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Holomorphic two-spheres in complex Grassmann manifold $G(2,4)$

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Abstract. In this paper, we use the harmonic sequence to study the linearly full holomorphic two-spheres in complex Grassmann manifold $G(2,4)$. We show that if the Gaussian curvature K (with respect to the induced metric) of a non-degenerate holomorphic two-sphere satisfies $K \leq 2$ (or $K \geq 2$), then K must be equal to 2. Simultaneously, we show that one class of the holomorphic two-spheres with constant curvature 2 is totally geodesic. Concerning the degenerate holomorphic two-spheres, if its Gaussian curvature $K \leq 1$ (or $K \geq 1$), then $K = 1$. Moreover, we prove that all holomorphic two-spheres with constant curvature 1 in $G(2,4)$ must be $U(4)$-equivalent.

Keywords. Gaussian curvature; holomorphic map; totally geodesic.

1. Introduction

Chern and Wolfson [1], using ∂-transform and $\bar{\partial}$-transform, obtained the following two harmonic sequences associated to f:

$$f = f_0 \xrightarrow{\partial} f_1 \xrightarrow{\partial} f_2 \xrightarrow{\partial} \cdots \xrightarrow{\partial} f_j \xrightarrow{\partial} \cdots,$$

$$f = f_0 \xrightarrow{\bar{\partial}} f_{-1} \xrightarrow{\bar{\partial}} f_{-2} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} f_{-j} \xrightarrow{\bar{\partial}} \cdots,$$

in which f is a harmonic map from a Riemann surface M into complex Grassmann manifold $G(k,n)$, $f_{j+1} = \partial f_j$, $f_{-j-1} = \bar{\partial} f_{-j}$, $f_j\colon M \to G(k_j, n)$ and $f_{-j}\colon M \to G(k_{-j}, n)$ are harmonic maps. The positive integer k_j is called the *rank* of f_j. Map f_j is called *non-degenerate* (resp. *degenerate*) when $k_j = k_{j+1}$ (resp. $k_j > k_{j+1}$). We say that f_i and f_j are orthogonal if $f_i(x) \perp f_j(x)$ as linear subspace of $\mathbb{C}^n$ with respect to the standard Hermitian inner product for any $x \in M$, and the harmonic sequence is orthogonal if $f_i \perp f_j$ for any $i \neq j$. Clearly, the $\bar{\partial}$-transform (resp. ∂-transform) is zero if and only if f is holomorphic (resp. anti-holomorphic). Particularly, by ∂-transform, the holomorphic map f will generate a finite and orthogonal harmonic sequence. This sequence is called *pseudo-holomorphic sequence*, and f_j is called *pseudo-holomorphic map* generated by f.

The map f is called *linearly full* if $\underline{\mathrm{Im}}\, f = \{(x, f(x)) | x \in M, f(x) \in G(2,4)\}$ is not contained in any trivial subbundle of $M \times \mathbb{C}^4$. We just consider the linearly full holomorphic maps from S^2 into $G(2,4)$ in this paper.

It is obvious that a linearly full holomorphic map from S^2 into complex Grassmann manifold $G(2,4)$ will generate one of the following two harmonic sequences:

$$f = f_0 \xrightarrow{\partial} f_1 \xrightarrow{\partial} 0, \tag{1.1}$$

where $f_1 \colon S^2 \to G(2,4)$ is an anti-holomorphic map, and

$$f = f_0 \xrightarrow{\partial} f_1 \xrightarrow{\partial} f_2 \xrightarrow{\partial} 0, \tag{1.2}$$

where $f_1, f_2 \colon S^2 \to \mathbb{C}P^3$ are harmonic maps. The holomorphic map f in harmonic sequence (1.1) (resp. (1.2)) is non-degenerate (resp. degenerate).

