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Quantum random walks and their convergence to Evans–Hudson flows

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Abstract. Using coordinate-free basic operators on toy Fock spaces, quantum random walks are defined following the ideas of Attal and Pautrat. Extending the result for one dimensional noise, strong convergence of quantum random walks associated with bounded structure maps to Evans–Hudson flow is proved under suitable assumptions. Starting from the bounded generator of a given uniformly continuous quantum dynamical semigroup on a von Neumann algebra, we have constructed quantum random walks which converges strongly and the strong limit gives an Evans–Hudson dilation for the semigroup.

Keywords. Quantum dynamical semigroup; Evans–Hudson flow; quantum random walk.

1. Introduction

The aim of this article is to investigate convergence of random walks on von Neumann algebra to Evans–Hudson flows. Here the random walks and Evans–Hudson flows are generalizations of classical Markov chains and Markov processes [1, 6, 7, 15]. In the algebraic language of Accardi, Frigerio and Lewis [1], Lindsay and Parthasarathy [15] reformulated the notion of a quantum random walk as a family of $*$-homomorphisms from an initial algebra $\mathcal{B}(\mathbf{h})$ to $\mathcal{B}(\mathbf{h}\otimes\Gamma(L^2(\mathbb{R}_+))$ with an approximation of four quantum stochastic increment on the symmetric Fock space $\Gamma(L^2(\mathbb{R}_+))$. With each $h > 0$ associating a quantum random walk $\{p_t^{(h)}: t \geq 0\}$ they proved its weak convergence, as h tends to 0, to a $*$-homomorphic flow j_t satisfying Evans–Hudson equation [7] which dilate a uniformly continuous quantum dynamical semigroup on $\mathcal{B}(\mathbf{h})$. By a quantum dynamical semigroup (QDS) $\{T_t\}$ on a von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathbf{h})$, we mean a strongly continuous semigroups of normal completely positive (CP) contractive maps. An Evans–Hudson (E–H) dilation of $\{T_t\}$ is a family $\{j_t\}$ of normal $*$-homomorphism from $\mathcal{A}$ into $\mathcal{A}\otimes\mathcal{B}(\Gamma(L^2(\mathbb{R}_+, \mathbf{k})))$, where $\mathbf{k}$ is a Hilbert space (noise space), satisfying Evans–Hudson equation and its expectation semigroup is $\{T_t\}$. For details, see [19, 11]. Since the $\{j_t\}$ is a $*$-homomorphic family, for each $t \geq 0$ and $x \in \mathcal{B}(\mathbf{h})$ the family of operators $\{p_t^{(h)}(x)\}_{h\geq 0}$ converges strongly to $j_t(x)$.

Attal and Pautrat [2] revived the idea of quantum random walk with infinite dimensional noise using 'toy Fock space' and showed the weak convergence of a family of unitaries $\{U_t^{(h)}\}_h$ to a unitary process U_t satisfying Hudson–Parthasarathy equation [13] with bounded operator coefficients. Since the limit $\{U_t\}$ is a unitary family the convergence hold in strong operator topology of $\mathcal{B}(\mathbf{h} \otimes \Gamma(L^2(\mathbb{R}_+, \mathbf{k})))$. They also observed the convergence of quantum random walks, implemented by these unitary family $\{U_t^{(h)}\}_h$ parameterize by h. In both the papers [15, 2], the authors proved the weak convergence, and hence strong convergence knowing that the limit is a $*$-homomorphic or unitary family. In [22], Sinha showed the strong convergence of $p_t^{(h)}(x)$ improving the result in [15], thus obtaining that the limit is $*$-homomorphic.

Existence of solution of E–H flow equation, with bounded structure maps, is studied by many authors, for example in [12, 10, 16–18], and in some situation, it is proved that the solution is $*$-homomorphic. Here the above approximation problem is discussed when structure maps are bounded and noise space is infinite dimensional. Following [2], [12] and extending the result of [22], under certain assumptions, strong convergence of quantum random walk is proved. Moreover, we have given a construction of quantum random walk starting from the bounded generator of uniformly continuous QDS on a unital von Neumann algebra which converges strongly. This gives an alternative proof for $*$-homomorphic property of the solution of E–H flow equation and we obtain an E–H dilation of the QDS.

As our aim is to use this quantum random walk approach to construct E–H dilation of strongly continuous QDS (not necessarily uniformly continuous), we mention some attempts made in this direction. In [21] quantum random walks are constructed, starting from the unbounded generators of a particular class of strongly continuous completely positive semigroup on uniformly hyperfinite C^*-algebras satisfying some analytic assumptions, and their weak convergence is shown. As a weak limit of $*$-homomorphisms the limit is a completely positive flow. A purely algebraic construction of quantum random walk is given in [9] which may be useful for construction of E–H dilation for strongly continuous QDS.

Recently similar problem have been studied in [8] for Lévy process on quantum groups and in [3, 4] for vacuum adapted process.

2. Preliminaries

2.1 *Toy Fock spaces*

Let $\mathcal{K} = L^2(\mathbb{R}_+, \mathbf{k})$ where $\mathbf{k}$ is a Hilbert space, not necessarily separable. We note that, for any $n \geq 0$, the n-fold symmetric tensor product of $\mathcal{K}$ and their direct sum can canonically be embedded in the symmetric Fock space $\Gamma := \Gamma(\mathcal{K})$. For any subset $\mathcal{D} \subseteq \mathcal{K}$, let $\mathcal{E}(\mathcal{D})$ be the subspace of Γ, spanned by a set of exponential vectors $\left\{\mathbf{e}(f) := \oplus_{n\geq 0} \frac{f^{\otimes n}}{\sqrt{n!}} : f \in \mathcal{D}\right\}$. For $0 \leq s < t < \infty$, let $1_{s]}$, $1_{[s,t)}$ and $1_{[t}$ are the characteristic functions of interval $[0, s)$, $[s, t)$ and $[t, \infty)$. We denote the functions $1_{s]}\, f$, $1_{[s,t)}\, f$ and $1_{[t}\, f$ by $f_{s]}$, $f_{[s,t)}$ and $f_{[t}$. The Hilbert spaces $\mathcal{K}$ and Γ can be decomposed as $\mathcal{K} = \mathcal{K}_{s]} \oplus \mathcal{K}_{[s,t)} \oplus \mathcal{K}_{[t}$ and $\Gamma = \Gamma_{s]} \otimes \Gamma_{[s,t)} \otimes \Gamma_{[t}$ where $\mathcal{K}_{s]} = 1_{s]}\, \mathcal{K}$, $\mathcal{K}_{[s,t)} = 1_{[s,t)}\, \mathcal{K}$, $\mathcal{K}_{[t} = 1_{[t}\, \mathcal{K}$ and $\Gamma_{s]} = \Gamma(\mathcal{K}_{s]})$, $\Gamma_{[s,t)} = \Gamma(\mathcal{K}_{[s,t)})$, $\Gamma_{[t} = \Gamma(\mathcal{K}_{[t})$. We can write $\mathbf{e}(f)$ as $\mathbf{e}(f_{s]}) \otimes \mathbf{e}(f_{(s,t]}) \otimes \mathbf{e}(f_{[t})$.

For a partition $S \equiv (0 = t_0 < t_1 < t_2 \cdots)$ of $\mathbb{R}_+$, the Fock space $\Gamma(\mathcal{K})$ can be viewed as the infinite tensor product $\otimes_{n\geq 1}\Gamma_n$ of symmetric Fock spaces $\{\Gamma_n = \Gamma_{(t_{n-1},t_n]}\}_{n\geq 1}$ with respect to the stabilizing sequence of vacuum vectors $\{\Omega_n : n \geq 1\}$. Let us denote the interval $(t_{n-1}, t_n]$ by $[n]$ and the othogonal projection of Γ_n onto the m-particle space by $P_m[n]$.

For $n \geq 1$, consider the subspace $\hat{\mathbf{k}}_n = \mathbb{C}\, \Omega_n \oplus \mathbf{k}_n$ of Γ, where $\mathbf{k}_n = \{1_{[n]}\phi : \phi \in \mathbf{k}\} \subseteq \mathcal{K}_n$. The spaces $\hat{\mathbf{k}}_n$ are isomorphic with $\hat{\mathbf{k}} := \mathbb{C} \oplus \mathbf{k}$.

DEFINITION 2.1

The toy Fock space associated with the partition S of $\mathbb{R}_+$ is defined to be the subspace $\Gamma(S) := \otimes_{n\geq 1}\hat{\mathbf{k}}_n$ with respect to the stabilizing sequence $(\Omega_n)_{n\geq 1}$.

Let $P(S)$ be the orthogonal projection of Γ onto the toy Fock space $\Gamma(S)$. Now onwards let us consider toy Fock spaces $\Gamma(S_h)$ associated with regular partition $S_h \equiv (0, h, \dots)$ for $h > 0$ and denote the orthogonal projection by P_h. The projection P_h is given by

$$P_h(\Omega) = \Omega,$$

$$P_h f = \sum_{m\geq 1} \frac{1_{[m]}}{h} \int_{[m]} f(s)\mathrm{d}s,$$

$$P_h\mathbf{e}(f) = \Omega \oplus \oplus_{n\geq 1} \frac{1}{\sqrt{n!}} \left[\sum_{1\leq m_1 < m_2 \cdots < m_n} \otimes_{l=1}^{n} \frac{1}{h} 1_{[m_l]} \int_{[m_l]} f(s)\mathrm{d}s \right].$$

Moreover,

$$P_h\mathbf{e}(f) = P_h\mathbf{e}(f_{(k-1)h]}) P_h\mathbf{e}(f_{[k]}) P_h\mathbf{e}(f_{[kh})$$

and

$$P_h\mathbf{e}(f_{[k]}) = \Omega_k \oplus \frac{1}{\sqrt{h}} 1_{[k]} \int_{[k]} f(s)\mathrm{d}s.$$

Denote the restriction of the orthogonal projection P_h to Γ_k by $P_h[k]$, $P_h = \otimes_{k\geq 1} P_h[k]$. Let us consider the subspace $\mathcal{M} = \{f \in L^2(\mathbb{R}_+, \mathbf{k})\colon f \in \mathcal{C}_c^1(\mathbb{R}_+, \mathbf{k})\} \subseteq L^2(\mathbb{R}_+, \mathbf{k})$. Clearly $\mathcal{M}$ is a dense subspace and hence the algebraic tensor product $\mathbf{h} \otimes \mathcal{E}(\mathcal{M})$ is dense in $\mathbf{h} \otimes \Gamma$. For $f \in \mathcal{M}$, define $c_f := \sup_\tau \|f'(\tau)\|$ where f' denotes the first derivative of the function f. We have the following estimates which will be needed later.

Lemma 2.2. *For any* $f \in \mathcal{M}, k \geq 1$, $\|(1 - P_h[k])\mathbf{e}(f_{[k]})\| \leq h(c_f + \|f\|_\infty)\|\mathbf{e}(f_{[k]})\|$.

