 1, let R_{++}^n be the set of n tuples $a = (a_1, a_2, \ldots, a_n)$ of real numbers $a_1, a_2, \ldots, a_n$, each > 0. The r-th power-mean of $a = (a_1, a_2, \ldots, a_n)$ has been considered in [2, 7, 9] and its first, second and third symmetric means of k degree (pp. 50, 51 of [2]) are defined by

$$P_n^{[k]}(a) = \left[\binom{n}{k}^{-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k} a_{i_j}\right]^{\frac{1}{k}};$$

$$(AG)_n^{[k]}(a) = \binom{n}{k}^{-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \left(\prod_{j=1}^{k} a_{i_j}\right)^{\frac{1}{k}};$$

$$(GA)_n^{[k]}(a) = \left[\prod_{1 \le i_1 < \cdots < i_k \le n} \left(k^{-1} \sum_{j=1}^{k} a_{i_j}\right)\right]^{\frac{1}{\binom{n}{k}}}.$$

In particular, $M_n^{[1]}(a) = P_n^{[1]}(a) = (AG)_n^{[1]}(a) = (GA)_n^{[n]}(a) = A_n(a) = (a_1 + \cdots + a_n)/n$, $M_n^{[0]}(a) = P_n^{[n]}(a) = (AG)_n^{[n]}(a) = (GA)_n^{[1]}(a) = G_n(a) = \sqrt[n]{a_1 \ldots a_n}$, where $A_n(a), G_n(a)$ are the arithmetic and geometric means of $a = (a_1, a_2, \ldots, a_n), \binom{n}{k} = \frac{n!}{k!(n-k)!}$.

It has been pointed out in [1] that means are central to the study of inequalities and have a number of applications. The large amount of work in the literature on means give them a special place in the research on inequalities. By now a number of beautiful results have been obtained.

Let $a \in R_{++}^n$. We then have

$$A_n(a) = P_n^{[1]}(a) \geq \cdots \geq P_n^{[k]}(a) \geq \cdots \geq P_n^{[n]}(a) = G_n(a); \quad \text{pp. 65, 66 of [3, 9]} \quad (1.1)$$

$$A_n(a) = (AG)_n^{[1]}(a) \geq \cdots \geq (AG)_n^{[k]}(a) \geq \cdots \geq (AG)_n^{[n]}(a) = G_n(a); \quad [7] \quad (1.2)$$

$$A_n(a) = (GA)_n^{[n]}(a) \geq \cdots \geq (GA)_n^{[k]}(a) \geq \cdots \geq (GA)_n^{[1]}(a) = G_n(a); \quad [7] \quad (1.3)$$

In [11], the inequalities

$$M_n^{[p]}(a) \leq P_n^{[k]}(a) \leq M_n^{[q]}(a), \quad (2 \leq k \leq n-1) \quad (1.4)$$

is shown to hold, where the maximum of p is 0, and the minimum of q is $\frac{2(\ln n - \ln(n-1))}{\ln n - \ln(n-2)}$ as also the inequality

$$(G_n(a))^{1-p}(A_n(a))^p \leq P_n^{[k]}(a) \leq (1-q)G_n(a) + qA_n(a) \quad (2 \leq k \leq n-1) \quad (1.5)$$

hold, where the maximum of p is $\frac{n-k}{k(n-1)}$, and the minimum of q is $\frac{n}{n-1}\sqrt[k]{1-\frac{k}{n}}$ (pp. 65, 66 of [3], [12]).

Let $f(x) = -\ln x$. Then Corollary 1 of [13] shows that for $0 < a_1 \leq \cdots \leq a_n$, $2 \leq k \leq n-1$, $p \leq \frac{n-k}{k(n-1)}\left(\frac{a_1}{a_n}\right)^2$ and $q \geq \frac{n-k}{k(n-1)}\left(\frac{a_n}{a_1}\right)^2$, we have

$$(G_n(a))^{1-p}(A_n(a))^p \leq (GA)_n^{[k]}(a) \leq (G_n(a))^{1-q}(A_n(a))^q. \quad (1.6)$$

The optimal values of the real λ have been discussed in refs [8, 10, 16] which make the inequality

$$(M_n^{[\alpha]}(a))^{1-\lambda}(M_n^{[\beta]}(a))^{\lambda} \leq (M_n^{[\theta]}(a)) \quad (\alpha \leq \theta \leq \beta) \quad (1.7)$$

and its reverse hold.

