Proc. Indian Acad. Sci. (Math. Sci.) Vol. 118, No. 3, August 2008, pp. 357–370.

Decomposition and removability properties of John domains

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MS received 14 May 2007; revised 9 February 2008

Abstract. In this paper we characterize John domains in terms of John domain decomposition property. In addition, we also show that a domain D in $\mathbb{R}^n$ is a John domain if and only if $D\backslash P$ is a John domain, where P is a subset of D containing finitely many points of D. The best possibility and an application of the second result are also discussed.

Keywords. John domain; John domain decomposition property; John disk; uniform domain; hyperbolic and quasihyperbolic metrics; distance condition; removability property.

1. Introduction and main results

We assume throughout that D is a proper subdomain of the Euclidean n-space $\mathbb{R}^n$, $n \geq 2$ and $\mathbb{B}^n(x, r) = \{y \in \mathbb{R}^n\colon |x - y| < r\}$ denotes the Euclidean ball at x of radius r. For such a domain D, we define the *quasi-hyperbolic metric* k_D by

$$k_D(z_1, z_2) = \inf_{\alpha} \ell_k(\alpha), \quad \ell_k(\alpha) = \int_{\alpha} \operatorname{dist}(z, \partial D)^{-1}\, \mathrm{d}s,$$

where the infimum is taken over all rectifiable arcs α joining z_1 and z_2 in D. An arc α from z_1 to z_2 is a quasi-hyperbolic geodesic or briefly a geodesic if $\ell_k(\alpha) = k_D(z_1, z_2)$. Many of the basic properties of this metric can be found in [4]. Important contributions on quasi-hyperbolic geodesics in domains in $\mathbb{R}^n$ were obtained by Martin [10]. For example, he proved that the geodesics are C^1 smooth.

As in [11], a domain $D \subset \mathbb{R}^n$ is called a *c-John domain* if for every pair of points z_1, $z_2 \in D$, there is a rectifiable arc $\alpha \subset D$ joining them with

$$\min_{j=1,2} \ell(\alpha[z_j, z]) \leq c \operatorname{dist}(z, \partial D) \quad \text{for all } z \in \alpha,$$

where c is a positive constant, $\alpha[z_j, z]$ denotes the subcurve of α between z_j and z, and $\ell(\alpha[z_j, z])$ denotes the Euclidean length of $\alpha[z_j, z]$, $j = 1, 2$. We call a domain D a John domain if it is a c-John domain for some positive constant c. A simply connected c-John domain $D \subset \mathbb{C}$ with at least two boundary points will be called a *c-John disk*.

A domain $D \subset \mathbb{R}^n$ is said to be *uniform* if there exist constants a and c such that each pair of points $z_1, z_2 \in D$ can be joined by a rectifiable arc $\alpha \subset D$ for which

$$\begin{cases} \ell(\alpha) \le a|z_1 - z_2| \text{ and} \\ \min_{j=1,2} \ell(\alpha[z_j, z]) \le c\,\mathrm{dist}(z, \partial D) \quad \text{for all } z \in \alpha. \end{cases} \tag{1.1}$$

Here $\ell(\alpha)$ denotes the length of α. The first condition says that $\ell(\alpha)$ is comparable to $|z_1 - z_2|$ whereas the second says that away from z_1, z_2, α stays away from the boundary ∂D. It is easy to see that $\mathbb{B}^n(0, 1)$ is a uniform domain. In Corollary 2 of [4], Gehring and Osgood have shown that a domain is uniform if and only if (1.1) holds for each quasi-hyperbolic geodesic.

The classes of John domains and of uniform domains in Euclidean space enjoy an important role in many areas of modern mathematical analysis, see [9, 11, 13]. Martio and Sarvas [9] were the first who introduced uniform domains in connection with approximation and injectivity properties of functions in $\mathbb{R}^n$. Moreover Gehring's article [3] lists many interesting properties of uniform domains and has an extensive bibliography. Since then its importance along with John domains throughout the function theory is well-documented and they have appeared quite naturally in the quasiconformal theory, see [3, 13]. In [7], Jones used uniform domains to obtain many interesting applications, particularly in the theory of BMO functions.

