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The L_p-curvature images of convex bodies and L_p-projection bodies

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Abstract. Associated with the L_p-curvature image defined by Lutwak, some inequalities for extended mixed p-affine surface areas of convex bodies and the support functions of L_p-projection bodies are established. As a natural extension of a result due to Lutwak, an L_p-type affine isoperimetric inequality, whose special cases are L_p-Busemann–Petty centroid inequality and L_p-affine projection inequality, respectively, is established. Some L_p-mixed volume inequalities involving L_p-projection bodies are also established.

Keywords. p-Curvature image; p-affine surface area; L_p-projection body; L_p-mixed volume; L_p-dual mixed volume.

0. Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex sets with non-empty interiors) in $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors, write $\mathcal{K}_0^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$. Let B denote the centered (centrally symmetric with respect to the origin) unit ball in $\mathbb{R}^n$, and write ω_n for $V(B)$, the n-dimensional volume of B. For $K \in \mathcal{K}^n$, the support function of K, $h_K = h(K, \cdot)$: $S^{n-1} \to \mathbb{R}$ is defined by $h(K, u) = \max\{u \cdot x: x \in K\}$, where $u \in S^{n-1}$ and $u \cdot x$ denotes the standard inner product of u and x in $\mathbb{R}^n$. If $K \in \mathcal{K}_0^n$, the polar body of K, K^*, is defined by $K^* = \{x \in \mathbb{R}^n: x \cdot y \leq 1$, for all $y \in K\}$. It is easily seen that $(K^*)^* = K$.

Let $\mathcal{F}^n$, $\mathcal{F}_0^n$ denote the set of all bodies in $\mathcal{K}^n$ and $\mathcal{K}_0^n$, respectively, that have a positive continuous curvature function. The subset of $\mathcal{F}_0^n$ containing the centered bodies will be denoted by $\mathcal{F}_e^n$. Let $f_p(K, \cdot)$ denote the positive continuous p-curvature function of $K \in \mathcal{F}_0^n$ (see §1 for details of notations).

The radial function $\rho_K = \rho(K, \cdot)$: $S^{n-1} \to [0, \infty)$ of a compact star-shaped (about the origin) $K \subset \mathbb{R}^n$, is defined for $u \in S^{n-1}$ by $\rho(K, \cdot) = \max\{\lambda \geq 0: \lambda u \in K\}$. If ρ_K is positive and continuous, call K a star body. Write $\mathcal{S}_o^n$ for the set of star bodies in $\mathbb{R}^n$ and $\mathcal{S}_e^n$ the set of centered star bodies. For $K \in \mathcal{K}_0^n$, it is easily seen that $\rho(K^*, \cdot) = 1/h(K, \cdot)$ and $h(K^*, \cdot) = 1/\rho(K, \cdot)$.

For $p \geq 1$, the p-curvature image of $K \in \mathcal{F}_0^n$, $\Lambda_p K \in \mathcal{S}_o^n$, was defined by Lutwak [10] as follows:

$$f_p(K, \cdot) = \frac{\omega_n}{V(\Lambda_p K)} \rho(\Lambda_p K, \cdot)^{n+p}. \tag{0.1}$$

For the case $p = 1$, the subscript in Λ_p will often be suppressed. Lutwak noted that, for $p = 1$, this definition of curvature image differs from the definition used by him in [6–8].

The p-curvature image plays an important role in Lutwak's Brunn–Minkowski–Firey theory. With it, Lutwak [10] generalized many affine isoperimetric inequalities from their classical forms, due to Petty [12–15] and Leichtweiß [5] to their L_p analogs.

The ideas and techniques in this paper are mainly from Lutwak [10].

One main aim is to establish some inequalities between extended mixed p-affine surface areas of convex bodies and the support functions of L_p-projection bodies using the p-curvature image. Firstly, we use the weak solutions to p-Minkowski problem to give the bijectivity of the mapping Λ_p. Based on this we get an equivalent definition of extended p-affine surface area (Theorem 1), which was introduced by Lutwak in [10] (see §1). From Theorem 1, we obtain an inequality between extended mixed p-affine surface areas of convex bodies and the support functions of L_p-projection bodies (Theorem 2), which is an extension of the mononicity connection between the projections of convex bodies and the affine surface areas, due to Zhang [18].

Our next aim is to establish an L_p-type affine isoperimetric inequality whose special cases include the L_p-Busemann–Petty centroid inequality and the L_p analog of affine projection inequality of Petty [12]. To demonstrate this, the following definitions and results should be given.