Proof. We have

$$\begin{aligned}
&\|(1 - P_h[k])\mathbf{e}(f_{[k]})\| \\
&\quad = \|(P_0 + P_1 - P_h)\mathbf{e}(f_{[k]}) + [1 - P_0 - P_1]\mathbf{e}(f_{[k]})\| \\
&\quad \leq \|f_{[k]} - P_h f_{[k]}\| + \|[1 - P_0 - P_1]\mathbf{e}(f_{[k]})\|.
\end{aligned}$$

It is clear that $\|[1 - P_0 - P_1]\mathbf{e}(f_{[k]})\| \leq h\|f\|_\infty^2 \|\mathbf{e}(f_{[k]})\|$. Let us consider the first term

$$\begin{aligned}
&\|f_{[k]} - P_h f_{[k]}\|^2 \\
&\quad = \|1_{[k]} \left(f - \frac{1}{h} \int_{[k]} f(s)\mathrm{d}s \right) \|^2 \\
&\quad = \int_{[k]} \left\| f(r) - \frac{1}{h} \int_{[k]} f(s)\mathrm{d}s \right\|^2 \mathrm{d}r
\end{aligned}$$

$$= \frac{1}{h^2} \int_{[k]} \left\| \int_{[k]} [f(r) - f(s)] \mathrm{d}s \right\|^2 \mathrm{d}r$$

$$\leq \frac{1}{h^2} \int_{[k]} \mathrm{d}r \left[\int_{[k]} h \sup \| f'(\tau) \| \mathrm{d}s \right]^2$$

for some $\tau \in [k]$. Thus the required estimate follows. □

By this lemma it can be proved that the family of othogonal projections P_h converges strongly to identity operator in Γ as h tends to 0.

2.2 *Coordinate-free basic operators*

Here we define basic operators associated with toy Fock space $\Gamma(S_h)$ using the fundamental processes in coordinate-free language of quantum stochastic calculus developed in [12] and obtained some useful estimates. Let $k \geq 1$, for $S \in \mathcal{B}(\mathbf{h})$, $R \in \mathcal{B}(\mathbf{h}, \mathbf{h} \otimes \mathbf{k})$, $Q \in \mathcal{B}(\mathbf{h} \otimes \mathbf{k}, \mathbf{h})$ and $T \in \mathcal{B}(\mathbf{h} \otimes \mathbf{k})$. Let us define four operators

$$\begin{aligned} N_S^1[k] &:= P_0[k] \frac{\Lambda_S^1[k]}{h} = S P_0[k], \\ N_Q^2[k] &:= \frac{\Lambda_Q^2[k]}{\sqrt{h}} P_1[k], \\ N_R^3[k] &:= P_1[k] \frac{\Lambda_R^3[k]}{\sqrt{h}}, \\ N_T^4[k] &:= P_1[k] (\Lambda_T^4[k]) P_1[k] P_h[k], \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \Lambda_S^1[k] &= \mathcal{I}_S[k] = h \otimes S \, 1_\Gamma, \\ \Lambda_Q^2[k] &= a_Q[k], \\ \Lambda_R^3[k] &= a_R^\dagger[k], \\ \Lambda_T^4[k] &= \lambda_T[k]. \end{aligned} \tag{2.2}$$

Here all these operators act nontrivially only on Γ_n. For definition of coordinate-free fundamental processes Λ's we refer to [12]. Here, we note that in the notation of [12], the annihilation process $a_Q[k]$ appeared above is $a_{Q^*}[k]$. It is clear that these operators $N^l, l = 1, \ldots, 4$ are bounded and leave the subspace $\Gamma(S_h)$ invariant and $(N_S^1[k])^* = N_{S^*}^1[k]$, $(N_Q^2[k])^* = N_{Q^*}^3[k]$ and $(N_T^4[k])^* = N_{T^*}^4[k]$. Moreover, the maps $\mathcal{B}(\mathbf{h}) \ni S \mapsto \Lambda_S^1[k]$, $\mathcal{B}(\mathbf{h} \otimes \mathbf{k}, \mathbf{h}) \ni Q \mapsto \Lambda_Q^2[k]$, $\mathcal{B}(\mathbf{h}, \mathbf{h} \otimes \mathbf{k}) \ni R \mapsto \Lambda_R^3[k]$ and $\mathcal{B}(\mathbf{h} \otimes \mathbf{k}) \ni T \mapsto \Lambda_T^4[k]$ are linear, and hence $S \mapsto N_S^1[k]$, $Q \mapsto N_Q^2[k]$, $R \mapsto N_R^3[k]$ and $T \mapsto N_T^4[k]$. For $u \in \mathbf{h}$, $f \in L^2(\mathbb{R}_+, \mathbf{k})$,

$$N_S^1[k] u \mathbf{e}(f_{[k]}) = S u \otimes \Omega_{[k]},$$

$$
\begin{aligned}
N_Q^2[k]u\mathbf{e}(f_{[k]}) &= \frac{a_Q[k]}{\sqrt{h}} u \otimes f_{[k]} \\
&= \frac{1}{\sqrt{h}} \int_{[k]} Q(uf(s))\mathrm{d}s\Omega_{[k]}, \\
N_R^3[k]u\mathbf{e}(f_{[k]}) &= \frac{a_R^\dagger[k]}{\sqrt{h}} u \otimes \Omega_{[k]} \\
&= \frac{1_{\mathbf{h}} \otimes 1_{[k]}}{\sqrt{h}}\ Ru \\
N_T^4[k]u\mathbf{e}(f_{[k]}) &= \lambda_T[k])P_h[k]f_{[k]} \\
&= (1_{\mathbf{h}} \otimes 1_{[k]})Tu \otimes P_h f(\cdot).
\end{aligned}
\tag{2.3}
$$

For any $S_1, S_2 \in \mathcal{B}(\mathbf{h})$, $Q \in \mathcal{B}(\mathbf{h} \otimes \mathbf{k}, \mathbf{h})$, $R \in \mathcal{B}(\mathbf{h}, \mathbf{h} \otimes \mathbf{k})$ and $T_1, T_2 \in \mathcal{B}(\mathbf{h} \otimes \mathbf{k})$ we observe the following simple but useful identities, which are easy to derive

- $(N_Q^2[k])^2 = (N_R^3[k])^2 = 0, \qquad N_S^1[k] + N_{S\otimes 1_{\mathbf{k}}}^4[k] = S \otimes P_h[k],$
- $N_{S_1}^1[k]\ N_{S_2}^1[k] = N_{S_1 S_2}^1[k], \qquad N_Q^2[k]\ N_R^3[k] = N_{QR}^1[k],$
- $N_S^1[k]\ N_Q^2[k] = N_{SQ}^2[k], \qquad N_Q^2[k]\ N_T^4[k] = N_{QT}^2[k],$
- $N_R^3[k]\ N_S^1[k] = N_{RS}^3[k], \qquad N_T^4[k]\ N_R^3[k] = N_{TR}^3[k],$
- $N_R^3[k]\ N_Q^2[k] = N_{RQ}^4[k], \qquad N_{T_1}^4[k]\ N_{T_2}^4[k] = N_{T_1 T_2}^4[k].$

From (2.3) we have

$$
\begin{aligned}
&\|N_S^1[k]u\mathbf{e}(f_{[k]})\| = \|Su\|, \\
&\|N_Q^2[k]u\mathbf{e}(f_{[k]})\| \le \sqrt{h}\|Q\|\ \|u\|\ \|f\|_\infty, \\
&\|N_R^3[k]u\mathbf{e}(f_{[k]})\| \le \|Ru\| \\
&\|N_T^4[k]u\mathbf{e}(f_{[k]})\| \le \sqrt{h}\|T\|\|u\|\|f\|.
\end{aligned}
\tag{2.4}
$$

Here we also note the following which can be verified easily using Lemma 2.12 and Lemma 2.14 in [12],

$$
\begin{aligned}
&\|\Lambda_R^3[k]u\mathbf{e}(f_{[k]})\|^2 \\
&\quad = \|(1_{\mathbf{h}\otimes 1_{[k]}})Ru\|^2\|\mathbf{e}(f_{[k]})\|^2 + \|\int_{[k]} R^*(uf(s))\mathrm{d}s\|^2\|\mathbf{e}(f_{[k]})\|^2, \\
&\|\Lambda_T^4[k]u\mathbf{e}(f_{[k]})\|^2 \\
&\quad = \int_{[k]} \|Tuf(s)\|^2\mathrm{d}s\|\mathbf{e}(f_{[k]})\|^2 + \|\int_{[k]} \langle f(s), T_{f(s)}\rangle \mathrm{d}su\mathbf{e}(f_{[k]})\|^2.
\end{aligned}
\tag{2.5}
$$

In the above expression $T_{f(s)} \in \mathcal{B}(\mathbf{h}, \mathbf{h} \otimes \mathbf{k})$ and $\langle f(s), T_{f(s)}\rangle \in \mathcal{B}(\mathbf{h})$ defined by

$$T_{f(s)}u = Tu \otimes f(s), \forall u \in \mathbf{h}$$

and

$$\langle\langle f(s), T_{f(s)}\rangle u, v\rangle = \langle T_{f(s)}u, v\otimes f(s)\rangle, \forall u, v\in \mathbf{h}.$$

For the basic operators N^l's we have the following estimates.

Lemma 2.3.

(a) *For any* $k\geq 1$ *and* $u\in\mathbf{h}$, $f\in\mathcal{M}$,

1. $\|\{h\ N_S^1[k]-\Lambda_S^1[k]\}u\mathbf{e}(f_{[k]})\|\leq h^{\frac{3}{2}}\|Su\|\|f\|_\infty\|\mathbf{e}(f_{[k]})\|$,
2. $\|\{\sqrt{h}\ N_Q^2[k]-\Lambda_Q^2[k]\}u\mathbf{e}(f_{[k]})\|\leq h^{\frac{3}{2}}\|Q\|\ \|u\|\|f\|_\infty^2\|\mathbf{e}(f_{[k]})\|$,
3. $\|\{\sqrt{h}\ N_R^3[k]-\Lambda_R^3[k]\}u\mathbf{e}(f_{[k]})\|\leq 2h\|Ru\|\|f\|_\infty\|\mathbf{e}(f_{[k]})\|$,
4. $\|\{N_T^4[k]-\Lambda_T^4[k]\}u\mathbf{e}(f_{[k]})\|\leq 2h\|T\|(c_f+\|f\|_\infty^2)\|u\mathbf{e}(f_{[k]})\|$.

(b) *For any* $k\geq 1$ *and* $u, v\in\mathbf{h}$, $f, g\in\mathcal{M}$, *we have*

1. $|\langle v\mathbf{e}(g_{[k]}), \{h\ N_S^1[k]-\Lambda_S^1[k]\}u\mathbf{e}(f_{[k]})\rangle|\leq h^{\frac{3}{2}}\|Su\|\|f\|_\infty\|\mathbf{e}(f_{[k]})\|\|v\mathbf{e}(g_{[k]})\|$,
2. $|\langle v\mathbf{e}(g_{[k]}), \{\sqrt{h}\ N_Q^2[k]-\Lambda_Q^2[k]\}u\mathbf{e}(f_{[k]})\rangle|\leq h^{\frac{3}{2}}\|Q\|\ \|u\|\|f\|_\infty^2\|g\|_\infty\ \|\mathbf{e}(f_{[k]})\|\ \|v\mathbf{e}(g_{[k]})\|$,
3. $|\langle v\mathbf{e}(g_{[k]}), \{\sqrt{h}\ N_R^3[k]-\Lambda_R^3[k]\}u\mathbf{e}(f_{[k]})\rangle|\leq 2h^2\|Ru\|\ \|v\|\ \|f\|_\infty\ \|g\|_\infty\ \|\mathbf{e}(f_{[k]})\|^2\ \|\mathbf{e}(g_{[k]})\|^2$,
4. $|\langle v\mathbf{e}(g_{[k]}), \{N_T^4[k]-\Lambda_T^4[k]\}u\mathbf{e}(f_{[k]})\rangle|\leq h^2[(\|f\|_\infty+c_f)\|g\|_\infty]^2\|T\|\ \|u\|\ \|v\|\ \|\mathbf{e}(f_{[k]})\|^2\ \|\mathbf{e}(g_{[k]})\|^2$.

Proof.