In [4], the inequalities

$$(G_n(a))^{1-p}(A_n(a))^p \leq (GA)_n^{[n-1]}(a) \leq (1-q)G_n(a) + qA_n(a) \quad (2 \leq k \leq n-1) \quad (1.8)$$

hold, where the maximum of the real p is $\frac{n-2}{n-1}$, and the minimum of q is $n(n-2)^{1-\frac{1}{n}}(n-1)^{\frac{1}{n}-2}$.

The purpose of this paper is to generalize the inequalities (1.8). More precisely, our main result is as follows.

Theorem 1.1. *Suppose $a \in R^n_{++}$, $2 \le k \le n-1$, then*

$$(G_n(a))^{1-p}(A_n(a))^p \le (GA)^{[k]}_n(a) \le (1-q)G_n(a) + qA_n(a), \tag{1.9}$$

where the maximum of the real p is $\frac{k-1}{n-1}$, and the minimum of q is $\frac{n}{n-1}\left(\frac{k-1}{k}\right)^{\frac{k}{n}}$. The equalities hold if and only if $a_1 = \cdots = a_n$.

The inequalities in (1.9) appear in a variety of contexts, for instance, in matrix theory (see [4, 5]). Our proof of Theorem 1.1 is based on the descending dimension method developed in [6, 14].

2. Several lemmas

To prove Theorem 1.1, the following preliminary result will be needed.

Lemma 2.1. *If $0 \le p, q \le 1$, $2 \le k \le n-1$, and $a = (a_1, \ldots, a_n)$ $(a_i > 0, i = 1, \ldots, n)$ is a critical point of function $F_r(a)$ on D, then any $n-1$ numbers of $a_1, \ldots, a_n$ must be equal, where*

$$D = \{a | a \in R^n_{++}, a_1 + a_2 + \cdots + a_n = n\},$$

$$F_1(a) = \frac{1}{\binom{n}{k}} \sum_{1 \le i_1 < \cdots < i_k \le n} \ln \frac{a_{i_1} + \cdots + a_{i_k}}{k} - (1-p) \cdot \frac{\ln a_1 + \cdots + \ln a_n}{n},$$

$$F_2(a) = \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \frac{a_{i_1} + \cdots + a_{i_k}}{k} \right)^{\frac{1}{\binom{n}{k}}} - (1-q)\sqrt[n]{a_1 \ldots a_n} - q.$$

Proof. Suppose on the contrary three of the numbers $a_1, a_2, \ldots, a_n$ are pairwise unequal, and, without loss of generality, assume that these are a_1, a_2, a_3.

Let Lagrange function $L_r = F_r(a) + \lambda_r(a_1 + \cdots + a_n - n)$, then

$$\frac{\partial L_r}{\partial a_j} = \frac{\partial F_r(a)}{\partial a_j} + \lambda_r = 0, \quad j = 1, \ldots, n. \tag{2.1}$$

When $r = 1$,

$$\frac{\partial F_1(a)}{\partial a_1} = \frac{1}{\binom{n}{k}} \sum_{2 \le i_1 < \cdots < i_{k-1} \le n} \frac{1}{a_1 + a_{i_1} + \cdots + a_{i_{k-1}}} - \frac{1-p}{n} \cdot \frac{1}{a_1}.$$

When $r = 2$,

$$\frac{\partial F_2(a)}{\partial a_1} = \frac{(GA)^{[k]}_n(a)}{\binom{n}{k}} \sum_{2 \le i_1 < \cdots < i_{k-1} \le n} \frac{1}{a_1 + a_{i_1} + \cdots + a_{i_{k-1}}} - \frac{1-q}{n} \cdot \frac{G_n(a)}{a_1}.$$

Let

$$M_r(a) = \begin{cases} \frac{1}{\binom{n}{k}}, & r = 1, \\ (GA)_n^{[k]}(a)/\binom{n}{k}, & r = 2, \end{cases}$$

and

$$N_r(a) = \begin{cases} \frac{1-p}{n}, & r = 1, \\ (1-q)G_n(a)/n, & r = 2. \end{cases}$$