For each $p \geq 1$, define $c_{n,p}$ by

$$c_{n,p} = \frac{\omega_{n+p}}{\omega_2 \omega_n \omega_{p-1}}.$$

By simple calculation, we have $nc_{n-2,p} = (n+p)c_{n,p}$.

For $p \geq 1$, the L_p-centroid body, $\Gamma_p M$ of $M \in \mathcal{S}_o^n$ is defined by

$$h(\Gamma_p M, u)^p = \frac{1}{c_{n,p} V(M)} \int_M |u \cdot x|^p \mathrm{d}x. \tag{0.2}$$

For $p \geq 1$, define the L_p-projection body, $\Pi_p K$ of $K \in \mathcal{K}_0^n$ to be a centered convex body by (see [11])

$$h(\Pi_p K, u)^p = \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |u \cdot v|^p \mathrm{d}S_p(K, v), \tag{0.3}$$

where $u, v \in S^{n-1}$, and $S_p(K, v)$ is a positive Borel measure on S^{n-1}, called the p-surface area measure of K. As usual, denote by $\Pi_p^* K$ the polar body of $\Pi_p K$.

In [11], Lutwak, Yang and Zhang proved the following important results. (Theorem B was also obtained by Campi and Gronchi [1] in a different way.)

Theorem A (L_p-Petty projection inequality). *If $K \in \mathcal{K}_0^n$, then for $1 \leq p < \infty$,*

$$V(K)^{(n-p)/p} V(\Pi_p^* K) \leq \omega_n^{n/p},$$

with equality if and only if K is a centered ellipsoid.

Theorem B (L_p-Busemann–Petty centroid inequality). *For $M \in \mathcal{S}_o^n$, and $1 \leq p < \infty$,*

$$V(\Gamma_p M) \geq V(M),$$

with equality if and only if M is a centered ellipsoid.

We will work with Theorem A to give an L_p-type inequality (Theorem 3). Then, we will make use of the characterizations of the p-curvature images of convex bodies and the L_p-projection bodies to show that Theorem B and the L_p-affine projection inequality (Corollary 5) both are special cases of Theorem 3.

In the same spirit of studying the support functions of L_p-projection bodies, we will establish some L_p-mixed volume inequalities associated to L_p-projection bodies in §4.

1. Preliminaries

For quick reference, we recall some basic properties of L_p-mixed and dual mixed volumes, and mixed p-affine surface areas and their extensions.

For $p \geq 1$, $K, L \in \mathcal{K}_0^n$, and $\varepsilon > 0$, the Firey L_p-combination (see [3]) $K +_p \varepsilon \cdot L \in \mathcal{K}_0^n$ is defined by

$$h(K +_p \varepsilon \cdot L, \cdot)^p = h(K, \cdot)^p + \varepsilon h(L, \cdot)^p.$$

For $p \geq 1$, the L_p-mixed volume, $V_p(K, L)$ of $K, L \in \mathcal{K}_0^n$ was defined in [9] by

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

It was shown in [9], that for $K \in \mathcal{K}_0^n$, there is a positive Borel measure, $S_p(K, \cdot)$, on S^{n-1} such that

$$V_p(K, Q) = \frac{1}{n} \int_{S^{n-1}} h(Q, u)^p \mathrm{d}S_p(K, u), \tag{1.1}$$

for each body $Q \in \mathcal{K}_0^n$. The measure $S_1(K, \cdot)$ is just the classical surface area measure of K, and will be written by $S_K = S(K, \cdot)$ as usual.

A body $K \in \mathcal{K}_0^n$ is called a p-zonoid if its support function h_K can be expressed as

$$h(K, u)^p = \frac{1}{(n + p)\omega_n c_{n,p}} \int_{S^{n-1}} |u \cdot v|^p \mathrm{d}\mu_K(v),$$

with an even positive finite Borel measure μ_K on S^{n-1}. In view of eq. (0.3), an L_p-projection body is a p-zonoid. Conversely, every centered p-zonoid is an L_p-projection body (cf. [17]). Denote by $\mathcal{Z}_p^n$ the set of L_p-projection bodies (centered p-zonoids).

The L_p analog of the classical Minkowski inequality (see [9]) states that for $K, L \in \mathcal{K}_0^n$, and $p \geq 1$,

$$V_p(K, L)^n \geq V(K)^{n-p} V(L)^p, \tag{1.2}$$

with equality if and only if K and L are dilates.