(a) (1) It is clear from the definition that

$$\begin{aligned}\|\{h\ N_S^1[k]-\Lambda_S^1[k]\}u\mathbf{e}(f_{[k]})\| &= h\|Su(\Omega_{[k]}-\mathbf{e}(f_{[k]}))\|\\ &= h\|Su\|\|\Omega_{[k]}-\mathbf{e}(f_{[k]})\|\\ &\leq h^{\frac{3}{2}}\|Su\|\|f\|_\infty\|\mathbf{e}(f_{[k]})\|.\end{aligned}$$

(2) From the definitions, we have

$$\begin{aligned}\|\{\sqrt{h}\ N_Q^2[k]-\Lambda_Q^2[k]\}u\mathbf{e}(f_{[k]})\| &= \left\|\int_{[k]} Q(uf(s))\mathrm{d}s\ (\Omega_{[k]}-\mathbf{e}(f_{[k]}))\right\|\\ &\leq \int_{[k]}\|Q(uf(s))\|\mathrm{d}s\ \|(\Omega_{[k]}-\mathbf{e}(f_{[k]}))\|\\ &\leq h^{\frac{3}{2}}\|Q\|\ \|u\|\|f\|_\infty^2\|\mathbf{e}(f_{[k]})\|.\end{aligned}$$

(3) We have

$$\begin{aligned}\|\{\sqrt{h}\ N_R^3[k]-\Lambda_R^3[k]\}u\mathbf{e}(f_{[k]})\|^2 &= \|(1_{\mathbf{h}}\otimes 1_{[k]})\ Ru-\Lambda_R^3[k]u\mathbf{e}(f_{[k]})\|^2\\ &= \|(1_{\mathbf{h}}\otimes 1_{[k]})\ Ru\|^2+\|\Lambda_R^3[k]u\mathbf{e}(f_{[k]})\|^2\\ &\quad -2\mathcal{R}e\langle(1_{\mathbf{h}}\otimes 1_{[k]})\ Ru, \Lambda_R^3[k]u\mathbf{e}(f_{[k]})\rangle.\end{aligned}$$

Now using (2.5) and the definition of Λ_R^3 the above quantity is equal to

$$
\begin{aligned}
&\|(1_{\mathbf{h}} \otimes 1_{[k]})Ru\|^2 + \|(1_{\mathbf{h}} \otimes 1_{[k]})Ru\|^2 \|\mathbf{e}(f_{[k]})\|^2 \\
&\quad + \left\| \int_{[k]} R^*(uf(s))\mathrm{d}s \right\|^2 \|\mathbf{e}(f_{[k]})\|^2 - 2\|(1_{\mathbf{h}} \otimes 1_{[k]})Ru\|^2 \\
&= \|(1_{\mathbf{h}} \otimes 1_{[k]})Ru\|^2 [\|\mathbf{e}(f_{[k]})\|^2 - 1] + \left\| \int_{[k]} R^*(uf(s))\mathrm{d}s \right\|^2 \|\mathbf{e}(f_{[k]})\|^2 \\
&\le 2h^2 \|R\|^2 \|u\|^2 \|f\|_\infty^2 \|\mathbf{e}(f_{[k]})\|^2.
\end{aligned}
$$

(4) We have

$$
\begin{aligned}
&\|\{N_T^4[k] - \Lambda_T^4[k]\}u\mathbf{e}(f_{[k]})\|^2 \\
&\quad = \|(1_{\mathbf{h}} \otimes 1_{[k]})T(u \otimes P_h f(\cdot))\|^2 + \|\Lambda_T^4[k]u\mathbf{e}(f_{[k]})\|^2 \\
&\qquad - 2\mathcal{R}e\langle (1_{\mathbf{h}} \otimes 1_{[k]})T(u \otimes P_h f(\cdot)), \Lambda_T^4[k]u\mathbf{e}(f_{[k]})\rangle.
\end{aligned}
$$

By the definition of Λ_T^4 (see [12])

$$
\begin{aligned}
&\langle (1_{\mathbf{h}} \otimes 1_{[k]})T(u \otimes P_h f(\cdot)), \Lambda_T^4[k]u\mathbf{e}(f_{[k]})\rangle \\
&\quad = \langle (1_{\mathbf{h}} \otimes 1_{[k]})T(u \otimes P_h f(\cdot)), a^\dagger(T_{f_{[k]}}^{[k]})u\mathbf{e}(f_{[k]})\rangle \\
&\quad = \langle (1_{\mathbf{h}} \otimes 1_{[k]})T(u \otimes P_h f(\cdot)), T_{f_{[k]}}^{[k]}(u\Omega_{[k]})\rangle \\
&\quad = \int_{[k]} \langle T(uP_h(f)(s)), T(uf(s))\rangle \, \mathrm{d}s.
\end{aligned}
$$

Thus using (2.5) we obtain

$$
\begin{aligned}
&\|\{N_T^4[k] - \Lambda_T^4[k]\}u\mathbf{e}(f_{[k]})\|^2 \\
&\quad = \int_{[k]} \|T(uP_h(f)(s))\|^2 \mathrm{d}s \\
&\qquad + \int_{[k]} \|T(uf(s))\|^2 \mathrm{d}s \|\mathbf{e}(f_{[k]})\|^2 + \left\| \int_{[k]} \langle f(s), T_{f(s)}\rangle \, \mathrm{d}s \; u\mathbf{e}(f_{[k]}) \right\|^2 \\
&\qquad - 2\mathcal{R}e \int_{[k]} \langle T(uP_h(f)(s)), T(uf(s))\rangle \mathrm{d}s \\
&\quad = \int_{[k]} \|Tuf(s)\|^2 \mathrm{d}s (\|\mathbf{e}(f_{[k]})\|^2 - 1) + \left\| \int_{[k]} \langle f(s), T_{f(s)}\rangle \, \mathrm{d}s \; u\mathbf{e}(f_{[k]}) \right\|^2 \\
&\qquad + \int_{[k]} \|T(u \otimes (1 - P_h)(f)(s))\|^2 \mathrm{d}s \\
&\quad \le 2h^2 \|T\|^2 \|f\|_\infty^4 \|u\mathbf{e}(f_{[k]})\|^2 + \|T\|^2 \|u\|^2 \int_{[k]} \|(1 - P_h)(f)(s))\|^2 \mathrm{d}s \\
&\quad \le 2h^2 \|T\|^2 \|f\|_\infty^4 \|u\mathbf{e}(f_{[k]})\|^2 + \|T\|^2 \|u\|^2 \|(1 - P_h)(f_{[k]})\|^2.
\end{aligned}
$$

By estimate of $\|(1-P_h)\mathbf{e}(f_{[k]})\|$ in Lemma 2.2 the requirement follows.
(b) The estimates (1) and (2) follow directly from (a).
(3) From the definitions

$$
\begin{aligned}
&\langle v\mathbf{e}(g_{[k]}), \{\sqrt{h}\ N_R^3[k]-\Lambda_R^3[k]\}u\mathbf{e}(f_{[k]})\rangle \\
&\quad = \langle v\mathbf{e}(g_{[k]}), \sqrt{h}\ N_R^3[k]u\mathbf{e}(f_{[k]})\rangle - \langle v\mathbf{e}(g_{[k]}), \Lambda_R^3[k]u\mathbf{e}(f_{[k]})\rangle \\
&\quad = \langle v\mathbf{e}(g_{[k]}), (1_{\mathbf{h}}\otimes 1_{[k]})\ Ru\rangle - \langle \Lambda_R^2[k]v\mathbf{e}(g_{[k]}), u\mathbf{e}(f_{[k]})\rangle \\
&\quad = \int_{[k]} \langle Ru, vg(s)\rangle \mathrm{d}s(1-\langle \mathbf{e}(g_{[k]}), \mathbf{e}(f_{[k]})\rangle).
\end{aligned}
$$

Thus we have obtained the required estimate,

$$
\begin{aligned}
&|\langle v\mathbf{e}(g_{[k]}), \{\sqrt{h}\ N_R^3[k]-\Lambda_R^3[k]\}u\mathbf{e}(f_{[k]})\rangle| \\
&\quad \leq h^2\|Ru\|\ \|v\|\ \|f\|_\infty\|g\|_\infty\|^2\mathbf{e}(f_{[k]})\|^2\|\mathbf{e}(g_{[k]})\|^2.
\end{aligned}
$$

(4) By definition of N_T^4 and Λ_T^4,

$$
\begin{aligned}
&\langle v\mathbf{e}(g_{[k]}), \{N_T^4[k]-\Lambda_T^4[k]\}u\mathbf{e}(f_{[k]})\rangle \\
&\quad = \langle v\mathbf{e}(g_{[k]}), N_T^4[k]u\mathbf{e}(f_{[k]})\rangle - \langle v\mathbf{e}(g_{[k]}), \Lambda_T^4[k]u\mathbf{e}(f_{[k]})\rangle \\
&\quad = \langle v\mathbf{e}(g_{[k]}), (1_{\mathbf{h}}\otimes 1_{[k]})T(uP_hf(\cdot))\rangle - \langle v\mathbf{e}(g_{[k]}), a^\dagger(T_{f_{[k]}}^{[k]})u\mathbf{e}(f_{[k]})\rangle \\
&\quad = \int_{[k]} \langle vg(s), T(u(P_hf)(s))\rangle \mathrm{d}s \\
&\qquad - \int_{[k]} \langle vg(s), T(uf(s))\rangle \mathrm{d}s\langle \mathbf{e}(g_{[k]}), \mathbf{e}(f_{[k]})\rangle. \\
&\quad = \int_{[k]} \langle vg(s), T[u((P_h-1)f)(s)]\rangle \mathrm{d}s \\
&\qquad + \int_{[k]} \langle vg(s), T(uf(s))\rangle \mathrm{d}s[1-\langle \mathbf{e}(g_{[k]}), \mathbf{e}(f_{[k]})\rangle].
\end{aligned}
$$

So we get

$$
\begin{aligned}
&|\langle v\mathbf{e}(g_{[k]}), \{N_T^4[k]-\Lambda_T^4[k]\}u\mathbf{e}(f_{[k]})\rangle| \\
&\quad \leq \left(\int_{[k]} \|vg(s)\|^2\mathrm{d}s\right)^{\frac{1}{2}} \left(\int_{[k]} \|T[u((P_h-1)f_{[k]}(s))]\|^2\mathrm{d}s\right)^{\frac{1}{2}} \\
&\qquad + \int_{[k]} \|vg(s)\|\|T(uf(s))\|\mathrm{d}s\ \|1-\langle \mathbf{e}(g_{[k]}), \mathbf{e}(f_{[k]})\rangle\| \\
&\quad \leq h\|v\|\|g\|_\infty\|T\|\|u\|\|(P_h[k]-1)f_{[k]}\| \\
&\qquad + h\|v\|\|g\|_\infty\|T\|\|u\|\|f\|_\infty\ \|1-\langle \mathbf{e}(g_{[k]}), \mathbf{e}(f_{[k]})\rangle\|.
\end{aligned}
$$

Using the estimates of $\|(P_h[k]-1)f_{[k]}\|$ and $\|1-\langle \mathbf{e}(g_{[k]}), \mathbf{e}(f_{[k]})\rangle\|$ the required estimate follows. □

Remark 2.4. The estimates in the above lemma will also hold if we replace the initial Hilbert space $\mathbf{h}$ by $\mathbf{h}' := \mathbf{h} \otimes \mathbf{k}^{\otimes m} \otimes \Gamma_{(k-1)h]}$ and take $S \in \mathcal{B}(\mathbf{h}')$, $R \in \mathcal{B}(\mathbf{h}', \mathbf{h}' \otimes \mathbf{k})$, $Q \in \mathcal{B}(\mathbf{h}' \otimes \mathbf{k}, \mathbf{h}')$ and $T \in \mathcal{B}(\mathbf{h}' \otimes \mathbf{k})$.