A *quasidisk* is the image of a disk or a half-plane under a quasiconformal mapping of the entire plane. We refer to the books of Väisälä [12] and Vuorinen [14] for the definition of quasiconformality and basic facts concerning quasiconformal mappings. In the planar case, there is a strong interplay between the notion of a uniform domain and that of a quasidisk. For example, it has been known that a simply connected planar domain D is a quasidisk if and only if D is a uniform domain (see Lemma 6.4 of [5]). In particular, a decomposition theorem follows as we see below in Theorem A. A Jordan domain D is a quasidisk if and only if both D and $D^* := \mathbb{C}\backslash\bar{D}$ are John disks, and every quasidisk is a John disk (see [8]). Hence John disks can be thought of as 'one-sided quasidisks'. A K-quasidisk is the image of $\mathbb{B}^2(0, 1)$ under a K-quasiconformal homeomorphism f of the Riemann sphere. $f(\partial\mathbb{B}^n(0, 1))$ is called a quasicircle. Observe that since a quasidisk is homeomorphic to a disk, it is necessarily a Jordan domain. Moreover, Martio and Sarvas [9] have shown that uniform domains are invariant under quasiconformal mappings of $\mathbb{R}^n$.

A domain D in $\mathbb{R}^2$ is said to be *quasiconformally decomposable* if there exists a constant K with the following property. For each pair $x_1, x_2 \in D$, there exists a subdomain D_0 of D such that $x_1, x_2 \in \bar{D}_0$ and ∂D_0 is a K-quasiconformal circle, i.e., the image of the unit circle under a K-quasiconformal mapping of $\bar{\mathbb{R}}^2$ onto itself, here $\bar{\mathbb{R}}^2 = \mathbb{R}^2 \cup \{\infty\}$.

Now, we recall the result of Gehring and Osgood (Theorem 5 of [4]).

Theorem A. *A domain $D \subset \mathbb{R}^2$ is a uniform domain if and only if it is quasiconformally decomposable.*

One of the aims of this paper is to obtain an analog of Theorem A for John domains.

DEFINITION 1.2

A domain $D \subset \mathbb{R}^n$ is said to have the John domain decomposition property if there exists a positive constant c with the following property: for each pair $z_1, z_2 \in D$, there exists a subdomain D_0 of D such that $z_1, z_2 \in \bar{D}_0$ and D_0 is a simply connected c-John domain.

We now state our first result.

Theorem 1.3. *A domain $D \subset \mathbb{R}^n$ is a John domain if and only if it has the John domain decomposition property.*

About the removability property of John domains, we obtain the following.

Theorem 1.4. *A domain $D \subset \mathbb{R}^n$ $(n \geq 2)$ is a John domain if and only if $G = D \setminus P$ is also a John domain, where $P = \{p_1, p_2, \ldots, p_m\}$ and $p_i \in D$ $(i = 1, 2, \ldots, m)$.*

The proofs of Theorems 1.3 and 1.4 will be presented in §2 and §3, respectively. We include below an example to show that the finiteness of P in Theorem 1.4 is not superfluous.

Example 1.5. Let $D = \mathbb{R}^2 \setminus [0, +\infty)$. For any $\delta > 0$, let $x_m = -m\delta (m = 1, 2, \ldots)$ and $P = \{x_1, x_2, \ldots\}$. Then D is a John domain, but $G = D \setminus P$ is not.

See §4 for basic information including precise definitions, notations and terminology concerning the metrics h_D, j_D and a'_D that we use below.

Theorem B (Corollary 2.3 of [15]). *Suppose that $D \subset \mathbb{R}^2$ is simply connected and that there is a constant c such that*

$$h_D(z_1, z_2) \leq c\, a'_D(z_1, z_2)$$

for all $z_1, z_2 \in D$. Then D is a b-John disk with $b = b(c)$.

In the same paper the authors have also constructed two examples, one for a bounded John disk and the other for an unbounded John disk, to show that the converse of Theorem B does not hold. These examples settled the conjecture of Broch in [2]. It is natural to raise the following.

Question 1.6. Does there exist a constant c such that if D is a b-John domain, then for all $z_1, z_2 \in D$,

$$j_D(z_1, z_2) \leq c\, a'_D(z_1, z_2),$$

where the constants b and c depend only on each other?

As an application of Theorem 1.4, we get the following result which provides a counterexample to Question 1.6.

Theorem 1.7. *Suppose that $D \subset \mathbb{R}^2$ is a b-John domain and p is a point in D. Then the following hold*:

(1) *$G = D \setminus \{p\}$ is a b_1-John domain.*
(2) *There does not exist a constant c such that for all $z_1, z_2 \in D$,*

$$j_G(z_1, z_2) \leq c\, a'_G(z_1, z_2),$$

where the constants b, b_1 and c depend only on each other.

The proof of Theorem 1.7 will be presented in §4.