Thus, for $p \geq 1$ and $K \in \mathcal{K}_0^n$,

$$V_p(K, K) = V(K). \tag{1.3}$$

A body $K \in \mathcal{K}_0^n$ is said to have a p-curvature function $f_p(K, \cdot)\colon S^{n-1} \to \mathbb{R}$, if the integral representation

$$V_p(K, Q) = \frac{1}{n} \int_{S^{n-1}} h(Q, u)^p f_p(K, u) \mathrm{d}S(u), \tag{1.4}$$

holds for all $Q \in \mathcal{K}_0^n$. When $p = 1$, the classical curvature function of $K \in \mathcal{K}^n$ is usually denoted by $f_K = f(K, \cdot)$.

Suppose $K \in \mathcal{K}_0^n$ and $Q \in \mathcal{S}_o^n$. For $p \geq 1$, define $V_p(K, Q^*)$ by (see [10])

$$V_p(K, Q^*) = \frac{1}{n} \int_{S^{n-1}} \rho(Q, u)^{-p} \mathrm{d}S_p(K, u). \tag{1.5}$$

Since $h_{L^*} = 1/\rho_L$ for $L \in \mathcal{K}_0^n$, it follows that, if Q happens to belong to $\mathcal{K}_0^n$, definition (1.5) agrees with (1.1).

The p-affine surface area, $\Omega_p(K)$ of $K \in \mathcal{F}_0^n$ can be defined by

$$\Omega_p(K) = \int_{S^{n-1}} f_p(K, u)^{\frac{n}{n+p}} \mathrm{d}S(u). \tag{1.6}$$

In [10], Lutwak gave an L_p extension of Leichtweiß's definition (see [5]) of extended affine surface area as follows: For $p \geq 1$, $K \in \mathcal{K}_0^n$, define $\Omega_p(K)$ by

$$n^{-\frac{p}{n}} \Omega_p(K)^{\frac{n+p}{n}} = \inf\{n V_p(K, Q^*) V(Q)^{\frac{p}{n}} \colon Q \in \mathcal{S}_o^n\}. \tag{1.7}$$

When $p = 1$, the subscript will often be suppressed.

For $p \geq 1$, the mixed p-affine surface area of $K, L \in \mathcal{F}_0^n$, $\Omega_{-p}(K, L)$, can be defined by

$$\Omega_{-p}(K, L) = \int_{S^{n-1}} f_p(K, u) f_p(L, u)^{-\frac{p}{n+p}} \mathrm{d}S(u). \tag{1.8}$$

Thus from (1.6), it follows that for $p \geq 1$ and $K \in \mathcal{F}_0^n$,

$$\Omega_{-p}(K, K) = \Omega_p(K). \tag{1.9}$$

Since for any $K \in \mathcal{K}_0^n$, the L_p-surface area measure, $S_p(K, \cdot)$, is well-defined, we can give a natural extension of eq. (1.8) of the L_p-mixed affine surface area Ω_{-p} from $\mathcal{F}_0^n \times \mathcal{F}_0^n$ to $\mathcal{K}_0^n \times \mathcal{F}_0^n$. Specifically, for $K \in \mathcal{K}_0^n$ and $L \in \mathcal{F}_0^n$, let

$$\Omega_{-p}(K, L) = \int_{S^{n-1}} f_p(L, u)^{-\frac{p}{n+p}} \mathrm{d}S_p(K, u). \tag{1.10}$$

It is well-known that for $K \in \mathcal{F}_0^n$, $\mathrm{d}S_p(K, \cdot) = f_p(K, \cdot)\mathrm{d}S(\cdot)$. Thus (1.10) boils down to (1.8) for $K \in \mathcal{F}_0^n$. Note that the case $p = 1$ was studied by Lutwak in [8].

For star bodies $K, L \in \mathcal{S}_o^n$, and $p \geq 1$, the L_p-dual mixed volume $\tilde{V}_{-p}(K, L)$ of K, L, can be defined by (see [10, 11])

$$\tilde{V}_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n+p} \rho(L, u)^{-p} \mathrm{d}S(u). \tag{1.11}$$

From (1.11) it follows immediately that for each $K \in \mathcal{S}_o^n$,

$$\tilde{V}_{-p}(K, K) = V(K). \tag{1.12}$$

2. The *p*-curvature image and the extended mixed *p*-affine surface area

Let us start with the solutions to the p-Minkowski problem ($p \geq 1$).