Zheng [6] and Chi and Zheng [2] studied the holomorphic maps from S^2 into $G(2,4)$, and classified those with Gaussian curvature $K = 2$ into two classes, up to unitary equivalence, in which a curve from one class is not congruent to the one from the other. Jiao and Peng [4] classified all linearly full holomorphic spheres in complex Grassmann manifolds $G(2,5)$ with the induced constant curvatures $K = 4, 2, 4/3, 1$ and $4/5$ into some classes, up to unitary equivalence. They also studied the pseudo-holomorphic curves in complex Grassmann manifolds in [5], and gave several pinching theorems about the Gaussian curvature and Kahler angle.

In this paper, non-degenerate and degenerate holomorphic two-spheres in $G(2,4)$ are studied by means of moving frames. Some fundamental formulas are given in §2. In §3, we present two pinching theorems about Gaussian curvature K, and also prove that f is totally geodesic when $K = 2$ and $\xi \equiv \eta$ on S^2. In §4, we prove that the Gaussian curvature K of a degenerate holomorphic two-sphere satisfies $K \leq 1$ (or $K \geq 1$), then $K = 1$, and all the holomorphic two-spheres with constant curvature 1 in $G(2,4)$ must be $U(4)$-equivalent.

2. Preliminaries

The complex Grassmann manifold $G(k,n)$ is the set of all k-dimensional complex linear subspace of $\mathbb{C}^n$ through the origin. Particularly, $G(1,n)$ is the complex projective space $\mathbb{C}P^{n-1}$.

Throughout this paper we will agree on the following ranges of indices:

$$1 \leq A, B, \cdots \leq n, \;\; 1 \leq i, j, \cdots \leq k, \;\; k+1 \leq \alpha, \beta, \cdots \leq n.$$

Let $\{\omega_{AB}\}$ be the Maurer–Cartan forms of $U(n)$ (these one-forms restricted onto $G(k,n)$ is still denoted by $\{\omega_{AB}\}$). They are skew-Hermitian, i.e.

$$\omega_{AB} + \bar{\omega}_{BA} = 0, \tag{2.1}$$

and satisfy the Maurer–Cartan structure equations of $U(n)$:

$$\mathrm{d}\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB}. \tag{2.2}$$

An element $\mathbb{C}^k$ of $G(k,n)$ can be defined by the multivector $e_1 \wedge e_2 \wedge \cdots \wedge e_k \neq 0$, defined up to a factor. The vectors $\{e_i\}$ and their orthogonal vectors $\{e_\alpha\}$ are defined up to a transformation of $U(k)$ and $U(n-k)$ respectively, so that $G(k, n)$ has a G-structure, with $G = U(k) \times U(n-k)$. In particular, the form

$$\mathrm{d}s_G^2 = \omega_{i\alpha}\bar{\omega}_{i\alpha} \tag{2.3}$$

is a positive definite Hermitian form, which defines a Hermitian metric on $G(k, n)$.

Let M be an oriented Riemannian surface and $f: M \to G(k, n)$ an isometric immersion. The induced Riemannian metric on M is denoted by

$$\mathrm{d}s_M^2 = f^*\mathrm{d}s_G^2 = \theta\bar{\theta}, \tag{2.4}$$

where θ is a complex-valued one-form, defined up to a complex factor of absolute value 1. The structure equations of M with respect to the induced metric can be written as

$$\mathrm{d}\theta = -\sqrt{-1}\omega \wedge \theta, \tag{2.5}$$

$$\mathrm{d}(\sqrt{-1}\omega) = \frac{K}{2}\theta \wedge \bar{\theta}, \tag{2.6}$$

where the real one-form ω is the connection form and K is the Gaussian curvature of M, with respect to induced metric $\mathrm{d}s_M^2$.