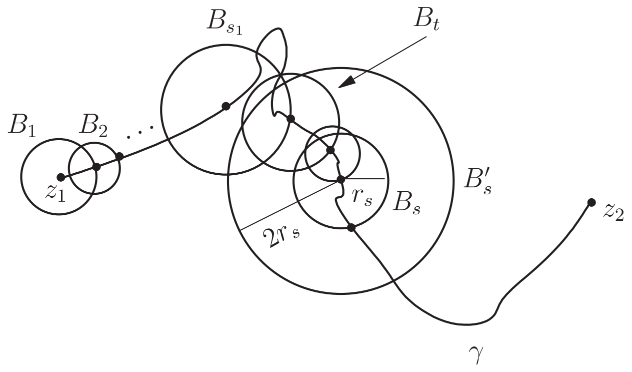

Figure 1. B_t the first intersection ball with B_s, and B_{s_1} the first intersection ball with B'_s.

2. Proof of theorem 1.3

First, we prove the following lemmas.

Lemma 2.1. *Suppose that $D \subset \mathbb{R}^n$ is a domain. If γ is a rectifiable curve joining z_1 and z_2 in D, then there exists a simply connected domain $D_0 = \bigcup_{i=1}^k B_i \subset D$ such that*

(1) $z_1, z_2 \in \bar{D}_0$;
(2) $\bar{B}_i \cap \bar{B}_j = \emptyset (|i-j| \geq 2)$;
(3) $r_i + r_{i+1} - |x_i - x_{i+1}| \geq \frac{1}{16\times 3^{4n}} \min\{r_i, r_{i+1}\} (1 \leq i \leq k-1)$.

Here $B_i = B^n(x_i, r_i)$, $r_i \geq \frac{1}{2}\,\mathrm{dist}(x_i, \partial D)$, $x_i \in \gamma$ but $x_i \notin B_{i-1}$.

Proof. Let $x_1 = z_1$, and let x_2 be the last intersection point of γ from z_1 to z_2 with the sphere ∂B_1, where $B_1 = \mathbb{B}^n(x_1, r_1)$ and $r_1 = \frac{1}{2}\,\mathrm{dist}(x_1, \partial D)$.

We let x_3 be the last intersection point of γ from z_1 to z_2 with the sphere ∂B_2, where $B_2 = \mathbb{B}^n(x_2, r_2)$ and $r_2 = \frac{1}{2}\,\mathrm{dist}(x_2, \partial D)$.

We would not repeat this procedure when $\bar{B}_s \cap \bar{B}_i \neq \emptyset$ $(1 \leq i \leq s-2)$, $s \geq 3$.

Let B_t be the first intersection ball of $\bar{B}_j$ $(1 \leq j \leq s-2)$ with B_s from B_1 to B_{s-2} $(s \geq 3)$ (see figure 1). Clearly $t \leq s-2$.

We choose $B_i = B_i$ $(1 \leq i \leq t)$ and $B_{t+1} = B_s$ whenever

$$r_t + r_s - |x_t - x_s| \geq \frac{1}{16 \times 3^{4n}} \min\{r_t, r_s\}.$$

Obviously (3) follows in this case. On the other hand, in the case of

$$r_t + r_s - |x_t - x_s| < \frac{1}{16 \times 3^{4n}} \min\{r_t, r_s\},$$

we consider the ball $B'_s = \mathbb{B}^n(x_s, 2r_s)$. Let B_{s_1} be the first intersection ball of $\bar{B}_j$ with $\bar{B}'_s$ from B_1 to B_t (see figure 1). Denote $d_i = \mathrm{dist}(\partial B_i, \partial B_s)$ $(s_1 \leq i \leq t)$. Clearly $d_t = 0$.

Now we divide the proof into two cases.

Case I. Suppose that $d_{s_1} \leq \frac{15 r_s}{16}$.

Then we take $B_i = B_i$ $(1 \le i \le s_1)$ and $B_{s_1+1} = B'_s$, so that (3) follows.

Case II. Next we consider $d_{s_1} > \frac{15r_s}{16}$.

Let $\delta_1 = d_{s_1}$ and δ_2 be the first d_i $(s_1 \le i \le t)$ satisfying $d_i \le \delta_1$. Clearly $\delta_1 \ge \delta_2$. By repeating the procedure, we get

$$\delta_1 \ge \delta_2 \ge \cdots \ge \delta_m = 0 \ (2 \le m \le t).$$

We note that $\delta_1 > \frac{15r_s}{16}$ and $\delta_m = 0$. Define $\varepsilon_i = \delta_i - \delta_{i+1}$ $(1 \le i \le m-1)$.