If $p = 1$, it is well-known that the weak solution to the Minkowski problem states that if a Borel measure μ on S^{n-1} is not concentrated on a great sphere and has property that $\int_{S^{n-1}} u \mathrm{d}\mu(u) = 0$, then there exists a body $L \in \mathcal{K}^n$, unique up to translation, such that $S_L = \mu$. Thus, given a continuous function $f: S^{n-1} \to (0, \infty)$, satisfying $\int_{S^{n-1}} uf(u)\mathrm{d}S(u) = 0$, there is a body $L \in \mathcal{F}^n$, unique up to translation, such that $f_L = f$.

If $n \neq p > 1$, it was shown in [9] that the weak solution to the p-Minkowski problem with even data gives: If μ is an even positive Borel measure on S^{n-1}, which is not concentrated on a great sphere of S^{n-1}, there exists an unique centered $L \in \mathcal{K}_e^n$, such that $S_p(L, \cdot) = \mu$. Thus, given a continuous function $f: S^{n-1} \to (0, \infty)$, such that $f(\cdot)S(\cdot)$ is an even positive Borel measure on S^{n-1}, there exists a unique body $L \in \mathcal{F}_e^n$, such that $f_p(L, \cdot) = f(\cdot)$.

Now we restrict the domain of Λ_p on $\mathcal{F}_e^n$. Then it is easily verified that its range is $\mathcal{S}_e^n$. Let $\Lambda_p|\mathcal{F}_e^n$ denote the restriction of Λ_p on $\mathcal{F}_e^n$. More accurately, we have the following:

PROPOSITION 1

For $n \neq p \geq 1$, the mapping $\Lambda_p|\mathcal{F}_e^n: \mathcal{F}_e^n \to \mathcal{S}_e^n$ is bijective.

Proof. The injectivity of Λ_p ($p \geq 1$) on $\mathcal{F}_0^n$ was proved by Lutwak [10]. So we need only to show the surjectivity of Λ_p on $\mathcal{F}_e^n$.

For the case $p = 1$, suppose $K \in \mathcal{S}_e^n$. Since $K \in \mathcal{S}_e^n$ is equivalent to

$$\int_{S^{n-1}} u\rho(K, u)^{n+1}\mathrm{d}S(u) = 0$$

(see p. 250 of [2]), let $f(u) = \omega_n \rho(K, u)^{n+1}/V(K)$. There is a body $L \in \mathcal{F}^n$, unique up to translation, satisfying $f_L = f$. But f is an even positive continuous function, $f(L, \cdot)S(\cdot) = S(L, \cdot)$ is an even measure, and this occurs if and only if $L \in \mathcal{F}_e^n$. Now it is easy to verify from (0.1) and the injectivity of Λ that $K = \Lambda L$.

On the other hand, if $n \neq p > 1$, and $K \in \mathcal{S}_e^n$, let $f(\cdot) = \omega_n \rho(K, \cdot)^{n+p}/V(K)$. Then f is an even positive continuous function, and there must exist a unique centered $L \in \mathcal{F}_e^n$, such that $f_p(L, \cdot) = f(\cdot)$. From (0.1) and the injectivity of Λ_p, it is easily seen that $K = \Lambda_p L$. This completes the proof. □

The following result due to Lutwak will be needed.

PROPOSITION 2 [10]

If $p \geq 1$ and $K \in \mathcal{F}_0^n$, then

$$V_p(K, Q^*) = \omega_n \tilde{V}_{-p}(\Lambda_p K, Q)/V(\Lambda_p K), \tag{2.1}$$

for all $Q \in \mathcal{S}_o^n$.

From (1.10), (0.1) and (1.5), it follows immediately that for $p \geq 1$, $K \in \mathcal{K}_0^n$, $L \in \mathcal{F}_0^n$,

$$\omega_n^p \Omega_{-p}(K, L)^{n+p} = n^{n+p} V(\Lambda_p L)^p V_p(K, \Lambda_p^* L)^{n+p}. \tag{2.2}$$

Note that $\Lambda_p^* L$ can be viewed as the 'polar body' of $\Lambda_p L \in \mathcal{S}_o^n$, defined by (1.5).