2.3 *Quantum random walk*

Let $\mathcal{A} \subseteq \mathcal{B}(\mathbf{h})$ be a unital von Neumann algebra. Let us consider the Hilbert von Neumann modules $\mathcal{A}\otimes\mathbf{k} \subseteq \mathcal{B}(\mathbf{h}, \mathbf{h}\otimes\mathbf{k})$ and $\mathcal{A}\otimes\mathbf{k}^* \subseteq \mathcal{B}(\mathbf{h}\otimes\mathbf{k}, \mathbf{h})$ (for Hilbert module see [14, 11]). Suppose we are given with a family of $*$-homomorphisms $\{\beta(h)\}_{h>0}$ from $\mathcal{A}$ to $\mathcal{A}\otimes\mathcal{B}(\hat{\mathbf{k}})$. For $h > 0$, $\beta(h)$ can be written as

$$\beta(h, x) = \begin{pmatrix} \beta_1(h, x) & \beta_2(h, x) \\ \beta_3(h, x) & \beta_4(h, x) \end{pmatrix}, \forall x \in \mathcal{A},$$

where the components $\beta_l(h)$'s are contractive maps and $\beta_1(h) \in \mathcal{B}(\mathcal{A})$, $\beta_4(h) \in \mathcal{B}(\mathcal{A}, \mathcal{A}\otimes \mathcal{B}(\mathbf{k}))$ and $\beta_3(h) \in \mathcal{B}(\mathcal{A}, \mathcal{A}\otimes\mathbf{k})$. The $*$-homomorphic properties of $\beta(h)$ can be translated into the following relations:

- $\beta_1(h, xy) = \beta_1(h, x)\beta_1(h, y) + \beta_2(h, x)\beta_3(h, y)$,
- $\beta_2(h, xy) = \beta_1(h, x)\beta_2(h, y) + \beta_2(h, x)\beta_4(h, y)$,
- $\beta_3(h, xy) = \beta_3(h, x)\beta_1(h, y) + \beta_4(h, x)\beta_3(h, y)$,
- $\beta_4(h, xy) = \beta_3(h, x)\beta_2(h, y) + \beta_4(h, x)\beta_4(h, y)$.
- $\beta_l(h, x^*) = (\beta_l(h, x))^*$, for $l = 1, 4$ and $\beta_2(h, x) = (\beta_3(h, x^*))^*$.

Note that for any $x \in \mathcal{A}$, $\beta_2(h, x) \in \mathcal{B}(\mathbf{h} \otimes \mathbf{k}, \mathbf{h})$ and in fact $\beta_2(h) \in \mathcal{B}(\mathcal{A}, \mathcal{A} \otimes \mathbf{k}^*)$.
For bounded map

$$\alpha\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\hat{\mathbf{k}}),\ \alpha(x) = \begin{pmatrix} \alpha_1(x) & \alpha_2(x) \\ \alpha_3(x) & \alpha_4(x) \end{pmatrix},$$

we write $\Lambda_{\alpha(x)}[k]$ for $\sum_{l=1}^4 \Lambda^l_{\alpha_l(x)}[k]$. Here we note that the map $\sum_{i=1}^m x_i \otimes \mathbf{e}(f_i) \mapsto \Lambda_\alpha[k]\big(\sum_{i=1}^m x_i \otimes \mathbf{e}(f_i)\big)$ defined by

$$\Lambda_\alpha[k]\left(\sum_{i=1}^m x_i \otimes \mathbf{e}(f_i)\right) u = \sum_{i=1}^m \Lambda_{\alpha(x_i)}[k]u \otimes \mathbf{e}(f_i)$$

map $\mathcal{A} \otimes \mathcal{E}(\mathcal{K})$ into $\mathcal{A} \otimes \Gamma$ (for details, see [11, 12]). Now setting $N_{\alpha(x)}[k] := \sum_{l=1}^4 N^l_{\alpha_l(x)}[k]$ and defining $N_\alpha[k]$ as for $\Lambda_\alpha[k]$, we note that $N_\alpha[k]$ maps $\mathcal{A}\otimes\mathcal{E}(\mathcal{K})$ into $\mathcal{A}\otimes\Gamma$.

For $k \geq 1$, $\rho_k(h, x) := N_{\beta(h,x)}[k]$ defines a bounded linear map on $\mathbf{h} \otimes \Gamma$ which act nontrivially only on $\mathbf{h} \otimes \Gamma_k$. By the properties of the family $\{\beta_l(h)\}$ and $\{N^l[k]\}$, each $\rho_k(h)$ is a $*$-homomorphism mapping $\mathcal{A}$ into $\mathcal{A} \otimes \mathcal{B}(\Gamma)$.

Inductively, we define a family of maps $\mathcal{P}^{(h)}_t\colon \mathcal{A} \otimes \mathcal{E}(\mathcal{K}) \to \mathcal{A} \otimes \Gamma$ as follows. We subdivide the interval $[0, t]$ into $[k] \equiv ((k-1)h, kh]$, $1 \leq k \leq n$ so that $t \in ((n-1)h, nh]$ and set for $x \in \mathcal{A}$, $f \in \mathcal{K}$,

$$\left.\begin{aligned} &\mathcal{P}^{(h)}_0(x\mathbf{e}(f)) = x\mathbf{e}(f) \\ &\mathcal{P}^{(h)}_{kh}(x\mathbf{e}(f)) = \mathcal{P}^{(h)}_{(k-1)h} N_{\beta(h)}[k](x \otimes \mathbf{e}(f)) \end{aligned}\right\} \tag{2.6}$$

and $\mathcal{P}_t^{(h)} = \mathcal{P}_{nh}^{(h)}$. Now setting maps $p_t^{(h)}\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\Gamma)$, by $p_t^{(h)}(x)u\mathbf{e}(f) := \mathcal{P}_t^{(h)}(x\mathbf{e}(f))u, \forall u \in \mathbf{h}$, this defines a family of $*$-homomorphism and can be written as $p_{nh}^{(h)} = \rho_1(h)\cdots\rho_n(h)$. Moreover,

$$\left.\begin{array}{l} p_0^{(h)}(x)u\mathbf{e}(f) = xu\mathbf{e}(f) \\ p_t^{(h)}(x)u\mathbf{e}(f) = p_{nh}^{(h)}(x)u\mathbf{e}(f) = \sum_{l=1}^4 N^l_{p_{(n-1)h}^{(h)}(\beta_l(h,x))}[n]u\mathbf{e}(f) \end{array}\right\}. \tag{2.7}$$

As per our convention $p_{(n-1)h}^{(h)}$ which appear above are identified with their ampliation $p_{(n-1)h}^{(h)} \otimes 1_{\mathbf{k}}$, $p_{(n-1)h}^{(h)} \otimes 1_{\mathbf{k}^*}$ and $p_{(n-1)h}^{(h)} \otimes 1_{\mathcal{B}(\mathbf{k})}$ with necessary tensor flip. For $k \geq 1$, $l = 1, 2, 3$ and 4, $N^l_{p_{(k-1)h}^{(h)}(\beta_l(h,x))}[k]$ are defined in terms of

$$\Lambda^1_{(p_{(k-1)h}^{(h)}(\beta_1(h,x))}[k],\ \Lambda^2_{(p_{(k-1)h}^{(h)}\otimes 1_{\mathbf{k}^*})(\beta_2(h,x))}[k],\ \Lambda^3_{(p_{(k-1)h}^{(h)}\otimes 1_{\mathbf{k}})(\beta_3(h,x))}[k]$$

and $\Lambda^4_{(p_{(k-1)h}^{(h)}\otimes 1_{\mathcal{B}(\mathbf{k})})(\beta_4(h,x))}[k]$ where, for example $\Lambda^3_{(p_{(k-1)h}^{(h)}\otimes 1_{\mathbf{k}})(\beta_3(h,x))}[k]$ carries the meaning of $a^\dagger_{(p_{(k-1)h}^{(h)}\otimes 1_{\mathbf{k}})(\beta_3(h,x))}[k]$ with initial Hilbert space $\mathbf{h} \otimes \Gamma_{(k-1)h]}$.

DEFINITION 2.5

This family of $*$-homomorphisms $\{p_t^{(h)}\colon t \geq 0\}$ is said to be *quantum random walk* (in short QRW) associated with $\beta(h)$.

Under certain assumptions on $\beta(h)$ in the next section we shall prove strong convergence of associated quantum random walks to E–H flow. Finally in the last section we shall give a construction of $\beta(h)$ such that the associated QRW approximate an E–H flow dilating a given uniformly continuous QDS. Let us conclude this section with the following observations which will be needed later.

Lemma 2.6. *For any $t \geq 0, t \in [n] = ((n-1)h, nh]$ for some $n \geq 1$ (so as h tends to 0, n increases to ∞ and nh goes to t) and $x \in \mathcal{A}$, $u \in \mathbf{h}$ and $f \in \mathcal{K}$,*

$$\mathcal{P}_t^{(h)}(x\mathbf{e}(f))u = xu\mathbf{e}(f) + \sum_{k=1}^n \mathcal{P}_{(k-1)h}^{(h)} N_{\beta(h,x)-b(x)}[k]\mathbf{e}(f)u + F(h,x,u,f), \tag{2.8}$$

where $*$-homomorphism $\mathcal{A} \ni x \mapsto b(x) \in \mathcal{A} \otimes \mathcal{B}(\hat{\mathbf{k}})$ is given by

$$b(x) = \begin{pmatrix} b_1(x) & b_2(x) \\ b_3(x) & b_4(x) \end{pmatrix} := \begin{pmatrix} x & 0 \\ 0 & x \otimes 1_{\mathbf{k}} \end{pmatrix} = x \otimes 1_{\hat{\mathbf{k}}} \tag{2.9}$$

and $F(h,x,u,f) = -\sum_{k=1}^n \mathcal{P}_{(k-1)h}^{(h)}(x(1_\Gamma - P_h[k])\mathbf{e}(f))u$.

Moreover, for any $f \in \mathcal{M}$,

$$\|F(h,x,u,f)\|^2 \leq h\, c(f,t)\|x\|^2\, \|u\|^2, \tag{2.10}$$

where $c(f,t) = 2t(c_f + \|f\|_\infty)\|\mathbf{e}(f)\|$.