If $m \le 3^{4n}$, then there must exist an ε_i $(1 \le i \le m-1)$ such that $\varepsilon_i > 2r_s/(16 \times 3^{4n})$ (because otherwise,

$$\frac{15r_s}{16} < \delta_1 - \delta_m \le (m-1)\frac{2r_s}{16 \times 3^{4n}} < \frac{2r_s}{16}$$

which is a contradiction). We now consider the ball $B''_s := \mathbb{B}^n(x_s, r''_s)$ with

$$B_s \subset B''_s \subset B'_s \quad \text{and} \quad r''_s = r_s + \delta_{i+1} + \frac{2r_s}{16 \times 3^{4n}}, \quad 1 \le i \le m-1.$$

By the condition on ε_i, we see that $B''_s \cap B_i = \emptyset$ and hence $B''_s \cap B_j = \emptyset$ for all $j \le i$. Denote $B_w := B_{i+1}$. In order to check condition (3), we take $B_i = B_i$, $1 \le i \le w$ and $B_{w+1} = B''_s$. Since $r''_s \le 2r_s$, we see that (3) holds.

If $m > 3^{4n}$, then also there exists an ε_i $(1 \le i \le m-1)$ satisfying the condition $\varepsilon_i > 2r_s/(16 \times 3^{4n})$ and the proof is similar. Indeed, suppose on the contrary that $\varepsilon_i \le \frac{2r_s}{16\times 3^{4n}}$ for all i $(1 \le i \le m-1)$. Then we show that

$$V_n\left(\bigcup_{i=1}^{m-1} B_i \cap \{B'_s \backslash B_s\}\right) > V_n(B'_s \backslash B_s), \tag{2.2}$$

where $V_n(G)$ denotes the n-dimensional volume of G, which leads to a contradiction.

Consider the union $\bigcup_{j=m-3^{4n}}^{m} B_j$. We note that

$$\delta_{m-3^{4n}} - \delta_m = \varepsilon_{m-3^{4n}} + \cdots + \varepsilon_{m-1} \le \frac{2r_s}{16}.$$

If $B_j \not\subset \{B'_s \backslash B_s\}$ for some j, we denote by $B'_j \subset B'_s \backslash B_s$ the ball, with radius $r'_j = (r_s - \delta_j)/2 > r_s/4$, tangent to both B_j and B'_s (see figure 2).

Hence

$$V_n(B_j \cap \{B'_s \backslash B_s\}) > \Omega_n \times \frac{r_s^n}{4^n}, \tag{2.3}$$

where $\Omega_n := \frac{\pi^{n/2}}{\Gamma(1+\frac{n}{2})}$ is the volume of the unit ball. On the other hand, if $B_j := B_j(x_j, r_j) \subset \{B'_s \backslash B_s\}$ for some j then also we see that $r_j > r_s/4$. Because otherwise,

$$\frac{r_s}{4} \ge r_j = \frac{1}{2}\text{dist}(x_j, \partial D) \ge \frac{1}{2}(r_s - \delta_j - r_j) \ge \frac{5r_s}{16},$$

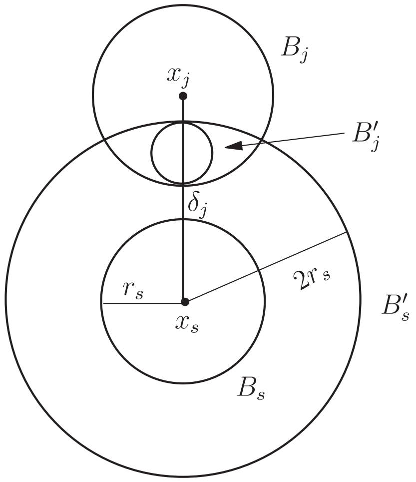

Figure 2. The ball $B'_j \subset B'_s \backslash B_s$ tangent to both B_j and B'_s

which is a contradiction. Thus for all j, $m - 3^{4n} \le j \le m - 1$, we have the relation (2.3). This observation yields that

$$V_n\left(\bigcup_{j=m-3^{4n}}^{m-1} B_j \cap \{B'_s \backslash B_s\}\right) \ge \frac{3^{4n}}{2} \times \Omega_n \times \frac{r_s^n}{4^n} > \Omega_n(2^n - 1)r_s^n,$$

which gives (2.2).