Take L for K in (2.2). From (1.9) and (2.1) we get

$$\omega_n^p \Omega_p(L)^{n+p} = n^{n+p} V(\Lambda_p L)^p V_p(L, \Lambda_p^* L)^{n+p}$$
$$= n^{n+p} \omega_n^{n+p} V(\Lambda_p L)^p \tilde{V}_{-p}(\Lambda_p L, \Lambda_p L)^{n+p} / V(\Lambda_p L)^{n+p}.$$

Thus by (1.12) we obtain

$$\Omega_p(L)^{n+p} = n^{n+p} \omega_n^n V(\Lambda_p L)^p. \tag{2.3}$$

This was obtained by Lutwak in Proposition 4.5 of [10].

Combine (2.2) and (2.3), and the result is as follows:

PROPOSITION 3

For $p \geq 1$ and $K \in \mathcal{K}_0^n$, $L \in \mathcal{F}_0^n$,

$$n^{-\frac{p}{n}} \Omega_{-p}(K, L) \Omega_p(L)^{\frac{p}{n}} = n V(\Lambda_p L)^{\frac{p}{n}} V_p(K, \Lambda_p^* L). \tag{2.4}$$

From Propositions 1 and 3, and (1.7), we obtain the following.

Theorem 1. *If $n \neq p \geq 1$, and $K \in \mathcal{K}_0^n$, then*

$$\Omega_p(K)^{\frac{n+p}{n}} = \inf\{\Omega_{-p}(K, L) \Omega_p(L)^{\frac{p}{n}} : L \in \mathcal{F}_e^n\}. \tag{2.5}$$

Corresponding to (1.7), Theorem 1 can be viewed as an equivalent definition of the extended p-affine surface area ($n \neq p \geq 1$) of arbitrary convex bodies $K \in \mathcal{K}_0^n$ (rather than in $\mathcal{F}_0^n$). From Theorem 1, the following result is obvious.

COROLLARY 1

If $n \neq p \geq 1$, and $K \in \mathcal{K}_0^n$, $L \in \mathcal{F}_e^n$, then

$$\Omega_{-p}(K, L)^n \geq \Omega_p(K)^{n+p}\, \Omega_p(L)^{-p}. \tag{2.6}$$

Note that if $p \geq 1$ and $K, L \in \mathcal{F}_0^n$, then from the Hölder inequality (see p. 140 of [4]), we obtain that equality holds in (2.6) if and only if K and L are dilates. It was shown in [8] that if $p = 1$, then (2.5) and (2.6) hold for all $K \in \mathcal{K}^n$ and $L \in \mathcal{F}^n$.

Before we give the next theorem, we first prove the following lemma for later use.

Lemma 1. *For $p \geq 1$ and $L, M \in \mathcal{K}_0^n$,*

$$V_p(L, \Pi_p M) = V_p(M, \Pi_p L).$$

Proof. From (1.1), (0.3) and Fubini theorem, we have

$$V_p(L, \Pi_p M) = \frac{1}{n} \int_{S^{n-1}} h(\Pi_p M, u)^p \mathrm{d}S_p(L, u)$$
$$= \frac{1}{n} \int_{S^{n-1}} \left[\frac{1}{n \omega_n c_{n-2,p}} \int_{S^{n-1}} |u \cdot v|^p \mathrm{d}S_p(M, v) \right] \mathrm{d}S_p(L, u)$$

$$= \frac{1}{n}\int_{S^{n-1}} \left[\frac{1}{n\omega_n c_{n-2,p}}\int_{S^{n-1}} |u\cdot v|^p \mathrm{d}S_p(L,u)\right]\mathrm{d}S_p(M,v)$$

$$= \frac{1}{n}\int_{S^{n-1}} h(\Pi_p L, v)^p \mathrm{d}S_p(M,v)$$

$$= V_p(M, \Pi_p L). \qquad \square$$

Now define a class

$$\mathcal{W}_p^n = \{Q \in \mathcal{F}_e^n\text{: there exists a } Z \in \mathcal{Z}_p^n \text{ with } f_p(Q,\cdot) = h(Z,\cdot)^{-(n+p)}\},$$

where $\mathcal{Z}_p^n$ is the set of centered p-zonoids (i.e., the set of L_p-projection bodies).