Set

$$f^*\omega_{AB} = a_{AB}\theta + b_{AB}\bar{\theta}. \tag{2.7}$$

Then from eq. (2.4), we have

$$\sum_{i,\alpha} a_{i\alpha} b_{i\alpha} = 0$$

and

$$\sum_{i,\alpha} (a_{i\alpha}\bar{a}_{i\alpha} + b_{i\alpha}\bar{b}_{i\alpha}) = 1. \tag{2.8}$$

Taking the exterior derivatives of (2.7), here $A = i$ and $B = \alpha$, together with (2.2) and (2.5), we get

$$Da_{i\alpha} \wedge \theta + Db_{i\alpha} \wedge \bar{\theta} = 0, \tag{2.9}$$

where

$$Da_{i\alpha} = da_{i\alpha} - \omega_{ij}a_{j\alpha} + a_{i\beta}\omega_{\beta\alpha} - \sqrt{-1}a_{i\alpha}\omega, \tag{2.10}$$

$$Db_{i\alpha} = db_{i\alpha} - \omega_{ij}b_{j\alpha} + b_{i\beta}\omega_{\beta\alpha} + \sqrt{-1}b_{i\alpha}\omega. \tag{2.11}$$

The quadratic form

$$\Pi_{i\alpha}^{\mathbb{C}} = Da_{i\alpha}\theta + Db_{i\alpha}\bar{\theta} \tag{2.12}$$

is called *complex second fundamental form* of the map f. Map f is called *totally geodesic* if $\Pi_{i\alpha}^{\mathbb{C}} = 0$ for all i and α.

The following lemma will be frequently used in this paper.

Lemma 2.1. *Let U be an open subset of Riemannian surface M, g a complex smooth function defined on U, and θ the unitary co-frame field restricted on U. Suppose that g satisfies*

$$dg \equiv g\varphi \mod \theta, \tag{2.13}$$

where φ is an imaginary valued one-form (i.e. $\bar{\varphi} = -\varphi$). Then

$$\Delta \log |g| \, \theta \wedge \bar{\theta} = 2\, d\varphi \tag{2.14}$$

away from its zeros, and Δ is the Laplace operator defined on M.

Proof. Without loss of generality, assume that U is a local complex coordinate system, with its coordinate z. Then the metric can be written as

$$ds_M^2 = \theta\bar{\theta} = \lambda^2 \, dz d\bar{z}, \tag{2.15}$$

where $\lambda > 0$ is a real function defined on U. Set $\partial = \frac{\partial}{\partial z}$ and $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$, hence the Laplace operator is

$$\Delta = \frac{4\partial\bar{\partial}}{\lambda^2}.$$

By (2.13), the following equation

$$dg \equiv g\varphi \mod dz, \tag{2.16}$$

is valid. We can write $\varphi = \mu dz - \bar{\mu} d\bar{z}$, since φ is an imaginary valued one-form. From (2.16), we get

$$\bar{\partial} \log g = -\bar{\mu}, \quad \partial \log \bar{g} = -\mu.$$

Thus

$$\begin{aligned} d\varphi &= d(\mu dz - \bar{\mu} d\bar{z}) \\ &= \bar{\partial}\mu d\bar{z} \wedge dz - \partial\bar{\mu} dz \wedge d\bar{z} \\ &= (\bar{\partial}\partial \log \bar{g} + \partial\bar{\partial} \log g) dz \wedge d\bar{z} \\ &= 2\, \partial\bar{\partial} \log |g| \, dz \wedge d\bar{z} \\ &= \frac{1}{2} \Delta \log |g| \, \theta \wedge \bar{\theta}. \end{aligned}$$

This completes the proof of Lemma 2.1.

3. Non-degenerate holomorphic two-sphere in $G(2,4)$

For convenience, we use 'Gaussian curvature K' instead of 'Gaussian curvature K of S^2 with respect to the induced metric'.

Let f be a non-degenerate holomorphic immersion from S^2 into $G(2,4)$, which can generate the harmonic sequence (1.1). A suitable local unitary frame $e = (e_1, e_2, e_3, e_4)$

can be chosen along f, which satisfies that $f = \text{span}\{e_1, e_2\}$ and $\partial f = \text{span}\{e_3, e_4\}$. Since e is uniquely determined up to $U(2) \times U(2)$-transformations, the pull back of the Maurer–Cartan forms can be

$$f^*(\omega_{AB}) = \begin{pmatrix} \omega_{11} & \omega_{12} & \xi\theta & 0 \\ \omega_{21} & \omega_{22} & 0 & \eta\theta \\ -\xi\bar{\theta} & 0 & \omega_{33} & \omega_{34} \\ 0 & -\eta\bar{\theta} & \omega_{43} & \omega_{44} \end{pmatrix}, \tag{3.1}$$

where $\xi,\ \eta$ are positive functions, $\xi^2 + \eta^2 = 1$, and θ is the unitary co-frame with respect to the induced metric.