By repeating the above procedure, we get a set of points $\{x_i\}_{i=1}^k$ on γ such that z_2 is contained in the closure of $\mathbb{B}^n(x_k, r_k) \subset D$, but not in the closure of $\mathbb{B}^n(x_{k-1}, r_{k-1}) \subset D$. The union of the B_i's for $0 \le i \le k$ is then a simply connected subdomain in D whose closure contains z_1 and z_2. ■

Lemma 2.4. *For any* $w_2 \in \mathbb{R}^n, r_1 \ge r_2 > 0$ *and* $w_1 \in \bar{\mathbb{R}}^n \backslash \mathbb{B}^n(w_2, r_2)$, *let* $Q = \mathbb{B}^n(w_1, r_1) \cup \mathbb{B}^n(w_2, r_2)$. *Suppose*

$$\mathrm{diam}(\mathbb{B}^n(w_1, r_1) \cap \mathbb{B}^n(w_2, r_2)) \ge \frac{r_2}{16 \times 3^{4n}}.$$

Then for any $y \in [w_1, w_2]$,

$$\mathrm{dist}(y, \partial Q) \ge \frac{r_2}{32 \times 3^{4n}} \quad \textit{and} \quad |y - w_2| \le r_2 + \mathrm{dist}(y, \partial Q). \tag{2.5}$$

Proof. Let $S = \partial\mathbb{B}^n(w_1, r_1) \cap \partial\mathbb{B}^n(w_2, r_2)$. If S contains at most one point, then the proof is obvious. On the other hand, if S contains at least two points, then we fix a point $x \in S$, and let $z \in [w_1, w_2]$ with

$$\min_{w \in [w_1, w_2]} |x - w| = |x - z|.$$

Then for all $y \in [w_1, w_2]$, we have

$$\mathrm{dist}(y, \partial Q) \ge |y - x| = |x - z| \ge \frac{r_2}{32 \times 3^{4n}}.$$

If $y \in [w_1, w_2] \cap \bar{\mathbb{B}}^n(w_2, r_2)$, then obviously

$$|y - w_2| \le r_2 + \mathrm{dist}(y, \partial Q).$$

If $y \in [w_1, w_2] \cap \{\bar{R}^n \backslash \mathbb{B}^n(w_2, r_2)\}$, then we let $[w_1, w_2] \cap \partial\mathbb{B}^n(w_2, r_2) = y_1$ so that

$$|y - w_2| = r_2 + |y - y_1| \le r_2 + \mathrm{dist}(y, \partial Q),$$

because $\mathrm{dist}(y, \partial Q) \ge |y - y_1|$. This completes the proof. ■

Proof of Theorem 1.3. We prove the necessary part as the sufficiency is obvious. Assume that D is a c-John domain. Then for every pair of points $z_1, z_2 \in D$, there is a rectifiable arc $\alpha \subset D$ joining them with

$$\min_{j=1,2} \ell(\alpha[z_j, z]) \le c\,\mathrm{dist}(z, \partial D) \quad \text{for all } z \in \alpha.$$

Thus, by Lemma 2.1, there exists a domain D_0 which is simply connected satisfying the desired properties (1)–(3) of Lemma 2.1. We need to prove that D_0 is a b-John domain, where b depends only on c and n.

If $y_1, y_2 \in D_0$ with $y_1, y_2 \in \mathbb{B}^n(x_i, r_i)$ for some i, then we take $\beta = [y_1, x_i] \cup [x_i, y_2]$, where $[x, z]$ denotes the Euclidean segment from x to z. In this case, we see that for any $y \in \beta$,

$$\min_{s=1,2} \ell(\beta[y_s, y]) \le \mathrm{dist}(y, \partial D). \tag{2.6}$$

If $y_1 \in \mathbb{B}^n(x_i, r_i)$ and $y_2 \in \mathbb{B}^n(x_j, r_j)$, where $i \ne j$, then, without loss of generality, we may assume that $i < j$ and set

$$\beta = [y_1, x_i] \cup [x_i, x_{i+1}] \cup \cdots \cup [x_{j-1}, x_j] \cup [x_j, y_2].$$

For any $y \in \beta$ with $y \in [y_1, x_i]$ or $[x_j, y_2]$, we also have

$$\min_{s=1,2} \ell(\beta[y_s, y]) \le \mathrm{dist}(y, \partial D). \tag{2.7}$$

If $y \in [x_m, x_{m+1}]$, where $i \le m \le j-1$, then we divide our discussions into the following two cases.

Case I. Let $\min\{\ell(\gamma[z_1, x_m]), \ell(\gamma[x_m, z_2])\} = \ell(\gamma[z_1, x_m])$.