Proof. Since for any $k \geq 1$,

$$N_{b(x)}[k] = \sum_{l=1}^{4} N^l_{b_l(x)}[k] = N^1_x[k] + N^4_{x\otimes 1_{\mathbf{k}}}[k] = x \otimes P_h[k],$$

we get

$$\begin{aligned}
&\mathcal{P}^{(h)}_t(x\mathbf{e}(f))u = \mathcal{P}^{(h)}_{nh}(x\mathbf{e}(f))u \\
&\quad = xu\mathbf{e}(f) + \sum_{k=1}^{n}(\mathcal{P}^{(h)}_{kh} - \mathcal{P}^{(h)}_{(k-1)h})(x\mathbf{e}(f))u \\
&\quad = xu\mathbf{e}(f) + \sum_{k=1}^{n}\mathcal{P}^{(h)}_{(k-1)h}N_{\beta(h,x)-b(x)}[k]\mathbf{e}(f)u \\
&\qquad - \sum_{k=1}^{n}\mathcal{P}^{(h)}_{(k-1)h}(x\otimes 1_\Gamma - N_{b(x)}[k])\mathbf{e}(f)u \\
&\quad = xu\mathbf{e}(f) + \sum_{k=1}^{n}\mathcal{P}^{(h)}_{(k-1)h}N_{\beta(h,x)-b(x)}[k]\mathbf{e}(f)u + F(h,x,u,f).
\end{aligned}$$

In order to obtain (2.10) let us consider the following. For any $1 \leq m \leq n$ setting $Z_m = \sum_{k=1}^m p^{(h)}_{(k-1)h}(x)(1 - P_h[k])$, we have

$$\begin{aligned}
&\|Z_m u\mathbf{e}(f_{mh]})\| \\
&\quad \leq \sum_{k=1}^{m}\|p^{(h)}_{(k-1)h}(x)u\mathbf{e}(f_{(k-1)h]})\|\ \|(1-P_h[k])\mathbf{e}(f_{[k]})\|\|\mathbf{e}(f_{(kh,mh]})\|.
\end{aligned}$$

Now using Lemma 2.2 and the fact that $p^{(h)}_{kh}$'s are homomorphisms,

$$\begin{aligned}
\|Z_m u\mathbf{e}(f_{mh]})\| &\leq \sum_{k=1}^{m} h(c_f + \|f\|_\infty)\|x\|\|u\mathbf{e}(f_{mh]})\| \\
&\leq t(c_f + \|f\|_\infty)\|x\|\|u\mathbf{e}(f_{mh]})\|.
\end{aligned}$$

We have

$$\begin{aligned}
&\|F(h,x,u,f)\|^2 \\
&\quad = \sum_{k=1}^{n}\|p^{(h)}_{(k-1)h}(x)u\mathbf{e}(f_{(k-1)h]})\|^2\|(1-P_h[k])\mathbf{e}(f_{[k]})\|^2\ \|\mathbf{e}(f_{[kh})\|^2 \\
&\qquad + 2\mathcal{R}e\sum_{k=1}^{n}\langle Z_{k-1}u\mathbf{e}(f_{(k-1)h]}),\, p^{(h)}_{(k-1)h}(x)u\mathbf{e}(f_{(k-1)h]})\rangle \\
&\qquad\quad \langle \mathbf{e}(f_{[k]}), (1-P_h[k])\mathbf{e}(f_{[k]})\rangle\ \|\mathbf{e}(f_{[kh})\|^2
\end{aligned}$$

$$\leq \sum_{k=1}^{n} \|x\|^2 \|u\mathbf{e}(f_{(k-1)h]})\|^2 \|(1 - P_h[k])\mathbf{e}(f_{[k]})\|^2 \ \|\mathbf{e}(f_{[kh})\|^2$$

$$+ 2 \sum_{k=1}^{n} \|Z_{k-1} u\mathbf{e}(f_{(k-1)h]})\| \ \|x\| \|u\mathbf{e}(f_{(k-1)h]})\|$$

$$\|(1 - P_h[k])\mathbf{e}(f_{[k]})\|^2 \ \|\mathbf{e}(f_{[kh})\|^2.$$

Using the uniform bound for $\|Z_{k-1}u\mathbf{e}(f_{(k-1)h]})\|$ and Lemma 2.2 the required estimate follows. □

By the above lemma and the definition $p_t^{(h)}$ we have

$$\mathcal{P}_t^{(h)}(x\mathbf{e}(f))u = p_t^{(h)}(x)u\mathbf{e}(f) = xu\mathbf{e}(f) + \sum_{k=1}^{n} N_{p_{(k-1)h}^{(h)}(\beta(h,x)-b(x))}[k]u\mathbf{e}(f) + F(h, x, u, f). \tag{2.11}$$

3. Convergence of quantum random walk

In this section we shall work under the following two assumptions and prove strong convergence of quantum random walk to Evans–Hudson flow extending the ideas in [22]. In §4 we shall concentrate on uniformly continuous QDS on von Neumann algebra and starting from its bounded generator give a construction of QRW which converges strongly.

3.1 *Assumption on structure map θ*

Let $\theta = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{pmatrix} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\hat{\mathbf{k}})$ is a completely bounded (CB) map.

By results from [16, 12] the quantum stochastic differential equation

$$J_t = id_{\mathcal{A}\otimes\Gamma} + \int_0^t J_s \Lambda_\theta(\mathrm{d}s) \tag{3.1}$$

admit a unique solution (regular adapted process) J_t mapping $\mathcal{A}\otimes\mathcal{E}(\mathcal{C})$ into $\mathcal{A}\otimes\Gamma$, where $\mathcal{C}$ is the set of all bounded continuous functions in $L^2(\mathbb{R}_+ \cdot \mathbf{k})$. Moreover, for any $f \in \mathcal{C}$ the map $J_t(\cdot \otimes \mathbf{e}(f))$: $\mathcal{A} \to \mathcal{A} \otimes \Gamma$ is a completely bounded map. For any $f \in \mathcal{C}$, it can be seen that [11]

$$\|\{J_t(x \otimes \mathbf{e}(f)\}u\|^2$$

$$\leq 2e^t(1 + \|f\|_\infty^2) \int_0^t \|[\hat{J}_s(\theta(x) \otimes \mathbf{e}(f))]_{\hat{f}(s)} u\|^2 \mathrm{d}s, \tag{3.2}$$

$$\leq 2e^t(1 + \|f\|_\infty^2) \sup_s \|J_s(\cdot \otimes \mathbf{e}(f))\|_{\mathrm{CB}}^2 \|\theta\|^2 \|x\|^2 \|u\|^2 \int_0^t \|\hat{f}(s)\|^2 \mathrm{d}s, \tag{3.3}$$

where $\hat{J}_s\colon \mathcal{A} \otimes \mathcal{B}(\hat{\mathbf{k}}) \otimes \mathbf{e}(f) \to \mathcal{A} \otimes \mathcal{B}(\hat{\mathbf{k}}) \otimes \Gamma$ is the ampliation of $J_s(\cdot \otimes \mathbf{e}(f))$, and $\hat{f}(s) = 1 \oplus f(s) \in \hat{\mathbf{k}}$. Here $\|\cdot\|_{\mathrm{CB}}$ denote the completely bounded norm. For details on completely bounded map, see [20].

Setting $\theta(h) := Z(\sqrt{h})\theta(\cdot)Z(\sqrt{h})$ where $Z(\sqrt{h}) := \begin{pmatrix} \sqrt{h}\,1_{\mathbf{h}} & 0 \\ 0 & 1_{\mathbf{h}\otimes\mathbf{k}} \end{pmatrix}$, $\theta(h)$ is a completely bounded map and it can be written as $\begin{pmatrix} h\theta_1 & \sqrt{h}\theta_2 \\ \sqrt{h}\theta_3 & \theta_4 \end{pmatrix}$.

Let $\{\beta(h)\}_{h>0}$ be a family of $*$-homomorphism : $\mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\hat{\mathbf{k}})$. Recall that $*$-homomorphism b is given in (2.9). Then the map $E(h) := \beta(h) - b - \theta(h) = \begin{pmatrix} \beta_1(h)-b_1-h\theta_1 & \beta_2(h)-\sqrt{h}\,\theta_2 \\ \beta_3(h)-\sqrt{h}\,\theta_3 & \beta_4(h)-b_4-\theta_4 \end{pmatrix}$ is completely bounded and so are its components. Where there is no confusion, we denote their ampliation by same symbol as the original map.

3.2 *Assumption on the $*$-homomorphic family $\beta(h)$*

There exists an $\varepsilon > 0$ and constant C independent of h such that

$$\|\beta_l(h) - b_l - h^{\varepsilon_l}\theta_l\|_{\mathrm{CB}} \le C h^{\varepsilon+\varepsilon_l}, \tag{3.4}$$

for $l = 1, \ldots, 4$, where $\varepsilon_1 = 1$, $\varepsilon_2 = \varepsilon_3 = \frac{1}{2}$ and $\varepsilon_4 = 0$.

In particular it says that for any $m \ge 1$,

$$\|\beta_l(h, X) - b_l(X) - h^{\varepsilon_l}\theta_l(X)\| \le C\|X\|h^{\varepsilon+\varepsilon_l}, \forall X \in \mathcal{A}\otimes\mathcal{B}(\hat{\mathbf{k}}^{\otimes m}). \tag{3.5}$$

From now on we assume that both **A** and **B** are satisfied. Let us consider the von Neumann algebra

$$\mathcal{B} = \mathcal{A} \otimes \oplus_{m\ge0}\mathcal{B}(\hat{\mathbf{k}}^{\otimes m}) = \oplus_{m\ge0}\mathcal{A} \otimes \mathcal{B}(\hat{\mathbf{k}}^{\otimes m}) \subseteq \mathcal{B}(h \otimes \Gamma_{\mathrm{fr}}(\hat{\mathbf{k}})),$$

where $\Gamma_{\mathrm{fr}}(\hat{\mathbf{k}})$ stands for free Fock space over $\hat{\mathbf{k}}$. Tensoring with identity on $\oplus_{m\ge0}\mathcal{B}(\hat{\mathbf{k}}^{\otimes m})$ and using necessary tensor flips consider the extensions of $\theta, \theta(h), \beta(h), b$ as bounded linear maps from $\mathcal{B}$ into itself and the extensions of $p_t^{(h)}$ and $j_t(\cdot \otimes \mathbf{e}(f))$ as map from $\mathcal{B}$ into $\mathcal{B} \otimes \mathcal{B}(\Gamma)$ and $\mathcal{B} \otimes \Gamma$ respectively. We denote these extensions by same symbols as the original maps. Here we have the following observations which will be needed later for proving the convergence of quantum random walk $p_t^{(h)}$.

Lemma 3.1. *For any $X \in \mathcal{A} \otimes \mathcal{B}(\hat{\mathbf{k}}^{\otimes m})$, $\xi \in \mathbf{h} \otimes \hat{\mathbf{k}}^{\otimes m}$ and $f \in \mathcal{M}$ we have*

1. $\|\sum_{k=1}^n N_{p^{(h)}_{(k-1)h}[\beta(h,X)-b(X)-\theta(h,X)]}[k]\xi\mathbf{e}(f)\| \le h^{\varepsilon}C_1(f,t)\|X\|\|\xi\|$,
2. $\|\sum_{k=1}^n [N_{p^{(h)}_{(k-1)h}(\theta(h,X))}[k] - \Lambda_{p^{(h)}_{(k-1)h}(\theta(X))}[k]]\xi\mathbf{e}(f)\|^2 \le h^{\frac{1}{2}}C_2(f,t)\|X\|^2\|\xi\|^2$,
3. $\|\sum_{k=1}^n p^{(h)}_{(k-1)h}(X)(1 - P_h[k])\xi\mathbf{e}(f))\|^2 \le h^{\frac{1}{2}}\ c(f,t)\|X\|^2\|\xi\|^2$,

where constants $c(f,t)$ is as in Lemma 2.6, $C_1(f,t) = t(1+\|f\|_\infty)\|\mathbf{e}(f)\|$ *and* $C_2(f,t) = (1+t)^2(\|f\|_\infty + \|f\|_\infty^2)^2(1+\|\theta\|_{\mathrm{CB}})^2\|\mathbf{e}(f)\|^2$.

Proof.