Then

$$\begin{aligned} \frac{\partial F_r(a)}{\partial a_1} &= M_r(a) \sum_{2\le i_1<\cdots<i_{k-1}\le n} \frac{1}{a_1 + a_{i_1} + \cdots + a_{i_{k-1}}} - \frac{N_r(a)}{a_1} \\ &= M_r(a)\left(\sum_{3\le i_1<\cdots<i_{k-1}\le n} \frac{1}{a_1 + \sum\limits_{j=1}^{k-1} a_{i_j}} \right. \\ &\quad \left. + \sum_{3\le i_1<\cdots<i_{k-2}\le n} \frac{1}{a_1 + a_2 + \sum\limits_{j=1}^{k-2} a_{i_j}} \right) - \frac{N_r(a)}{a_1} \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \frac{\partial F_r(a)}{\partial a_2} &= M_r(a)\left(\sum_{3\le i_1<\cdots<i_{k-1}\le n} \frac{1}{a_2 + \sum\limits_{j=1}^{k-1} a_{i_j}} \right. \\ &\quad \left. + \sum_{3\le i_1<\cdots<i_{k-2}\le n} \frac{1}{a_1 + a_2 + \sum\limits_{j=1}^{k-2} a_{i_j}} \right) - \frac{N_r(a)}{a_2}. \end{aligned} \tag{2.3}$$

When $n > 3$, by (2.1), (2.2) and (2.3), we have

$$\left(\frac{\partial F_r(a)}{\partial a_1} - \frac{\partial F_r(a)}{\partial a_2} \right) \Big/ (a_1 - a_2) = 0,$$

that is,

$$M_r(a) \cdot \sum_{3\le i_1<\cdots<i_{k-1}\le n} \frac{1}{\left(a_1 + \sum\limits_{j=1}^{k-1} a_{i_j}\right)\left(a_2 + \sum\limits_{j=1}^{k-1} a_{i_j}\right)} - \frac{N_r(a)}{a_1 a_2} = 0$$

which is the same as

$$M_r(a)\cdot\sum_{3\le i_1<\cdots<i_{k-1}\le n}\frac{a_1a_2}{\left(a_1+\sum_{j=1}^{k-1}a_{i_j}\right)\left(a_2+\sum_{j=1}^{k-1}a_{i_j}\right)}=N_r(a),$$

from which we have

$$\sum_{4\le i_1<\cdots<i_{k-1}\le n}\frac{a_1a_2}{\left(a_1+\sum_{j=1}^{k-1}a_{i_j}\right)\left(a_2+\sum_{j=1}^{k-1}a_{i_j}\right)}$$
$$+\sum_{4\le i_1<\cdots<i_{k-2}\le n}\frac{a_1a_2}{\left(a_1+a_3+\sum_{j=1}^{k-2}a_{i_j}\right)\left(a_2+a_3+\sum_{j=1}^{k-2}a_{i_j}\right)}=\frac{N_r(a)}{M_r(a)}. \tag{2.4}$$

In a similar manner, from $\left(\frac{\partial F_r(a)}{\partial a_1}-\frac{\partial F_r(a)}{\partial a_3}\right)/(a_1-a_3)=0$, we have

$$\sum_{4\le i_1<\cdots<i_{k-1}\le n}\frac{a_1a_3}{\left(a_1+\sum_{j=1}^{k-1}a_{i_j}\right)\left(a_3+\sum_{j=1}^{k-1}a_{i_j}\right)}$$
$$+\sum_{4\le i_1<\cdots<i_{k-2}\le n}\frac{a_1a_3}{\left(a_1+a_2+\sum_{j=1}^{k-2}a_{i_j}\right)\left(a_3+a_2+\sum_{j=1}^{k-2}a_{i_j}\right)}=\frac{N_r(a)}{M_r(a)}. \tag{2.5}$$

On subtracting (2.4) from (2.5), we get

$$\sum_{4\le i_1<\cdots<i_{k-1}\le n}\frac{a_1a_2\left(a_3+\sum_{j=1}^{k-1}a_{i_j}\right)-a_1a_3\left(a_2+\sum_{j=1}^{k-1}a_{i_j}\right)}{\left(a_1+\sum_{j=1}^{k-1}a_{i_j}\right)\left(a_2+\sum_{j=1}^{k-1}a_{i_j}\right)\left(a_3+\sum_{j=1}^{k-1}a_{i_j}\right)}$$
$$+\sum_{4\le i_1<\cdots<i_{k-2}\le n}\frac{a_1a_2\left(a_1+a_2+\sum_{j=1}^{k-2}a_{i_j}\right)-a_1a_3\left(a_1+a_3+\sum_{j=1}^{k-2}a_{i_j}\right)}{\left(a_1+a_2+\sum_{j=1}^{k-2}a_{i_j}\right)\left(a_1+a_3+\sum_{j=1}^{k-2}a_{i_j}\right)\left(a_2+a_3+\sum_{j=1}^{k-2}a_{i_j}\right)}=0. \tag{2.6}$$