Then, as D is a c-John domain and $\mathrm{dist}(x_m, \partial D) \le 2\,\mathrm{dist}(x_m, \partial D_0)$, we easily have

$$\ell(\gamma[z_1, x_m]) \le 2c\,\mathrm{dist}(x_m, \partial D_0) \tag{2.8}$$

so that

$$\begin{aligned} \ell(\beta[y_1, x_m]) &= |y_1 - x_i| + \ell(\beta[x_i, x_m]) \\ &\le 2\ell(\gamma[z_1, x_m]) \\ &\le 4c\,\mathrm{dist}(x_m, \partial D_0). \end{aligned} \tag{2.9}$$

Here we note that, if $i = m$ then $\ell(\beta[y_1, x_m]) = |y_1 - x_m| \le \mathrm{dist}(x_m, \partial D_0)$.

Subcase I. Assume that $\min\{\ell(\gamma[z_1, x_{m+1}]), \ell(\gamma[x_{m+1}, z_2])\} = \ell(\gamma[z_1, x_{m+1}])$.

Again, as D is a c-John domain, we obtain that

$$\ell(\gamma[z_1, x_{m+1}]) \le 2c\,\mathrm{dist}(x_{m+1}, \partial D_0). \tag{2.10}$$

If $r_m \le r_{m+1}$, then Lemma 2.4 and (2.9) show that

$$\begin{aligned}\min_{j=1,2} \ell(\beta[y_j, y]) &\le \ell(\beta[y_1, y]) \\ &\le 4c\,\mathrm{dist}(x_m, \partial D_0) + |y - x_m| \\ &\le (4c+1)\mathrm{dist}(x_m, \partial D_0) + \mathrm{dist}(y, \partial D_0) \\ &\le 64 \times 3^{4n}(2c+1)\,\mathrm{dist}(y, \partial D_0). \end{aligned} \tag{2.11}$$

If $r_m > r_{m+1}$, then Lemma 2.4, (2.9) and (2.10) yield

$$\begin{aligned}\min_{j=1,2} \ell(\beta[y_j, y]) &\le \ell(\beta[y_1, y]) \\ &\le 4c\,\mathrm{dist}(x_{m+1}, \partial D_0) \\ &\le 128 \times 3^{4n} c\,\mathrm{dist}(y, \partial D_0). \end{aligned} \tag{2.12}$$

Subcase II. Let $\min\{\ell(\gamma[z_1, x_{m+1}]), \ell(\gamma[x_{m+1}, z_2])\} = \ell(\gamma[z_2, x_{m+1}])$.

Then $\ell(\gamma[z_2, x_{m+1}]) \le 2c\,\mathrm{dist}(x_{m+1}, \partial D_0)$. A similar discussion as in the proof of (2.9) shows that

$$\ell(\beta[y_2, x_{m+1}]) \le 4c\,\mathrm{dist}(x_{m+1}, \partial D_0). \tag{2.13}$$

If $r_{m+1} \le r_m$, then by replacing w_j $(j = 1, 2)$ with x_{m+j-1} and r_j with $\mathrm{dist}(x_{m+j-1}, \partial G)$ in Lemmas 2.1, 2.4 and (2.13) imply that

$$\begin{aligned}\min_{j=1,2} \ell(\beta[y_j, y]) &\le \ell(\beta[y_2, y]) \\ &\le 4c\,\mathrm{dist}(x_{m+1}, \partial D_0) + |y - x_{m+1}| \\ &\le (4c+1)\,\mathrm{dist}(x_{m+1}, \partial D_0) + \mathrm{dist}(y, \partial D_0) \\ &\le 64 \times 3^{4n}(2c+1)\,\mathrm{dist}(y, \partial D_0). \end{aligned} \tag{2.14}$$

If $r_{m+1} > r_m$, then by Subcase I, we have

$$\min_{j=1,2} \ell(\beta[y_j, y]) \le \ell(\beta[y_1, y]) \le 64 \times 3^{4n}(2c+1)\mathrm{dist}(y, \partial D_0). \tag{2.15}$$

Case II. Finally, let $\min\{\ell(\gamma[z_1, x_m]), \ell(\gamma[x_m, z_2])\} = \ell(\gamma[z_2, x_m]) \le 2c\,\mathrm{dist}(x_m, \partial D_0)$.

Then, we have

$$\begin{aligned}&\min\{\ell(\gamma[z_1, x_{m+1}]), \ell(\gamma[x_{m+1}, z_2])\} \\ &\quad = \ell(\gamma[z_2, x_{m+1}]) \le 2c\,\mathrm{dist}(x_{m+1}, \partial D_0)\end{aligned}$$

and a reasoning similar to the proof of Case I shows that

$$\min_{j=1,2} \ell(\beta[y_j, y]) \le 64 \times 3^{4n}(2c+1)\,\mathrm{dist}(y, \partial D_0). \tag{2.16}$$

Thus, (2.6), (2.7), (2.11), (2.12), (2.14), (2.15) and (2.16) show that D_0 is a b-John domain with $b = 64 \times 3^{4n}(2c + 1)$. The proof is complete. ■

3. Proof of theorem 1.4

We begin by proving the sufficiency part of the theorem. For any pair of points $z_1, z_2 \in D$, we need to deal with three different situations.