(1) For any l we have

$$\left\|\sum_{k=1}^n N^l_{p^{(h)}_{(k-1)h}[\beta_l(h,X)-b_l(X)-h^{\varepsilon_l}\theta_l(X)]}[k]\xi\mathbf{e}(f)\right\|$$
$$\le \sum_{k=1}^n \|N^l_{p^{(h)}_{(k-1)h}[\beta_l(h,X)-b_l(X)-h^{\varepsilon_l}\theta_l(X)]}[k]\xi_{k-1}\mathbf{e}(f_{[k]})\|\|\mathbf{e}(f_{[kh})\|,$$

where $\xi_{k-1} = \xi \mathbf{e}(f_{(k-1)h]})$ is a vector in the initial Hilbert space $\mathbf{h} \otimes \Gamma_{\text{fr}}(\hat{\mathbf{k}}) \otimes \Gamma_{(k-1)h]}$. For any l, from (3.4) and contractivity of $p_t^{(h)}$, we get

$$\|p^{(h)}_{(k-1)h}[\beta_l(h,X) - b_l(X) - h^{\varepsilon_l}\theta_l(X)]\| \leq Ch^{\varepsilon+\varepsilon_l}\|X\|.$$

Hence by (2.4) the above quantity is dominated by $\sum_{k=1}^n h^{1+\varepsilon}C(1+\|f\|_\infty)\|X\| \, \|\xi \mathbf{e}(f)\|$ and the required estimate follows.
(2) By Lemma 2.3 the terms corresponding to $l = 1, 2$ can be estimated as

$$\begin{aligned}
&\left\| \sum_{k=1}^n [h^{\varepsilon_l} N^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k] - \Lambda^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k]]\xi \mathbf{e}(f) \right\| \\
&\quad \leq \sum_{k=1}^n \|[h^{\varepsilon_l} N^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k] - \Lambda^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k]]\xi_{k-1}\mathbf{e}(f_{[k]})\| \|\mathbf{e}(f_{[kh})\| \\
&\quad \leq \sum_{k=1}^n h^{\frac{3}{2}} \|p^{(h)}_{(k-1)h}(\theta_l(X))\| \|\xi_{k-1}\|(\|f\|_\infty + \|f\|^2_\infty)\|\mathbf{e}(f_{[(k-1)h})\| \\
&\quad \leq (\|f\|_\infty + \|f\|^2_\infty)\|\theta\|_{\text{CB}} \sum_{k=1}^n h^{\frac{3}{2}} \|X\| \|\xi \mathbf{e}(f)\|.
\end{aligned}$$

Thus the required estimate follows. Now consider the other two terms corresponding to $l = 3$ and 4. For $1 \leq m \leq n$,

$$Z_m = \sum_{k=1}^m [\sqrt{h} N^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k] - \Lambda^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k]],$$

and by Lemma 2.3(a), we have

$$\begin{aligned}
&\|Z_m u \mathbf{e}(f_{mh]})\| \\
&\quad \leq \sum_{k=1}^m \|[\sqrt{h} N^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k - \Lambda^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k]]\xi_{(k-1)}\mathbf{e}(f_{[k]})\| \|\mathbf{e}(f_{(kh,mh]})\| \\
&\quad \leq \sum_{k=1}^m h \|p^{(h)}_{(k-1)h}(\theta_l(X))\xi_{k-1}\| \, \|f\|_\infty \, \|\mathbf{e}(f_{[k]})\| \|\mathbf{e}(f_{(kh,mh]})\|.
\end{aligned}$$

Thus

$$\|Z_m u \mathbf{e}(f_{mh]})\| \leq t\|\theta\|_{\text{CB}} \, \|f\|_\infty \|X\| \|\xi \mathbf{e}(f_{mh]})\|. \tag{3.6}$$

We have the following equality:

$$\begin{aligned}
&\|Z_n \xi \mathbf{e}(f)\|^2 \\
&\quad = \sum_{k=1}^n \|[\sqrt{h} N^l_{(k-1)h^{(h)}(\theta_l(X))}[k] - \Lambda^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k]]\xi_{(k-1)}\mathbf{e}(f_{[k]})\|^2 \mathbf{e}(f_{[kh})\|^2 \\
&\qquad + 2\,\mathcal{R}e \sum_{k=1}^n \langle Z_{k-1}\xi_{k-1}\mathbf{e}(f_{[k]}), [\sqrt{h} N^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k] - \Lambda^l_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k]] \\
&\qquad\quad \xi_{k-1}\mathbf{e}(f_{[k]})\rangle \|\mathbf{e}(f_{[kh})\|^2.
\end{aligned}$$

By the estimate in Lemma 2.3,

$$\left\| \sum_{k=1}^{n} [\sqrt{h} N^{l}_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k] - \Lambda^{l}_{p^{(h)}_{(k-1)h}(\theta_l(X))}[k]] \xi \mathbf{e}(f) \right\|^2$$

$$\leq \sum_{k=1}^{n} h^2 \| p^{(h)}_{(k-1)h}(\theta_l(X)) \xi_{k-1} \|^2 \| f \|^2_\infty \| \mathbf{e}(f_{[(k-1)h)}) \|^2$$

$$+ 2 \sum_{k=1}^{n} h^2 \| Z_{k-1} \xi_{k-1} \| \| p^{(h)}_{(k-1)h}(\theta_l(X)) \| \, \| \xi_{k-1} \| \, \| f \|^2_\infty \| \mathbf{e}(f_{[(k-1)h)}) \|^2.$$

Now using (3.6), the above quantity is less than or equal to

$$\sum_{k=1}^{n} h^2 \| f \|^2_\infty \, \| X \|^2 \| \xi \mathbf{e}(f) \|^2 + 2 \sum_{k=1}^{n} h^2 t \| \theta \|^2_{\mathrm{CB}} \, \| f \|^3_\infty \| X \|^2 \| \xi \mathbf{e}(f) \|^2$$

and the required estimate follows.
(3) The proof is same as for estimate (2.10) in Lemma 2.6. □

Note that $J_t \colon \mathcal{A} \otimes \mathcal{E}(\mathcal{K}) \to \mathcal{A} \otimes \Gamma$ is the unique solution of the quantum stochastic differential equation

$$J_t = id_{\mathcal{A} \otimes \Gamma} + \int_0^t J_s \Lambda_\theta(\mathrm{d}s). \tag{3.7}$$

Since the above integral is a Riemann–Bochner integral [19, 11], it can be approximated by

$$\sum_{k=1}^{n} J_{(k-1)h} \Lambda_{\theta(\cdot)}[k]$$

for $t \in ((n-1)h, nh]$. The next lemma estimates the order of approximation,

Lemma 3.2. *For $\tau \geq 0$ and $f \in \mathcal{M}$ there exist a constant $C_{f,\tau}$, that depends only on f and τ, such that*

1. *For any $t \leq \tau$, $x \in \mathcal{A}$, $u \in \mathbf{h}$,*

$$\left\| \left[\int_0^t J_s \Lambda_\theta(ds) - \sum_{k=1}^{n} J_{(k-1)h} \Lambda_{\theta(\cdot)}[k] \right] (x\boldsymbol{e}(f))u \right\| \leq C_{f,\tau} \sqrt{h} \|x\| \|u\|.$$

2. *For any $t \leq \tau$, $X \in \mathcal{A} \otimes \mathcal{B}(\hat{\mathbf{k}}^{\otimes m})$ and $\xi \in \mathbf{h}_0 \otimes \hat{\mathbf{k}}^{\otimes m}$,*

$$\left\| \left[\int_0^t J_s \Lambda_\theta(\mathrm{d}s) - \sum_{k=1}^{n} J_{(k-1)h} \Lambda_{\theta(\cdot)}[k] \right] (X\boldsymbol{e}(f))\xi \right\| \leq C_{f,\tau} \sqrt{h} \|X\| \|\xi\|.$$

Proof.

(1) Let $0 \leq s \leq t \leq \tau$. Then,

$$\| J_t(x\mathbf{e}(f))u - J_s(x\mathbf{e}(f))u \|^2 = \left\| \int_s^t J_\tau \Lambda_\theta(\mathrm{d}r)(x\mathbf{e}(f))u \right\|^2.$$

Using the estimate (3.2) the above quantity is dominated by

$$2e^t(1+\|f\|_\infty^2)\int_s^t \|[\hat{J}_r(\theta(x)\mathbf{e}(f))]_{\hat{f}(r)}u\|^2 \mathrm{d}r$$
$$\leq c_{f,\tau}^2(t-s)\sup_r \|J_r(\cdot\otimes\mathbf{e}(f))\|_{\mathrm{CB}}\|\theta\|^2\|x\|^2(1+\|f\|_\infty^2)\|u\|^2.$$

This gives

$$\|J_t(x\mathbf{e}(f))u - J_s(x\mathbf{e}(f))u\| \leq \sqrt{t-s}\, D_{f,\tau}\|x\|\|u\|, \tag{3.8}$$

where $D_{f,\tau}$ is a constant independent of x and u.

Now let us consider

$$\left[\int_0^t J_s\Lambda_\theta(\mathrm{d}s) - \sum_{k=1}^n J_{(k-1)h}(\Lambda_{\theta(\cdot)}[k]\right](x\mathbf{e}(f))u$$
$$= \sum_{k=1}^n \int_{[k]}[J_{(k-1)h} - J_s]\Lambda_\theta(\mathrm{d}s)(x\mathbf{e}(f))u - \int_t^{nh} J_\tau\Lambda_\theta(\mathrm{d}\tau)(x\mathbf{e}(f))u.$$

Taking norm

$$\left\|\left[\int_0^t J_s\Lambda_\theta(\mathrm{d}s) - \sum_{k=1}^n J_{(k-1)h}(\Lambda_{\theta(\cdot)}[k]\right](x\mathbf{e}(f))u\right\|$$
$$\leq \sum_{k=1}^n \left\|\int_{[k]}[J_{(k-1)h} - J_s]\Lambda_\theta(\mathrm{d}s)(x\mathbf{e}(f))u\right\|$$
$$+\left\|\int_t^{nh} J_s\Lambda_\theta(\mathrm{d}s)(x\mathbf{e}(f))u\right\|. \tag{3.9}$$

Using the argument as in the previous estimate and the estimate (3.8)

$$\left\|\int_{[k]}[J_{(k-1)h} - J_s]\Lambda_\theta(\mathrm{d}s)(x\mathbf{e}(f))u\right\|^2 \leq h^{\frac{3}{2}}C_{f,\tau}\|x\|\|u\|$$

and

$$\left\|\int_t^{nh} J_s\Lambda_\theta(\mathrm{d}s)(x\mathbf{e}(f))u\right\| \leq \sqrt{h}C_{f,\tau}\|x\|\|u\|.$$

These along with inequality (3.9) give the required estimate.
(2) By the same argument as above the estimate follows. □

Now we are in position to prove the main result

Theorem 3.3. *Let* $\theta\colon \mathcal{A}\to\mathcal{A}\otimes\mathcal{B}(\hat{\mathbf{k}})$ *be a completely bounded map and* $\{\beta(h)\colon \mathcal{A}\to\mathcal{A}\otimes\mathcal{B}(\hat{\mathbf{k}})\colon h>0\}$ *be a family of* $*$*-homomorphisms such that assumptions* **A** *and* **B** *hold. Let* $p_t^{(h)}$ *be the quantum random walk associated with* $\beta(h)$*. Then for each* $x\in\mathcal{A}$ *and* $t\geq 0$, $p_t^{(h)}(x)$ *converges strongly to* $j_t(x)$*. Thus* $j_t\colon\mathcal{A}\to\mathcal{A}\otimes\mathcal{B}(\Gamma)$ *is a* $*$*-homomorphic flow.*

Proof. Let $\tau \geq 0$ and $f \in \mathcal{M}$ fixed. For $t \leq \tau, h > 0$ set linear maps

$$W_t^{(h)}: \mathcal{A} \to \mathcal{A} \otimes \Gamma$$

given by, for $x \in \mathcal{A}$ and $u \in \mathbf{h}$,

$$\begin{aligned} W_t^{(h)}(x)u &= p_t^{(h)}(x)u\mathbf{e}(f)) - j_t(x)u\mathbf{e}(f) \\ &= [\mathcal{P}_t^{(h)}(x\mathbf{e}(f)) - J_t(x\mathbf{e}(f)]u. \end{aligned}$$

Note that the maps $x \mapsto j_t(x \otimes \mathbf{e}(f))$ and $x \mapsto p_t^{(h)}(x \otimes \mathbf{e}(f))$ are completely bounded and hence they extend as bounded linear maps from $\mathcal{B}$ into $\mathcal{B} \otimes \Gamma$. Thus the map $W_t^{(h)}$ extend as a bounded linear map from $\mathcal{B}$ into $\mathcal{B} \otimes \Gamma$ and the map $Y_t^{(h)}$ given by $Y_t^{(h)}(x\mathbf{e}(f))u := [\mathcal{P}_t^{(h)}(x\mathbf{e}(f)) - J_t(x\mathbf{e}(f)]u$ extend as a linear map from $\mathcal{B} \otimes \mathcal{E}(\mathcal{M})$ into $\mathcal{B} \otimes \Gamma$.