On dividing both sides of (2.6) by $(a_2 - a_3)$, we get

$$\sum_{4\le i_1<\cdots<i_{k-1}\le n} \frac{a_1 \sum_{j=1}^{k-1} a_{i_j}}{\left(a_1+\sum_{j=1}^{k-1} a_{i_j}\right)\left(a_2+\sum_{j=1}^{k-1} a_{i_j}\right)\left(a_3+\sum_{j=1}^{k-1} a_{i_j}\right)}$$

$$+\sum_{4\le i_1<\cdots<i_{k-2}\le n} \frac{a_1^2 + a_1(a_2+a_3) + a_1\left(\sum_{j=1}^{k-2} a_{i_j}\right)}{\left(a_1+a_2+\sum_{j=1}^{k-2} a_{i_j}\right)\left(a_1+a_3+\sum_{j=1}^{k-2} a_{i_j}\right)\left(a_2+a_3+\sum_{j=1}^{k-2} a_{i_j}\right)} = 0. \tag{2.7}$$

Clearly, (2.7) gives a contradiction, since its left-hand side is necessarily > 0.

When $n = 3$, eqs (2.4) and (2.5) become

$$\frac{a_1 a_2}{(a_1+a_3)(a_2+a_3)} = \frac{N_r(a)}{M_r(a)}, \tag{2.8}$$

$$\frac{a_1 a_3}{(a_1+a_2)(a_3+a_2)} = \frac{N_r(a)}{M_r(a)}. \tag{2.9}$$

On subtracting (2.9) from (2.8), we obtain $a_2 = a_3$, which is absurd, thus completing the proof. □

Lemma 2.2. *If* $2 \le k \le n-1$, $p = \frac{k-1}{n-1}$, *then for* $t > 0$,

$$f_1(t) := \frac{k}{n}\ln\frac{t+k-1}{k} - \frac{1-p}{n}\ln t - p\ln\frac{t+n-1}{n} \ge 0. \tag{2.10}$$

Proof.

$$\begin{aligned} f_1'(t) &= \frac{k}{n}\cdot\frac{1}{t+k-1} - \frac{1-p}{n}\cdot\frac{1}{t} - p\cdot\frac{1}{t+n-1} \\ &= \frac{k}{n}\cdot\left(\frac{1}{t+k-1}-\frac{1}{k}\right) - \frac{1-p}{n}\cdot\left(\frac{1}{t}-1\right) - p\cdot\left(\frac{1}{t+n-1}-\frac{1}{n}\right) \\ &= \frac{k}{n}\cdot\frac{1-t}{k(t+k-1)} - \frac{1-p}{n}\cdot\frac{1-t}{t} - p\cdot\frac{1-t}{n(t+n-1)} \\ &= \frac{1-t}{n}\left(\frac{1}{t+k-1} - \frac{1-p}{t} - \frac{p}{t+n-1}\right) \\ &= \frac{1-t}{n}\cdot\frac{[1-k+p(n-1)]t-(1-p)(k-1)(n-1)}{t(t+k-1)(t+n-1)} \qquad \left(\text{where } p = \frac{k-1}{n-1}\right) \\ &= \frac{(n-k)(k-1)(n-1)(t-1)}{n(n-1)t(t+k-1)(t+n-1)} \end{aligned} \tag{2.11}$$

from which it follows that, $f(t)$ is strictly monotone decreasing on the interval (0,1], and that it is strictly monotone increasing on the interval $[1,+\infty)$. Thus we infer

$$f_1(t) := \frac{k}{n}\ln\frac{t+k-1}{k} - \frac{1-p}{n}\ln t - p\ln\frac{t+n-1}{n} \geq f_1(1) = 0.$$

□

Lemma 2.3. *If* $2 \leq k \leq n-1$, $q = \frac{n}{n-1}\cdot\left(\frac{k-1}{k}\right)^{\frac{k}{n}}$, *then for* $t > 0$,

$$f_2(t) := \left(\frac{t+k-1}{k}\right)^{\frac{k}{n}} - (1-q)t^{\frac{1}{n}} - q\cdot\frac{t+n-1}{n} \leq 0. \tag{2.12}$$

Proof. When $t = 1$, equality holds in (2.12).