Case I. First we assume that $z_1, z_2 \in G$.

Since $G = D\backslash P$ is a John domain, there is a rectifiable curve $\gamma \subset G$ connecting z_1 and z_2 such that for any $z \in \gamma$,

$$\min_{j=1,2}\{\ell(\gamma[z_j, z])\} \le c\,\mathrm{dist}(z, \partial G),$$

where $c \ge 1$. Consequently,

$$\min_{j=1,2}\{\ell(\gamma[z_j, z])\} \le c\,\mathrm{dist}(z, \partial D). \tag{3.1}$$

Case II. Next we let $z_1, z_2 \in D\backslash G$.

Set $\min_{j=1,2}\mathrm{dist}(z_j, \partial D) = 4t$ and $\delta = \min\{|p_i - p_j| : i \neq j\}$. Choose $z_1' \in \mathbb{B}^n(z_1, s) \cap G$ and $z_2' \in \mathbb{B}^n(z_2, s) \cap G$ with $s = \min\left\{t, \frac{\delta}{4}\right\}$. Then there must exist a curve $\gamma \subset G$ joining z_1' with z_2' such that for any $z \in \gamma$,

$$\min_{j=1,2}\ell(\gamma[z_j', z]) \le c\,\mathrm{dist}(z, \partial G).$$

Let $\beta = [z_1, z_1'] \cup \gamma \cup [z_2', z_2]$. For any $z \in \beta$, if $z \in [z_1, z_1']$ or $z \in [z_2', z_2]$, we have

$$\min_{j=1,2}\ell(\beta[z_j, z]) \le \ell(\beta[z_1,\ z]) \le 3\,\mathrm{dist}(z, \partial D). \tag{3.2}$$

If $z \in \gamma$ and $\min_{j=1,2}\ell(\gamma[z_j', z]) \le t$, then we have

$$\begin{aligned}\min_{j=1,2}\ell(\beta[z_j, z]) &\le t + \min_{j=1,2}\ell(\gamma[z_j', z])\\ &\le 2t\\ &\le \mathrm{dist}(z, \partial D),\end{aligned} \tag{3.3}$$

because $\mathrm{dist}(z, \partial D) \ge 4t - \min_{j=1,2}\ell(\beta[z_j, z]) \ge 2t$.

If $z \in \gamma$ and $\min_{j=1,2}\ell(\gamma[z_j', z]) > t$, then

$$\begin{aligned}\min_{j=1,2}\ell(\beta[z_j, z]) &\le t + \min_{j=1,2}\ell(\gamma[z_j', z])\\ &\le 2\min_{j=1,2}\ell(\gamma[z_j', z])\\ &\le 2c\,\mathrm{dist}(z, \partial D).\end{aligned} \tag{3.4}$$

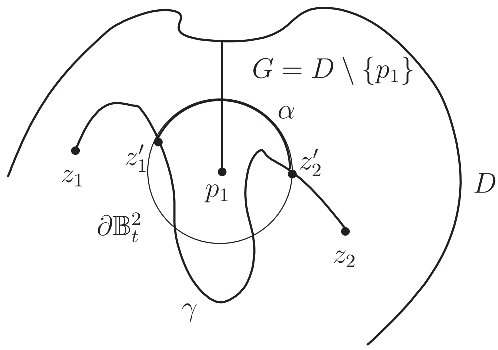

Figure 3. The circle $\partial\mathbb{B}_t^2$ passing through z_1' and z_2' with center p_1; and the shortest arc $\alpha \subset \partial\mathbb{B}_t^2$ joining z_1' and z_2'.

Case III. Finally, we let $z_1 \in G$ and $z_2 \in D \backslash G$.

A discussion similar to the proof of Case II shows that there is a rectifiable curve $\gamma \subset D$ connecting z_1 and z_2 such that for any $z \in \gamma$,

$$\min_{j=1,2} \{\ell(\gamma[z_j, z])\} \leq 3c \,\mathrm{dist}(z, \partial D). \tag{3.5}$$

The proof of sufficiency is complete.

For the proof of necessity, it suffices to consider the case $P = \{p_1\}$.