In order to prove the result we shall show that $\|W_t^{(h)}\|$ converges to 0 as h tends to 0. For any $X \in \mathcal{A} \otimes \mathcal{B}(\hat{\mathbf{k}}^{\otimes m})$ and $\xi \in \mathbf{h}_0 \otimes \hat{\mathbf{k}}^{\otimes m}$ by (2.11), we have

$$\begin{aligned} & W_t^{(h)}(X)\xi \\ &= \mathcal{P}_t^{(h)}(X\mathbf{e}(f))\xi - \left\{ id + \sum_{k=1}^n J_{(k-1)h}(\Lambda_{\theta(\cdot)}[k] \right\} (X\mathbf{e}(f))\xi \\ &\quad + \left\{ \sum_{k=1}^n J_{(k-1)h}(\Lambda_{\theta(\cdot)}[k] - \int_0^t J_s \Lambda_\theta(\mathrm{d}s) \right\} (X\mathbf{e}(f))\xi \\ &= \sum_{k=1}^n [N_{p^{(h)}_{(k-1)h}(\beta(h,X)-b(X))}[k] - \Lambda_{j_{(k-1)h}(\theta(X))}[k]]\xi\mathbf{e}(f) \\ &\quad - \sum_{k=1}^n p^{(h)}_{(k-1)h}(X)(1 - P_h[k])\xi\mathbf{e}(f) \\ &\quad + \left\{ \sum_{k=1}^n J_{(k-1)h}(\Lambda_{\theta(\cdot)}[k] - \int_0^t J_s \Lambda_\theta(\mathrm{d}s) \right\} (X\mathbf{e}(f))\xi \\ &= \sum_{k=1}^n [(N_{p^{(h)}_{(k-1)h}(\beta(h,X)-b(X))}[k] - N_{p^{(h)}_{(k-1)h}(\theta(h,X))}[k])\xi\mathbf{e}(f) \\ &\quad + (N_{p^{(h)}_{(k-1)h}(\theta(h,X))}[k] - \Lambda_{p^{(h)}_{(k-1)h}(\theta(X))}[k])\xi\mathbf{e}(f) \\ &\quad + (\Lambda_{p^{(h)}_{(k-1)h}(\theta(X))}[k] - \Lambda_{j_{(k-1)h}(\theta(X))}[k])\xi\mathbf{e}(f)] \\ &\quad - \sum_{k=1}^n p^{(h)}_{(k-1)h}(X)(1 - P_h[k])\xi\mathbf{e}(f) \\ &\quad + \left\{ \sum_{k=1}^n J_{(k-1)h}(\Lambda_{\theta(\cdot)}[k] - \int_0^t J_s \Lambda_\theta(\mathrm{d}s) \right\} (X\mathbf{e}(f))\xi. \end{aligned}$$

Using linearity of $N_{(\cdot)}[k]$ and $\Lambda_{(\cdot)}[k]$,

$$
\begin{aligned}
&\| W_t^{(h)}(X)\xi \|^2 \\
&\quad \le 5 \left(\left\| \sum_{k=1}^{n} N_{p^{(h)}_{(k-1)h}(\beta(h,X)-b(X)-\theta(h,X))}[k]\xi_{k-1}\mathbf{e}(f_{[(k-1)h)}) \right\|^2 \right. \\
&\qquad + \left\| \sum_{k=1}^{n} [N_{p^{(h)}_{(k-1)h}(\theta(h,X))}[k] - \Lambda_{p^{(h)}_{(k-1)h}(\theta(X))}[k]]\xi_{k-1}\mathbf{e}(f_{[(k-1)h)}) \right\|^2 \\
&\qquad + \left\| \sum_{k=1}^{n} p^{(h)}_{(k-1)h}(X)(1 - P_h[k])\xi_{k-1}\mathbf{e}(f_{[(k-1)h)}) \right\|^2 \\
&\qquad + \left\| \left\{ \sum_{k=1}^{n} J_{(k-1)h}(\Lambda_{\theta(\cdot)}[k] - \int_0^t J_s \Lambda_\theta(\mathrm{d}s) \right\} (X\mathbf{e}(f))\xi \right\|^2 \\
&\qquad \left. + \left\| \sum_{k=1}^{n} \Lambda_{[p^{(h)}_{(k-1)h} - j_{(k-1)h}](\theta(X))}[k]\xi_{k-1}\mathbf{e}(f_{[(k-1)h)}) \right\|^2 \right) \\
&\quad = 5(I_1 + I_2 + I_3 + I_4 + I_5).
\end{aligned}
$$

By Lemmas 3.1 and 3.8 we have

$$
I_1 + I_2 + I_3 + I_4 \le \mathrm{const}(f, \tau)\|X\|^2\|\xi\|^2 h^{\varepsilon}.
$$

Now let us consider the terms in I_5,

$$
\begin{aligned}
&\left\| \sum_{k=1}^{n} \Lambda_{[p^{(h)}_{(k-1)h} - j_{(k-1)h}](\theta(X))}[k]\xi\mathbf{e}(f) \right\|^2 \\
&\quad = \left\| \sum_{k=1}^{n} Y^{(h)}_{(k-1)h}\Lambda_\theta[k]X\mathbf{e}(f)\xi \right\|^2 \\
&\quad = \left\| \int_0^{nh} Y_s^{(h)}\Lambda_\theta(\mathrm{d}s)X\mathbf{e}(f)\xi \right\|^2.
\end{aligned}
$$

Using the estimate (3.2),

$$
\begin{aligned}
I_5 &\le 2e^t(1 + \|f\|_\infty^2) \int_0^{nh} \|[Y_s^{(h)}(\theta(X)\mathbf{e}(f))]_{\hat{f}(s)}\xi\|^2 \mathrm{d}s, \\
&= 2e^t(1 + \|f\|_\infty^2) \int_0^{nh} \|W_s^{(h)}(\theta(X))\xi \otimes \hat{f}(s)\|^2 \mathrm{d}s \\
&\le 2e^t(1 + \|f\|_\infty^2)^2 \sum_{k=1}^{n} h\|W^{(h)}_{(k-1)h}\|^2\|\theta(X))\|^2\|\xi\|^2 \\
&\le c_f \sum_{k=1}^{n} h\|W^{(h)}_{(k-1)h}\|^2\|\theta\|^2_{\mathrm{CB}}\|X\|^2\|\xi\|^2.
\end{aligned}
$$

Combining all the above estimates, we obtained

$$\|W_t^{(h)}(X)\xi\|^2 \leq h^\varepsilon C'\|X\|^2\|\xi\|^2 + D\sum_{k=1}^{n} h\|W_{(k-1)h}^{(h)}\|^2\|X\|^2\|\xi\|^2 \tag{3.10}$$

for some constants C' and D independent of h. For any $X = \oplus_{m\geq 0}X_m \in \mathcal{B}$ and $\xi = \oplus_{m\geq 0}\xi_m \in \mathbf{h}\otimes\Gamma_{\mathrm{fr}}(\hat{\mathbf{k}})$, using the estimate (3.10) we have

$$\begin{aligned}
&\|W_t^{(h)}(X)\xi\|^2 \\
&= \sum_{m\geq 0}\|W_t^{(h)}(X_m)\xi_m\|^2 \\
&\leq h^\varepsilon C'\sum_{m\geq 0}\|X_m\|^2\|\xi_m\|^2 + D\sum_{k=1}^{n} h\|W_{(k-1)h}^{(h)}\|^2\sum_{m\geq 0}\|X_m\|^2\|\xi_m\|^2 \\
&\leq h^\varepsilon C'\|X\|^2\|\xi\|^2 + D\sum_{k=1}^{n} h\|W_{(k-1)h}^{(h)}\|^2\|X\|^2\|\xi\|^2.
\end{aligned}$$

Taking supremum over all $\xi \in \mathbf{h}\otimes\Gamma_{\mathrm{fr}}(\hat{\mathbf{k}})$, $X\in\mathcal{B}$ such that $\|\xi\|\leq 1$, $\|X\|\leq 1$ we get

$$\|W_t^{(h)}\|^2 = \|W_{nh}^{(h)}\|^2 \leq h^\varepsilon C' + hD\sum_{k=1}^{n}\|W_{(k-1)h}^{(h)}\|^2. \tag{3.11}$$

By definition $\|W_0^{(h)}\|^2 = 0$ and so (3.11) gives $\|W_h^{(h)}\|^2 \leq h^\varepsilon C'$ and

$$\|W_{2h}^{(h)}\|^2 \leq h^\varepsilon C' + hD\|W_h^{(h)}\|^2 \leq h^\varepsilon C'(1+hD).$$

Then by induction it follows that

$$\|W_t^{(h)}\|^2 = \|W_{nh}^{(h)}\|^2 \leq h^\varepsilon C'(1+hD)^{n-1} \leq h^\varepsilon C' e^{Dt}$$

and hence

$$\lim_{h\to 0}\|W_t^{(h)}\|^2 = 0, \quad \text{in particular} \quad \lim_{h\to 0}\|p_t^{(h)}(x)u\mathbf{e}(f)) - j_t(x)u\mathbf{e}(f)\| = 0,$$

which says that for any $u\in\mathbf{h}$ and $f\in\mathcal{M}$, $\{p_t^{(h)}(x)u\mathbf{e}(f))\colon h>0\}$ is a Cauchy sequence in $\mathbf{h}\otimes\Gamma$. Since $\|p_t^{(h)}(x)\|\leq\|x\|$ and algebraic tensor product $\mathbf{h}\otimes\mathcal{E}(\mathcal{M})$ is dense in $\mathbf{h}\otimes\Gamma$ it follows that $\{p_t^{(h)}(x)\xi\colon h>0\}$ is a Cauchy sequence for all $\xi\in\mathbf{h}\otimes\Gamma$ and hence for each $x\in\mathcal{A}$, $\{p_t^{(h)}(x)\}$ converges strongly to $j_t(x)$. Thus $j_t\colon\mathcal{A}\to\mathcal{A}\otimes\mathcal{B}(\Gamma)$ is a contractive $*$-homomorphic flow. □

4. Construction of QRW for uniformly continuous QDS on a unital von Neumann algebra

Let T_t be a uniformly continuous conservative QDS on a von Neumann algebra $\mathcal{A}\subseteq\mathcal{B}(\mathbf{h})$. Here the generator is a bounded conditionally completely positive (CCP) map and by [5] it admits a unique structure. Let us note down some important observations from [12].