Let $0 < t \neq 1$. From Bernoulli's inequality [15], we get

$$t^{\frac{1}{n}} - \frac{t+n-1}{n} < 0.$$

Consequently, the inequality (2.12) is the same as

$$q \geq \frac{\left(\frac{t+k-1}{k}\right)^{\frac{k}{n}} - t^{\frac{1}{n}}}{\frac{t+n-1}{n} - t^{\frac{1}{n}}} = \frac{\left(\frac{t+k-1}{kt^{\frac{1}{k}}}\right)^{\frac{k}{n}} - 1}{\frac{t+n-1}{nt^{\frac{1}{n}}} - 1} = \frac{g(t)}{h(t)}, \tag{2.13}$$

where $g(t) = \left(\frac{t+k-1}{kt^{\frac{1}{k}}}\right)^{\frac{k}{n}} - 1$ and $h(t) = \frac{t+n-1}{nt^{\frac{1}{n}}} - 1$. On noting that $g(1) = 0$, $h(1) = 0$, and that

$$g'(t) = \frac{k}{n}\cdot\left(\frac{t+k-1}{kt^{\frac{1}{k}}}\right)^{\frac{k}{n}-1}\cdot\frac{(1-\frac{1}{k})t^{-\frac{1}{k}} + (k-1)\left(-\frac{1}{k}\right)t^{-\frac{1}{k}-1}}{k}$$

$$= \frac{(k-1)}{kn}\cdot t^{-\frac{1}{k}-1}\cdot\left(\frac{t+k-1}{kt^{\frac{1}{k}}}\right)^{\frac{k}{n}-1}\cdot(t-1),$$

$$h'(t) = \frac{(n-1)t^{-\frac{1}{n}-1}\cdot(t-1)}{n^2},$$

we have

$$\frac{g'(t)}{h'(t)} = \frac{n(k-1)}{k(n-1)}\cdot t^{\frac{1}{n}-\frac{1}{k}}\cdot\left(\frac{t+k-1}{kt^{\frac{1}{k}}}\right)^{\frac{k}{n}-1} = \frac{n(k-1)}{k(n-1)}\cdot\left(\frac{t+k-1}{k}\right)^{-\frac{n-k}{n}}. \tag{2.14}$$

By the Cauchy mean value theorem, there is ξ between t and 1 ($0 < t < \xi < 1$ or $0 < 1 < \xi < t$, this is to say, $\xi > 0$) such that

$$\begin{aligned}\frac{g(t)}{h(t)} &= \frac{g(t)-g(1)}{h(t)-h(1)} = \frac{g'(\xi)}{h'(\xi)} \\ &= \frac{n(k-1)}{k(n-1)} \cdot \left(\frac{\xi+k-1}{k}\right)^{-\frac{n-k}{n}} \\ &\le \frac{n(k-1)}{k(n-1)} \cdot \left(\frac{k-1}{k}\right)^{-\frac{n-k}{n}} \\ &= \frac{n}{n-1} \cdot \left(\frac{k-1}{k}\right)^{\frac{k}{n}} = q.\end{aligned}$$

The equivalence of (2.12) and (2.13) completes the proof. □

3. The proof of Theorem 1.1

First, we consider the inequality

$$(G_n(a))^{1-p}(A_n(a))^p \le (GA)_n^{[k]}(a). \tag{3.1}$$

The first step: For $p = \frac{k-1}{n-1}$, we will show that (3.1) holds.

Because (3.1) is a homogeneous symmetric function of order one in $a_1, a_2, \ldots, a_n$, we may assume that $a \in D$, where D is as defined in Lemma 2.1. Let $F_1(a)$ be the function defined as in Lemma 2.1, and take the natural logarithm on both side of (3.1). Then (3.1) equals to

$$F_1(a) \ge F_1(e) = 0. \tag{3.2}$$

Case 1. When a is a critical point of $F_1(a)$ on D.