Theorem 2. *Let $p \geq 1$ and $L \in \mathcal{F}_e^n$. If $L \in \mathcal{W}_p^n$, then for all $K, M \in \mathcal{K}_0^n$,*

$$\frac{\Omega_{-p}(K,L)}{\Omega_{-p}(M,L)} \geq \min_{u\in S^{n-1}} \frac{h(\Pi_p K, u)^p}{h(\Pi_p M, u)^p}.$$

Proof. Let $f_p(L,u)^{\frac{-p}{n+p}} = h(Z,u)^p$ for some $Z \in \mathcal{Z}_p^n$. Since every centered p-zonoid is an L_p-projection body, let $Z = \Pi_p Q$ for some $Q \in \mathcal{K}_0^n$. From Lemma 1, we have

$$\frac{\Omega_{-p}(K,L)}{\Omega_{-p}(M,L)} = \frac{\int_{S^{n-1}} f_p(L,u)^{\frac{-p}{n+p}} \mathrm{d}S_p(K,u)}{\int_{S^{n-1}} f_p(L,u)^{\frac{-p}{n+p}} \mathrm{d}S_p(M,u)} = \frac{\int_{S^{n-1}} h(Z,u)^p \mathrm{d}S_p(K,u)}{\int_{S^{n-1}} h(Z,u)^p \mathrm{d}S_p(M,u)} = \frac{V_p(K,Z)}{V_p(M,Z)}$$

$$= \frac{V_p(K,\Pi_p Q)}{V_p(M,\Pi_p Q)} = \frac{V_p(Q,\Pi_p K)}{V_p(Q,\Pi_p M)} = \frac{\int_{S^{n-1}} h(\Pi_p K,u)^p \mathrm{d}S_p(Q,u)}{\int_{S^{n-1}} h(\Pi_p M,u)^p \mathrm{d}S_p(Q,u)}$$

$$\geq \min_{u\in S^{n-1}} \frac{h(\Pi_p K,u)^p}{h(\Pi_p M,u)^p}.$$

This proves the theorem. $\square$

The following result is an immediate consequence of Theorem 2.

COROLLARY 2

Let $p \geq 1$, $L \in \mathcal{F}_e^n$. If $L \in \mathcal{W}_p^n$, then

$$\Omega_{-p}(M,L) \leq \Omega_{-p}(K,L)$$

holds for all $K, M \in \mathcal{K}_0^n$ satisfying $\Pi_p M \subseteq \Pi_p K$.

COROLLARY 3

If $n \neq p \geq 1$, $L \in \mathcal{W}_p^n$ and $M \in \mathcal{K}_0^n$, then

$$\left[\frac{\Omega_p(L)}{\Omega_p(M)}\right]^{\frac{n+p}{n}} \geq \min_{u\in S^{n-1}} \frac{h(\Pi_p L,u)^p}{h(\Pi_p M,u)^p}.$$

Proof. Take L for K in Theorem 2. By (1.9) and Corollary 1 we have

$$\frac{\Omega_{-p}(L,L)}{\Omega_{-p}(M,L)} \le \frac{\Omega_p(L)}{\Omega_p(M)^{\frac{n+p}{n}}\Omega_p(L)^{\frac{-p}{n}}} = \left[\frac{\Omega_p(L)}{\Omega_p(M)}\right]^{\frac{n+p}{n}}.$$

This proves the corollary. □

The above corollary implies that for $n \neq p \ge 1$, $L \in \mathcal{W}_p^n$ and $M \in \mathcal{K}_0^n$ if

$$\Pi_p M \subseteq \Pi_p L.$$

Then

$$\Omega_p(M) \le \Omega_p(L).$$

Theorem 2 and Corollary 3 for the case $p = 1$ and $K, M \in \mathcal{F}_e^n$ were established by Zhang in [18].

3. An L_p-type affine isoperimetric inequality

In [6], Lutwak proved an affine isoperimetric inequality whose special cases include a general version of the Busemann–Petty centroid inequality and the affine projection inequality of Petty. In this section, we will establish the L_p analog of Lutwak's result by using the p-curvature image and starting with the L_p-Petty projection inequality. We note again that for $p \ge 1$, the definition of p-curvature image is not an extension of the definition used in [6–8].

If we apply the Hölder inequality to the functions $\rho(K,u)^{\frac{-pn}{n+p}} f_p(L,u)^{\frac{n}{n+p}}$ and $\rho(K,u)^{\frac{np}{n+p}}$, then from (1.6), (1.5), and the polar coordinate formula of volume, we get

Lemma 2. *If $p \ge 1$, and $L \in \mathcal{F}_0^n$, $K \in \mathcal{S}_e^n$, then*

$$\Omega_p(L)^{n+p} \le n^{n+p} V_p(L, K^*)^n V(K)^p,$$

with equality if and only if L and $\Lambda_p^{-1} K$ are dilates.