Theorem 4.1. *Given a uniformly continuous conservative quantum dynamical semigroup T_t on $\mathcal{A}$*

1. *There exists a Hilbert space* $\mathbf{k}$, *$*$-representation π of $\mathcal{A}$ in $\mathbf{h} \otimes \mathbf{k}$ and maps* $(\mathcal{L}, \delta, \delta^\dagger, \sigma)$: *$\mathcal{L}$ is the generator of T_t, $\delta \in \mathcal{B}(\mathcal{A}, \mathcal{A} \otimes \mathbf{k})$ is a π-derivation, and $\sigma \in \mathcal{B}(\mathcal{A}, \mathcal{A} \otimes \mathcal{B}(\mathbf{k}))$ given by, for $x \in \mathcal{A}$*

$$\mathcal{L}(x) = R(x \otimes 1_{\mathbf{k}})R - \frac{1}{2}R^*Rx - \frac{1}{2}xR^*R + i[H, x], \tag{4.1}$$

$$\delta(x) = \Sigma^*((x \otimes 1_{\mathbf{k}})R - Rx), \delta^\dagger(x) = \delta(x^*)^*, \tag{4.2}$$

$$\sigma(x) = \pi(x) - x \otimes 1_{\mathbf{k}} = \Sigma^*(x \otimes 1_{\mathbf{k}})\Sigma - x \otimes 1_{\mathbf{k}}, \tag{4.3}$$

where $R \in \mathcal{B}(\mathbf{h}, \mathbf{h} \otimes \mathbf{k})$, H is a self-adjoint element in $\mathcal{A}$, Σ is a partial isometry in $\mathbf{h} \otimes \mathbf{k}$: $\Sigma\Sigma^*$ commute with $\mathcal{A} \otimes 1_{\mathbf{k}}$ (so $\pi(\mathcal{A}) \subseteq \mathcal{A} \otimes \mathcal{B}(\mathbf{k})$) such that $\mathcal{L}(xy) - x\mathcal{L}(y) - \mathcal{L}(x)y = \delta^\dagger(x)\delta(y)$, $\forall x, y \in \mathcal{A}$.

2. *The map $\theta = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{pmatrix} := \begin{pmatrix} \mathcal{L} & \delta^\dagger \\ \delta & \sigma \end{pmatrix}: \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\hat{\mathbf{k}})$, is a bounded CCP map.*
3. *As mentioned earlier there exists a unique solution J_t of the equation,*

$$J_t = id_{\mathcal{A}\otimes\Gamma} + \int_0^t J_s \Lambda_\theta(\mathrm{d}s), 0 \le t \le \tau \tag{4.4}$$

as a regular adapted process mapping $\mathcal{A} \otimes \mathcal{E}(\mathcal{C})$ into $\mathcal{A} \otimes \Gamma$ such that for any $f \in \mathcal{C}$, $J_t(\cdot \otimes \mathbf{e}(f)): \mathcal{A} \to \mathcal{A} \otimes \Gamma$ is completely bounded and $\sup_{t\le\tau} \|J_t(x \otimes \mathbf{e}(f))u\| \le C_f \|x\| \, \|u\|$.

We shall construct a family of $*$-homomorphism $\{\beta(h)\}_{h>0}: \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\hat{\mathbf{k}})$ satisfying Assumption B of the previous section.

Theorem 4.2. *There exists a $*$-homomorphic family $\{\beta(h)\}_{h>0}: \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\hat{\mathbf{k}})$ such that for any l,*

$$\|\beta_l(h) - b_l - h^{\varepsilon_l}\theta_l\|_{\mathrm{CB}} \le Mh^{1+\varepsilon_l}, \tag{4.5}$$

for some constant M independent of h.

Proof. Set $\widetilde{R} = \begin{pmatrix} 0 & -R^* \\ R & 0 \end{pmatrix}$ from $\mathbf{h} \otimes \hat{\mathbf{k}}$ into itself. It is clear that $\widetilde{R}$ is a bounded skew symmetric operator. Thus it generate a one-parameter unitary group $\{e^{t\widetilde{R}}\}$. For $h > 0$, we consider the unitary operator $W(h) = e^{\sqrt{h}\widetilde{R}}$ which can be written as $\begin{pmatrix} \cos(\sqrt{h}|R|) & -\sqrt{h}D(h)R^* \\ \sqrt{h}RD(h) & \cos(\sqrt{h}|R^*|) \end{pmatrix}$ where $D(h) = \sum_{n\ge0}(-1)^n \frac{(\sqrt{h}|R|)^{2n}}{(2n+1)!}$ and when $|R|$, the positive square root of R^*R, is invertible, $D(h) = \sin(\sqrt{h}|R|)(\sqrt{h}|R|)^{-1}$. Consider the partial isometry $V(h) = \begin{pmatrix} e^{-ihH} & 0 \\ 0 & \Sigma(e^{-ihH}\otimes 1_{\mathbf{k}}) \end{pmatrix}$ in $\mathcal{B}(\mathbf{h} \otimes \hat{\mathbf{k}})$.

It can easily be observed that for some constant C, independent of h

$$\|\cos(\sqrt{h}|R|) - 1_{\mathbf{h}} + \frac{h}{2}|R|^2\| \le h^2C\|R\|^4,$$

$$\|\cos(\sqrt{h}|R|) - 1_{\mathbf{h}}\| \le hC\|R\|^2,$$

$$\|\cos(\sqrt{h}|R^*|) - 1_{\mathbf{h}\otimes\mathbf{k}}\| \le hC\|R\|^2,$$

$$
\begin{aligned}
&\|D(h)-1_{\mathbf{h}}\| \le hC\|R\|^2,\\
&\|\cos(\sqrt{h}|R|)\| \le 1,\\
&\|D(h)\| \le 1,\\
&\|e^{ihH}-1_{\mathbf{h}}-ihH\| \le h^2C\|H\|^2,\\
&\|e^{ihH}-1_{\mathbf{h}}\| \le hC\|H\|. \qquad (4.6)
\end{aligned}
$$

Now setting a map $\beta(h), \beta(h,x) \equiv \beta(h)(x) := (V(h))^*(U(h))^*(x \otimes 1_{\hat{\mathbf{k}}})U(h)V(h)$, $\forall x \in \mathcal{A}$, β is a $*$-homomorphism from $\mathcal{A}$ to $\mathcal{A} \otimes \mathcal{B}(\hat{\mathbf{k}})$. So for any $x \in \mathcal{A}$, we have

$$
\beta(h,x) = \begin{pmatrix} \beta_1(h,x) & \beta_2(h,x) \\ \beta_3(h,x) & \beta_4(h,x) \end{pmatrix} = (V(h))^*\eta(h,x)V(h),
$$

where

$$
\begin{aligned}
&\eta(h,x)\\
&\quad = \begin{pmatrix} \eta_1(h,x) & \eta_2(h,x) \\ \eta_3(h,x) & \eta_4(h,x) \end{pmatrix}\\
&\quad = \begin{pmatrix} \begin{matrix}\{\cos(\sqrt{h}|R|)x\cos(\sqrt{h}|R|)\\ +hD(h)R^*(x\otimes 1_{\mathbf{k}})RD(h)\}\end{matrix} & \begin{matrix}\{-\sqrt{h}\cos(\sqrt{h}|R|)xD(h)R^*\\ +\sqrt{h}D(h)R^*(x\otimes 1_{\mathbf{k}})\cos(\sqrt{h}|R^*|)\}\end{matrix} \\[2ex] \begin{matrix}\{-\sqrt{h}RD(h)x\cos(\sqrt{h}|R|)\\ +\sqrt{h}\cos(\sqrt{h}|R^*|)(x\otimes 1_{\mathbf{k}})RD(h)\}\end{matrix} & \begin{matrix}\{hRD(h)xD(h)R^*\\ +\cos(\sqrt{h}|R^*|)(x\otimes 1_{\mathbf{k}})\cos(\sqrt{h}|R^*|)\}\end{matrix} \end{pmatrix}.
\end{aligned}
$$

From the above structure all these maps $\eta_l(h), l = 1, \ldots, 4$ are completely bounded. Now estimate the following. We have

$$
\begin{aligned}
&\beta_1(h,x) - x - h\theta_1(x)\\
&\quad = \{e^{ihH}\eta_1(h,x)e^{-ihH} - \eta_1(h,x) - ih[H,x]\}\\
&\qquad + \left\{\eta_1(h,x) - x - h\left(R^*(x\otimes 1_{\mathbf{k}})R - \frac{1}{2}|R|^2x - \frac{1}{2}x|R|^2\right)\right\}.
\end{aligned}
$$

Let us look at the first term

$$
\begin{aligned}
&e^{ihH}\eta_1(h,x)e^{-ihH} - \eta_1(h,x) - ih[H,x]\\
&\quad = (e^{ihH} - 1_{\mathbf{h}} - ihH)\eta_1(h,x)e^{-ihH} + \eta_1(h,x)(e^{-ihH} - 1 + ihH)\\
&\qquad + ihH\eta_1(h,x)(e^{-ihH} - 1) + ihH(\eta_1(h,x) - x) + ihH(x - \eta_1(h,x)).
\end{aligned}
$$

We also have

$$
\begin{aligned}
\eta_1(h,x) - x = {} & [\cos(\sqrt{h}|R|) - 1_{\mathbf{h}}]x\cos(\sqrt{h}|R|)\\
& + x[\cos(\sqrt{h}|R|) - 1_{\mathbf{h}}] + hD(h)R^*(x\otimes 1_{\mathbf{k}})RD(h).
\end{aligned}
$$

The second term can be rewritten as

$$\eta_1(h,x) - x - h\left(R^*(x\otimes 1_{\mathbf{k}})R - \frac{1}{2}|R|^2 x - \frac{1}{2}x|R|^2\right)$$
$$= \left[\cos(\sqrt{h}|R|) - 1_{\mathbf{h}} + \frac{h}{2}|R|^2\right] x \cos(\sqrt{h}|R|)$$
$$+ x\left[\cos(\sqrt{h}|R|) - 1_{\mathbf{h}} + \frac{h}{2}|R|^2\right] + \frac{h}{2}|R|^2 x[1_{\mathbf{h}} - \cos(\sqrt{h}|R|)]$$
$$+ h[D(h) - 1_{\mathbf{h}}]R^*(x\otimes 1_{\mathbf{k}})RD(h) + hR^*(x\otimes 1_{\mathbf{k}})R[D(h) - 1_{\mathbf{h}}].$$

We note that CB norm of maps of the form $x \mapsto AxB$ and $x \mapsto Y(x\otimes 1_{\mathbf{k}})Z$ are dominated by $\|A\|\ \|B\|$ and $\|Y\|\ \|Z\|$ respectively [20]. Then using (4.6) we get

$$\|\beta_1(h) - b_1 - h\theta_1\|_{\mathrm{CB}} \leq h^2 C_1, \tag{4.7}$$

for some h-independent constant C_1.

Similarly it can be shown that

$$\|\beta_2 - \sqrt{h}\theta_2\|_{\mathrm{CB}} = \|\beta_3 - \sqrt{h}\theta_3\|_{\mathrm{CB}} \leq h^{\frac{3}{2}} C_{23}, \tag{4.8}$$
$$\|\beta_4 - b_4 - \theta_4\|_{\mathrm{CB}} \leq hC_4 \tag{4.9}$$

for some h-independent constants C_{23} and C_4. □

Let $\{p_t^{(h)}\colon h > 0\}$ be the quantum random walks associated with the $*$-homomorphic family $\{\beta(h)\}$ constructed above from the uniformly continuous QDS on $\mathcal{A}$. Then by Theorem 3.3, for any $x \in \mathcal{A}$, $\{p_t^{(h)}(x)\colon h > 0\}$ converges strongly to $j_t(x)$ and thus j_t is a $*$-homomorphic family. This family j_t dilate the uniformly continuous QDS T_t.

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