Obviously, $0 < p = \frac{k-1}{n-1} < 1$. From Lemma 2.1, let $a = (u, \underbrace{v, \ldots, v}_{n-1})$, $\frac{u}{v} = t$. Then

$$u + (n-1)v = n, v = \left(\frac{t+n-1}{n}\right)^{-1}, \quad u > 0,\ v > 0,\ t > 0. \tag{3.3}$$

By (3.3), $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$, the logarithmic identity and Lemma 2.2, we get

$$\begin{aligned}F_1(a) &= \frac{1}{\binom{n}{k}} \sum_{1 \le i_1 < \cdots < i_j \le n} \ln \frac{a_{i_1} + \cdots + a_{i_k}}{k} - (1-p) \cdot \frac{\ln a_1 + \cdots + \ln a_n}{n} \\ &= \frac{1}{\binom{n}{k}} \left[\binom{n-1}{k-1} \ln \frac{u+(k-1)v}{k} + \binom{n-1}{k} \ln v\right] - (1-p) \cdot \frac{\ln u + (n-1)\ln v}{n} \\ &= \frac{1}{\binom{n}{k}} \cdot \binom{n-1}{k-1} \ln \frac{t+k-1}{k} - \frac{1-p}{n} \cdot \ln t + p \ln v\end{aligned}$$

$$= \frac{k}{n} \ln \frac{t+k-1}{k} - \frac{1-p}{n} \cdot \ln t - p \cdot \ln \frac{t+n-1}{n}$$

$$= f_1(t) \geq 0.$$

Case 2. When a is a boundary point of D.

We only need to prove that (3.1) holds. If a is a boundary point of D, then among $a_1, \ldots, a_n$ there is a_i such that $a_i \to 0$. For convenience, let $a_n \to 0$ or $a_n = 0$. Since $a_1 + \cdots + a_{n-1} = 0$, $0 \leq a_i \leq n$ $(i = 1, \ldots, n)$, inequality (3.1) becomes

$$0 \leq (GA)_n^{[k]}(a)|_{a_n=0},$$

which obviously holds.

Since $F_1(a)$ is continuous and differentiable on D, from Cases 1 and 2, we know that for each $a \in D$, inequality (3.2) holds, that is, for each $a \in R^n_{++}$, inequality (3.1) holds.

The second step: For $p \leq \frac{k-1}{n-1}$, we will show that (3.1) holds. Let $p^* = \frac{k-1}{n-1}$. Since $\frac{A_n(a)}{G_n(a)} \geq 1$, from the first step, we obtain

$$(G_n(a))^{1-p}(A_n(a))^p = G_n(a) \cdot \left(\frac{A_n(a)}{G_n(a)}\right)^p \leq G_n(a) \cdot \left(\frac{A_n(a)}{G_n(a)}\right)^{p^*} \leq (GA)_n^{[k]}(a).$$

The third step: We will show that if (3.1) holds, then $p \leq \frac{k-1}{n-1}$.

From the proof of the first step, we have that if inequality (3.1) holds, then for each $t > 0$, inequality (2.10) holds, that is,

$$\begin{aligned} f_1(t) &= \frac{k}{n} \ln \frac{t+k-1}{k} - \frac{1-p}{n} \ln t - p \ln \frac{t+n-1}{n} \\ &= \frac{k}{n} \ln \frac{1+(k-1)t^{-1}}{k} + \left(\frac{k}{n} - \frac{1-p}{n} - p\right) \ln t - p \ln \frac{1+(n-1)t^{-1}}{n} \\ &= \frac{k}{n} \ln \frac{1+(k-1)t^{-1}}{k} + \frac{k-1-(n-1)p}{n} \ln t - p \ln \frac{1+(n-1)t^{-1}}{n} \qquad (3.4) \\ &\geq 0. \end{aligned}$$

If $p \geq \frac{k-1}{n-1}$, let $t \to +\infty$ in (3.4), then we get that $-\infty \geq 0$, which is impossible.

From the above three steps, we know the maximum of the real p, which make inequality (3.1) hold, is $\frac{k-1}{n-1}$.

Second, we consider the following inequality:

$$(GA)_n^{[k]}(a) \leq (1-q)G_n(a) + qA_n(a). \qquad (3.5)$$

The first step: For $q = \frac{n}{n-1}\left(\frac{k-1}{k}\right)^{\frac{k}{n}}$, we will show that (3.5) holds.

Because (3.5) is a homogeneous symmetric function of order one in $a_1, a_2, \ldots, a_n$, we may assume that $a \in D$, where D is as defined in Lemma 2.1. Let function $F_2(a)$ be as in Lemma 2.1. Then (3.5) equals

$$F_2(a) \leq F_2(e) = 0. \qquad (3.6)$$

Case 1. When a is a critical point of $F_2(a)$ on D.