Here the existence of Λ_p^{-1} is ensured by Proposition 1 and the condition $K \in \mathcal{S}_e^n$, and the equality condition follows from the Hölder inequality.

Let $K \in \mathcal{K}_0^n$. Obviously $\Pi_p^* K \in \mathcal{S}_e^n$ and since $\Pi_p K \in \mathcal{K}_0^n$, $(\Pi_p^* K)^* = \Pi_p K$. Now take $\Pi_p^* K$ for K in Lemma 2, and we get: For $n \neq p \ge 1$, $L \in \mathcal{F}_0^n$, $K \in \mathcal{K}_0^n$,

$$\Omega_p(L)^{n+p} \le n^{n+p} V_p(L, \Pi_p K)^n V(\Pi_p^* K)^p, \tag{3.1}$$

with equality if and only if L and $\Lambda_p^{-1}\Pi_p^* K$ are dilates.

Combining Theorem A and (3.1), we get

Theorem 3. *If $p \ge 1$ and $L \in \mathcal{F}_0^n$, $K \in \mathcal{K}_0^n$, then*

$$V(K)^{n-p}\Omega_p(L)^{n+p} \le \omega_n^n n^{n+p} V_p(L, \Pi_p K)^n, \tag{3.2}$$

with equality if and only if L and K are dilated ellipsoids centered at the origin.

Note that the equality of (3.2) occurs if and only if K is a centered ellipsoid and L and $\Lambda_p^{-1}\Pi_p^* K$ are dilates. That K is a centered ellipsoid occurs if and only if $\Pi_p^* K$, and then $\Lambda_p^{-1}\Pi_p^* K$, is an ellipsoid dilated to K (cf. [6]). This gives the desired equality condition.

Two special cases of Theorem 3, are Theorem B and the L_p analog of affine projection inequality. In order to demonstrate this natural extension of Lutwak [6]'s result, we give the following lemmas.

Lemma 3. *If $p \geq 1$ and $L \in \mathcal{F}_0^n$, then*

$$\Pi_p L = \Gamma_p \Lambda_p L.$$

Proof. Since $nc_{n-2,p} = (n+p)c_{n,p}$, from (0.2), (0.1) and (0.3), we have

$$\begin{aligned} h(\Gamma_p \Lambda_p L, u)^p &= \frac{1}{c_{n,p} V(\Lambda_p L)} \int_{\Lambda_p L} |u \cdot x|^p \mathrm{d}x \\ &= \frac{1}{(n+p)c_{n,p} V(\Lambda_p L)} \int_{S^{n-1}} |u \cdot v|^p \rho(\Lambda_p L, v)^{n+p} \mathrm{d}S(v) \\ &= \frac{1}{\omega_n (n+p) c_{n,p}} \int_{S^{n-1}} |u \cdot v|^p f_p(L, v) \mathrm{d}S(v) \\ &= \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |u \cdot v|^p \mathrm{d}S_p(L, v) \\ &= h(\Pi_p L, u)^p. \end{aligned}$$

□

If we combine the results of Lemmas 1 and 3, we get

Lemma 4. *If $p \geq 1$ and $L \in \mathcal{F}_0^n$, $M \in \mathcal{K}_0^n$, then*

$$V_p(L, \Pi_p M) = V_p(M, \Gamma_p \Lambda_p L).$$

We are now in a position to show that the L_p-Busemann–Petty centroid inequality and the L_p-affine projection inequality both are special cases of Theorem 3.

COROLLARY 4

If $M \in \mathcal{S}_o^n$, then for $1 \leq p < \infty$,

$$V(\Gamma_p M) \geq V(M), \tag{3.3}$$

with equality if and only if M is a centered ellipsoid.

Proof. Suppose $K \in \mathcal{K}_0^n$ and $L \in \mathcal{F}_0^n$. If we combine (2.3) and Theorem 3, we get

$$V(K)^{n-p} V(\Lambda_p L)^p \leq V_p(K, \Pi_p L)^n, \tag{3.4}$$

with equality if and only if L and K are dilated ellipsoids centered at the origin. Hence, from Lemma 4 we can rewrite (3.4) as

$$V(K)^{n-p} V(\Lambda_p L)^p \leq V_p(K, \Gamma_p \Lambda_p L)^n.$$

Now let $M = \Lambda_p L$ and take $\Gamma_p M$ for K. Then from (1.3) we get

$$V(M) \leq V(\Gamma_p M).$$

To obtain the equality condition, we note that if L is an origin symmetric ellipsoid, then from Proposition 1, it is easily obtained from observation that $\Lambda_p L$ must also be an origin symmetric ellipsoid. That is exactly the equality condition desired. □

Remark. For $p = \infty$, the quantities on both sides of inequality (3.3) are equal.