Taking the exterior derivative of the equations $\omega_{13} = \xi\theta$, $\omega_{24} = \eta\theta$, $\omega_{14} = 0$, and $\omega_{23} = 0$, we get

$$d\xi \equiv \xi\,(\sqrt{-1}\omega + \omega_{11} - \omega_{33}) \qquad \text{mod } \theta, \tag{3.2}$$

$$d\eta \equiv \eta\,(\sqrt{-1}\omega + \omega_{22} - \omega_{44}) \qquad \text{mod } \theta \tag{3.3}$$

and

$$\omega_{12} \wedge \omega_{24} + \omega_{13} \wedge \omega_{34} = 0, \tag{3.4}$$

$$\omega_{21} \wedge \omega_{13} + \omega_{24} \wedge \omega_{43} = 0. \tag{3.5}$$

Due to Lemma 2.1, we get

$$\Delta \log \xi\, \theta \wedge \bar{\theta} = (K - 4\xi^2)\theta \wedge \bar{\theta} + \omega_{12} \wedge \omega_{21} - \omega_{34} \wedge \omega_{43}, \tag{3.6}$$

$$\Delta \log \eta\, \theta \wedge \bar{\theta} = (K - 4\eta^2)\theta \wedge \bar{\theta} - \omega_{12} \wedge \omega_{21} + \omega_{34} \wedge \omega_{43}, \tag{3.7}$$

by (2.2), (3.2) and (3.3). Combining (3.6) and (3.7) with $\xi^2 + \eta^2 = 1$,

$$\Delta \log(\xi\eta) = 2(K - 2), \tag{3.8}$$

where $\xi\eta$ is a globally defined positive invariant on S^2, for e is uniquely determined up to $U(2) \times U(2)$-transformations and θ is uniquely determined up to $U(1)$-transformations.

Therefore, we have the following pinching theorem about Gaussian curvature K.

Theorem 3.1. *Let f be a non-degenerate holomorphic immersion from S^2 into the complex Grassmann manifold $G(2,4)$ and K the Gaussian curvature. If $K \leq 2$ (or $K \geq 2$), then $K = 2$. Particularly, if K is a constant, then $K = 2$.*

Proof. Under the condition $K \leq 2$, together with eq. (3.8), there comes

$$\Delta \log(\xi\eta) = 2(K - 2) \leq 0$$

which implies $K = 2$, based on the maximum principle of Hopf.

Similarly, the result holds for $K \geq 2$ and $K = \text{const}$.

Remark.

(i) The result for $K \geq 2$ belongs to Zheng [6].
(ii) It is clear that ξ, η are positive constants when $K = 2$, since $\xi\eta$ is a constant and $\xi^2 + \eta^2 = 1$.

There is an analogous result for non-degenerate holomorphic two-spheres in $G(k, 2k)$.

COROLLARY 3.2

Let f be a non-degenerate holomorphic immersion from S^2 into the complex Grassmann manifold $G(k, 2k)$ and K the Gaussian curvature. If $K \leq \frac{4}{k}$ $\left(\text{or } K \geq \frac{4}{k}\right)$, then $K = \frac{4}{k}$.

Proof. It is similar to Theorem 3.1, but more complicated.

Now, we show that f is totally geodesic when $\xi \equiv \eta$ on S^2. Let h be a smooth function defined on S^2, then $\mathrm{d}h$ can be written as $\mathrm{d}h = h_{;1}\theta + h_{;2}\bar{\theta}$, where $h_{;1}$, $h_{;2}$ are smooth functions. As $\sqrt{-1}\omega$ is a purely imaginary one-form, we can set $\sqrt{-1}\omega = \mu\theta - \bar{\mu}\bar{\theta}$.