From Bernoulli's inequality [15], $(1+x)^\alpha < 1+\alpha x \quad (-1 < x \neq 0, 0 < a < 1)$, we get

$$0 < q = \frac{n}{n-1}\left(\frac{k-1}{k}\right)^{\frac{k}{n}} = \frac{n}{n-1}\left(1-\frac{1}{k}\right)^{\frac{k}{n}} < \frac{n}{n-1}\left(1-\frac{k}{n}\cdot\frac{1}{k}\right) = 1.$$

From (3.3), $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$ and Lemma 2.3, we infer

$$\begin{aligned}
F_2(a) &= \left(\prod_{1\le i_1<\cdots<i_k\le n} \frac{a_{i_1}+\cdots+a_{i_k}}{k}\right)^{1/\binom{n}{k}} - (1-q)\sqrt[n]{a_1\ldots a_n} - q \\
&= \left\{\left[\frac{u+(k-1)v}{k}\right]^{\binom{n-1}{k-1}} \times v^{\binom{n}{k}}\right\}^{1/\binom{n}{k}} - (1-q)\sqrt[n]{u\cdot v^{n-1}} - q\cdot\frac{u+(n-1)v}{n} \\
&= v\left[\left(\frac{t+k-1}{k}\right)^{\binom{n-1}{k-1}/\binom{n}{k}} - (1-q)\sqrt[n]{t} - q\cdot\frac{t+n-1}{n}\right] \\
&= v\left[\left(\frac{t+k-1}{k}\right)^{\frac{k}{n}} - (1-q)t^{\frac{1}{n}} - q\cdot\frac{t+n-1}{n}\right] \\
&= v\cdot f_2(t) \le 0.
\end{aligned}$$

Case 2. When a is a boundary point of D.

We only need to prove that (3.5) holds. If a is a boundary point of D, then among $a_1,\ldots,a_n$ there is a_i such that $a_i \to 0$. We may assume that $a_n \to 0$ or $a_n = 0$. Since $a_1+\cdots+a_{n-1} = 0, 0 \le a_i \le n$ $(i = 1,\ldots,n)$, inequality (3.5) equals

$$\left[\left(\prod_{1\le i_1<\cdots<i_k\le n-1} \frac{a_{i_1}+\cdots+a_{i_k}}{k}\right)\cdot\left(\prod_{1\le i_1<\cdots<i_{k-1}\le n-1} \frac{a_{i_1}+\cdots+a_{i_{k-1}}}{k}\right)\right]^{1/\binom{n}{k}}$$
$$\le q\cdot\frac{a_1+\cdots+a_{n-1}}{n},$$

that is,

$$\left\{\left[(GA)^{[k]}_{n-1}(a)\right]^{\binom{n-1}{k}} \times \left(\frac{k-1}{k}\right)^{\binom{n-1}{k-1}} \times \left[(GA)^{[k-1]}_{n-1}(a)\right]^{\binom{n-1}{k-1}}\right\}^{1/\binom{n}{k}}$$
$$\le q\cdot\frac{n-1}{n}\cdot A_{n-1}(a), \frac{n}{n-1}\cdot\left(\frac{k-1}{n}\right)^{\frac{k}{n}}$$
$$\times\left[(GA)^{[k]}_{n-1}(a)\right]^{1-\frac{k}{n}} \times \left[(GA)^{[k-1]}_{n-1}(a)\right]^{\frac{k}{n}} \le q\cdot A_{n-1}(a). \tag{3.7}$$

From (1.3), we get

$$\frac{n}{n-1}\cdot\left(\frac{k-1}{n}\right)^{\frac{k}{n}}\times\left[(GA)_{n-1}^{[k]}(a)\right]^{1-\frac{k}{n}}\times\left[(GA)_{n-1}^{[k-1]}(a)\right]^{\frac{k}{n}}$$

$$\leq\frac{n}{n-1}\cdot\left(\frac{k-1}{n}\right)^{\frac{k}{n}}\times[A_{n-1}(a)]^{1-\frac{k}{n}}\times[A_{n-1}(a)]^{\frac{k}{n}}$$

$$=q\times A_{n-1}(a).$$

This shows that (3.7) holds, which implies that (3.6) holds.

Since the function $F_2(a)$ is continuous and differentiable on the area D, from Cases 1 and 2, we have for each $a\in D$, that the inequality (3.6) holds, that is, for each $a\in R_{++}^{n}$, inequality (3.5) holds.