COROLLARY 5 (L_p-affine projection inequality)

If $p \geq 1$ *and* $L \in \mathcal{F}_0^n$, *then*

$$\Omega_p(L)^{n+p} \leq n^{n+p}\omega_n^n V(\Pi_p L)^p, \tag{3.5}$$

with equality if and only if L is a centered ellipsoid.

Proof. For $L \in \mathcal{F}_0^n$, from Lemma 1 and (1.3), we have

$$V_p(L, \Pi_p(\Pi_p L)) = V_p(\Pi_p L, \Pi_p L) = V(\Pi_p L). \tag{3.6}$$

Now take $\Pi_p L$ for K in Theorem 3, and from (3.6) we get

$$V(\Pi_p L)^{n-p}\Omega_p(L)^{n+p} \leq n^{n+p}\omega_n^n V_p(L, \Pi_p(\Pi_p L))^n = n^{n+p}\omega_n^n V(\Pi_p L)^n.$$

This is just (3.5). The equality holds if and only if L and $\Pi_p L$ are dilated centered ellipsoids, i.e., if and only if L is a centered ellipsoid. □

4. L_p-mixed volume inequalities involving L_p-projection bodies

Similar to §2, we will show an inequality between L_p-mixed volumes of convex bodies and the support functions of L_p-projection bodies in this section.

Theorem 4. *Let* $p \geq 1$, $K \in \mathcal{K}_0^n$, *if* $K \in \mathcal{Z}_p^n$, *then for all* $L, M \in \mathcal{K}_0^n$,

$$\frac{V_p(L, K)}{V_p(M, K)} \geq \min_{u \in S^{n-1}} \frac{h(\Pi_p L, u)^p}{h(\Pi_p M, u)^p}, \tag{4.1}$$

with equality if and only if L and M are dilates.

Proof. Since K is an L_p-projection body, there exists a $Q \in \mathcal{K}_0^n$ such that $K = \Pi_p Q$. From Lemma 1 we have

$$\frac{V_p(L, K)}{V_p(M, K)} = \frac{V_p(L, \Pi_p Q)}{V_p(M, \Pi_p Q)} = \frac{V_p(Q, \Pi_p L)}{V_p(Q, \Pi_p M)} = \frac{\int_{S^{n-1}} h(\Pi_p L, u)^p \mathrm{d}S_p(Q, u)}{\int_{S^{n-1}} h(\Pi_p M, u)^p \mathrm{d}S_p(Q, u)}$$

$$\geq \min_{u \in S^{n-1}} \frac{h(\Pi_p L, u)^p}{h(\Pi_p M, u)^p}.$$

Obviously, if K and L are dilates, then the equality in (4.1) must hold. On the other hand, the equality in (4.1) holds only if $\Pi_p K$ and $\Pi_p L$ are dilates in view of the mean value theorem for integrals, that is, K and L are dilations of each other. This completes the proof. □

COROLLARY 6

If $n \neq p \geq 1$, $K \in \mathcal{Z}_p^n$ and $M \in \mathcal{K}_0^n$, then

$$\left[\frac{V(K)}{V(M)}\right]^{\frac{n-p}{n}} \geq \min_{u \in S^{n-1}} \frac{h(\Pi_p K, u)^p}{h(\Pi_p M, u)^p},$$

with equality if and only if K and M are dilates.

Proof. Take K for L in (4.1). From (1.3) and (1.2) we have

$$\frac{V_p(K,K)}{V_p(M,K)} \leq \frac{V(K)}{V(M)^{\frac{n-p}{n}} V(K)^{\frac{p}{n}}} = \left[\frac{V(K)}{V(M)}\right]^{\frac{n-p}{n}}.$$

Thus the equality condition follows from (1.2). □

The above corollary implies that for $M \in \mathcal{K}_0^n$ and K is an L_p-projection body, if

$$\Pi_p M \subseteq \Pi_p K,$$

then

$$V(M) \leq V(K) \quad \text{for } n > p \geq 1,$$

and

$$V(M) \geq V(K) \quad \text{for } n < p.$$

This is just the L_p version of the celebrated Petty–Schneider theorem, i.e., it is an affirmative answer to the L_p analog of Shephard problem (cf. [12, 16]).

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