Since ξ, η are constants, one gets

$$\xi_{;1} = \xi(a_{33} - a_{11} - \mu) = 0, \tag{3.9}$$

$$\xi_{;2} = \xi(b_{11} - b_{33} - \bar{\mu}) = 0, \tag{3.10}$$

by eqs (2.1), (2.7) and (3.2). Due to (2.7), (3.9) and (3.10), equation (2.10) implies

$$Da_{13} = 2\xi(a_{33} - a_{11} - \mu)\theta = 0 \tag{3.11}$$

and

$$Da_{14} = \xi(a_{34}\theta + b_{34}\bar{\theta}) - \eta(a_{12}\theta + b_{12}\bar{\theta}), \tag{3.12}$$

$$Da_{23} = \eta(a_{43}\theta + b_{43}\bar{\theta}) - \xi(a_{21}\theta + b_{21}\bar{\theta}). \tag{3.13}$$

Similarly, $Da_{24} = 2\eta(a_{44} - a_{22} - \mu)\theta = 0$. Moreover, $Db_{i\alpha} = 0$ as $b_{i\alpha} = 0$.

Equations (3.4) and (3.5) imply

$$\xi b_{34} = \eta b_{12}, \text{ or } \xi a_{43} = \eta a_{21}, \tag{3.14}$$

$$\xi b_{21} = \eta b_{43}, \text{ or } \xi a_{12} = \eta a_{34}, \tag{3.15}$$

by (2.7). Hence, it is clear that $Da_{14} = Da_{23} = 0$ when $\xi \equiv \eta$. In other words, the second fundamental form $\Pi^{\mathbb{C}}_{i\alpha} = 0$.

Theorem 3.3. *A non-degenerate holomorphic immersion f from S^2 into $G(2, 4)$ with Gaussian curvature $K = 2$ is totally geodesic if $\xi \equiv \eta$ on S^2.*

Remark. By the remark of Theorem 3.1, we know ξ, η are positive constants if $K = 2$. Then due to eqs (3.12)–(3.15), we know that the converse of this theorem is true when $f^*\omega_{12} \neq 0$ (or $f^*\omega_{34} \neq 0$) for one point on S^2. However there is no more information about $f^*\omega_{12}$ and $f^*\omega_{34}$.

4. Degenerate holomorphic two-sphere in $G(2,4)$

Let f be a degenerate holomorphic map from S^2 into $G(2,4)$, which can generate the harmonic sequence (1.2). We chose a suitable local unitary frame $e = (e_1, e_2, e_3, e_4)$ along f, such that

$$f = \mathrm{span}\{e_1, e_2\},\ \ker \partial = \mathrm{span}\{e_2\},\ f_1 = \mathrm{span}\{e_3\},\ f_2 = \mathrm{span}\{e_4\},$$

and the pull back of the Maurer–Cartan forms are

$$f^*(\omega_{AB}) = \begin{pmatrix} \omega_{11} & \omega_{12} & \theta & 0 \\ \omega_{21} & \omega_{22} & 0 & 0 \\ -\bar{\theta} & 0 & \omega_{33} & \omega_{34} \\ 0 & 0 & \omega_{43} & \omega_{44} \end{pmatrix}, \tag{4.1}$$

where θ is the unitary co-frame with respect to the induced metric.

Taking the exterior derivatives on both sides of the equations $\omega_{13} = \theta$, $\omega_{14} = 0$ and $\omega_{23} = 0$, together with the fact that $\sqrt{-1}\omega,\ \omega_{11}$ and ω_{33} are purely imaginary one-forms, we get

$$\sqrt{-1}\omega = \omega_{33} - \omega_{11} \tag{4.2}$$

and

$$\omega_{34} = a_{34}\theta, \tag{4.3}$$

$$\omega_{21} = a_{21}\theta, \tag{4.4}$$

where a_{34} and a_{21} are locally defined smooth functions, while $|a_{34}|$ and $|a_{21}|$ are globally defined on S^2, since they are independent of the choice of unitary co-frame θ and the unitary frame e is uniquely determined up to $U(1)\times U(1)\times U(1)\times U(1)$-transformations.