The second step: For $q\geq q^*=\frac{n}{n-1}\left(\frac{k-1}{k}\right)^{\frac{k}{n}}$, we will show that (3.5) holds.

Since $A_n(a)-G_n(a)\geq 0$, from the first step, we get

$$\begin{aligned}(1-q)G_n(a)+qA_n(a)&=G_n(a)+q[A_n(a)-G_n(a)]\\&\geq G_n(a)+q^*[A_n(a)-G_n(a)]\\&\geq(GA)_n^{[k]}(a).\end{aligned}$$

The third step: We will show that if (3.5) holds, then $q\geq\frac{n}{n-1}\left(\frac{k-1}{k}\right)^{\frac{k}{n}}$.

In (3.5), let $a_1=\cdots=a_{n-1}=\frac{n}{n-1}$ and $a_n=0$. We get that the inequality (3.7) holds. Then, we have

$$\frac{n}{n-1}\cdot\left(\frac{k-1}{n}\right)^{\frac{k}{n}}\cdot\left(\frac{n}{n-1}\right)^{1-\frac{k}{n}}\cdot\left(\frac{n}{n-1}\right)^{\frac{k}{n}}\leq q\cdot\frac{n}{n-1},$$

that is,

$$q\geq\frac{n}{n-1}\left(\frac{k-1}{k}\right)^{\frac{k}{n}}.$$

From the above three steps, we obtain that the minimum of the real q, which makes (3.5) hold, is $\frac{n}{n-1}\left(\frac{k-1}{k}\right)^{\frac{k}{n}}$.

From the above proof, we get (1.9) and equality holds if and only if $a_1=\cdots=a_n$. Thus the proof is complete. □

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References

[1] Bullen P S, Mitrinnovic D S and Vasic P M, Means and their inequalities (Dordrecht/Boston/Lancaster/Tokyo: Reidel) (1988)

[2] Kuang J C, Usual inequalities (the second version) (Changsha: Hunan Education Press) (1993)

[3] Kuang J C, Usual inequalities (the third version) (Jinan: Shandong Science and Technology Press) (2004)

[4] Luo Z, Zhang R X and Wen J J, Inequalities involving third symmetric mean and their applications, *J. Chengdu Univ. (Natural Science Edition)* **23(3)** (2004) 1–10

[5] Luo Z, Wen J J and Shi H N, A class of geometric inequalities involving k-Brocard point, *J. Sichuan Univ. (Natural Science Edition)* **39(6)** (2002) 971–976

[6] Luo Z, Lai L and Wen J J, Descending dimension method for overall situation optimal of symmtric functions, *J. Chengdu Univ. (Natural Science Edition)* **21(1)** (2002) 5–11

[7] Pecaric J and Svrtan D, New refinements of the Jensen inequalities based on amples with repetitions, *J. Math. Anal. Appl.* **222** (1998) 365–373

[8] Shi H N, Wen J J and Zhou B J, An inequality of power mean, *Practice and Cognition of Math.* **31(2)** (2001) 227–230

[9] Wang W L and Wang P F, A sort of inequalities of symmtric functions, *Acta. Math.* **27(4)** (1984) 485–497

[10] Wang W L, Wen J J and Shi H N, The optimal values of power mean inequalities, *Acta. Math.* **47(6)** (2004) 1053–1062

[11] Wen J J, Lai L and Luo Z, Separation between symmtric mean and power mean and their applications, *J. Xinan National Univ. (Natural Science Edition)* **27(3)** (2000) 244–250

[12] Wen J J and Shi H N, The optimal strengthen of Maclaurin inequality, *J. Chengdu Univ. (Natural Science Edition)*, **19(3)** (2000) 1–8

[13] Wen J J and Wang W L, On some refinements of Jensen inequality, *J. Chengdu Univ. (Natural Science Edition)* **21(2)** (2002) 1–4

[14] Wen J J, Wang W L and Lu Y J, The method of descending dimension of establishing inequalities, *J. Xinan National Univ. (Natural Science Edition)* **29(5)** (2003) 527–532

[15] Wen J J and Zhang R X, The strengthen and improvement of Hardy inequality, *Practice and Cognition of Math.* **32(2)** (2002) 475–482

[16] Wen J J, The optimal generalization of A-G-H inequality and its applications, *J. Shaanxi Normal Univ. (Nat. Sci. Ed.)* **32** (2004) 12–16