We claim $a_{21}, a_{43} \neq 0$. Indeed, if $a_{21} = 0$, the map $S^2 \to \mathbb{C}P^3$ defined by e_2 is not only holomorphic but also anti-holomorphic by reading (4.1), which implies e_2 is a constant vector in $\mathbb{C}^4$. It is a contradiction to the assumption that f is linearly full. Obviously, $a_{43} \neq 0$, since f can generate f_2.

Taking the exterior derivatives on both sides of (4.2), we have

$$K\theta \wedge \bar{\theta} = 4\theta \wedge \bar{\theta} - 2(|a_{34}|^2 + |a_{21}|^2)\theta \wedge \bar{\theta},$$

namely,

$$K = 4 - 2(|a_{34}|^2 + |a_{21}|^2), \tag{4.5}$$

which implies the Gaussian curvature $K < 4$.

One can take the exterior derivatives on both sides of equations (4.3) and (4.4) to get

$$\mathrm{d}a_{34} \equiv a_{34}(\sqrt{-1}\omega + \omega_{33} - \omega_{44}) \qquad \mathrm{mod}\ \theta, \tag{4.6}$$

$$\mathrm{d}a_{21} \equiv a_{21}(\sqrt{-1}\omega + \omega_{22} - \omega_{11}) \qquad \mathrm{mod}\ \theta. \tag{4.7}$$

Due to Lemma 2.1, we get

$$\Delta \log |a_{34}| = K + 2 - 4|a_{34}|^2, \tag{4.8}$$

$$\Delta \log |a_{21}| = K + 2 - 4|a_{21}|^2, \tag{4.9}$$

by (4.6) and (4.7).

Combining (4.8) and (4.9), together with (4.5),

$$\Delta \log |a_{34}a_{21}| = 4(K - 1). \tag{4.10}$$

In summary, the following theorem is valid.

Theorem 4.1. *Let f be a degenerate linearly full holomorphic map from S^2 into $G(2,4)$ and K is the Gaussian curvature. If $K \leq 1$ (or $K \geq 1$ or $K = const$), then $K = 1$.*

Proof. It is similar to Theorem 3.1 by eq. (4.10).

At the end of this section, we show that any two holomorphic two-sphere with constant curvature 1 in $G(2,4)$ must be $U(4)$-equivalent.

From the preceding proof we know that $|a_{21}|$, $|a_{34}|$ are constants when $K = 1$. Set $3\omega_{11} + \omega_{44} = p\theta - \bar{p}\bar{\theta}$ and $\omega_{22} + 3\omega_{33} = q\theta - \bar{q}\bar{\theta}$, where p, q are smooth complex-valued functions. A suitable local unitary frame e can be chosen along f, such that $a_{21} = |a_{21}|$ and $a_{34} = |a_{34}|$. Due to the computation by Jensen [3], we get

$$a_{21}p = (a_{21})_{;1} = 0, \quad a_{34}q = (a_{34})_{;1} = 0, \tag{4.11}$$

since a_{21}, a_{34} are constants. Obviously, eq. (4.11) implies $3\omega_{11} + \omega_{44} = \omega_{22} + 3\omega_{33} = 0$.

Theorem 4.2. *Any linearly full holomorphic two-sphere with constant Gaussian curvature 1 in $G(2,4)$ is $U(4)$-equivalent to $f\colon S^2 = \mathbb{C}P^1 \to G(2,4)$, in terms of homogeneous coordinate, f is defined by*

$$f([z_0, z_1]) = \begin{bmatrix} z_0^2\bar{z}_0 - 2z_0z_1\bar{z}_1 & -\sqrt{3}z_0^2\bar{z}_1 & 2z_0\bar{z}_0z_1 & \sqrt{3}\bar{z}_0z_1^2 \\ \sqrt{3}z_0^2z_1 & z_0^3 & \sqrt{3}z_0z_1^2 & z_1^3 \end{bmatrix}.$$

Proof. The details can be found in [3], P_{95-96}.

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