For convenience, we let $\mathrm{dist}(p_1, \partial D) = 2t$ and $\mathbb{B}_t^n = \mathbb{B}^n(p_1, t)$. For any pair of points $z_1, z_2 \in G = D \backslash \{p_1\}$, we again need to deal with three cases.

Case I. Let $z_1, z_2 \in D \backslash \mathbb{B}_t^n$.

Since D is a John domain, there must exist a curve γ joining z_1 with z_2 such that for all $z \in \gamma$,

$$\min_{j=1,2} \ell(\gamma[z_j, z]) \leq c \,\mathrm{dist}(z, \partial D),$$

where $c \geq 1$.

If $\gamma \subset D \backslash \mathbb{B}_t^n$, then we take $\beta = \gamma$. Also, we see that $\beta \subset G$ and for all $z \in \beta$,

$$\min_{j=1,2} \ell(\beta[z_j, z]) \leq 3c \,\mathrm{dist}(z, \partial G), \tag{3.6}$$

because $\mathrm{dist}(z, \partial D) \leq 3 \,\mathrm{dist}(z, \partial G)$.

If $\gamma \cap \mathbb{B}_t^n \neq \emptyset$, we let z_1' to be the first intersection point of γ from z_1 to z_2 with $\partial\mathbb{B}_t^n$ and z_2' to be the last intersection point of γ from z_1 to z_2 with $\partial\mathbb{B}_t^n$. Let $\mathbb{B}_t^2$ be the disk determined by z_1', z_2' and p_1 in $\mathbb{B}_t^n$. Then z_1' and z_2' divide $\partial\mathbb{B}_t^2$ into two subarcs. We denote the subarc with the shorter arclength by α (see figure 3). (If they have the same arclength, then we choose one of them to be α). Set $\beta = \gamma[z_1, z_1'] \cup \alpha \cup \gamma[z_2', z_2]$. Then for any $z \in \beta$ with $z \in \gamma[z_1, z_1']$, we have

$$\min_{j=1,2} \ell(\beta[z_j, z]) \leq \frac{\pi}{2} \min_{j=1,2} \ell(\gamma[z_j, z])$$

$$\leq \frac{\pi}{2} c \operatorname{dist}(z, \partial D)$$
$$\leq \frac{3\pi}{2} c \operatorname{dist}(z, \partial G), \tag{3.7}$$

because $\ell(\alpha) \leq \frac{\pi}{2}\, \ell(\gamma[z_1', z_2'])$ and $\operatorname{dist}(z, \partial D) \leq 3 \operatorname{dist}(z, \partial G)$.

If $z \in \gamma[z_2', z_2]$, then a similar reasoning as above shows that

$$\min_{j=1,2} \ell(\beta[z_j, z]) \leq \frac{3\pi}{2} c \operatorname{dist}(z, \partial G). \tag{3.8}$$

If $z \in \alpha$, then

$$\begin{aligned}\min_{j=1,2} \ell(\beta[z_j, z]) &\leq 2\pi t + \min\{\ell(\gamma[z_1, z_2']), \ell(\gamma[z_2', z_2])\} \\ &\leq 2\pi t + c \operatorname{dist}(z_2', \partial D) \\ &\leq (2\pi + 3c) \operatorname{dist}(z, \partial G),\end{aligned} \tag{3.9}$$

because $\operatorname{dist}(z, \partial G) = t$ and $\operatorname{dist}(z_2', \partial G) \leq 3t$.

Case II. Assume that $z_1, z_2 \in \bar{\mathbb{B}}_t^n \backslash \{p_1\}$.

Let z_1' be the intersection point of the ray starting from p_1 and passing through z_1 with $\partial \mathbb{B}_t^n$, and let z_2' be the intersection point of the ray starting from p_1 and passing through z_2 with $\partial \mathbb{B}_t^n$. We use $\mathbb{B}_t^2$ to denote the disk determined by z_1', z_2' and p_1 in $\mathbb{B}_t^n$. Then z_1' and z_2' divide $\partial \mathbb{B}_t^2$ into two subarcs. Let α denote the subarc with the shorter arclength. We set $\beta = [z_1, z_1'] \cup \alpha \cup [z_2', z_2]$. It is then easy to see that for any $z \in \beta$,

$$\min_{j=1,2} \ell(\beta[z_j, z]) \leq (\pi + 1) \operatorname{dist}(z, \partial G). \tag{3.10}$$

Case III. Finally, let $z_1 \in D \backslash \bar{\mathbb{B}}_t^n$ and $z_2 \in \mathbb{B}_t^n \backslash \{p_1\}$.

There must exist a curve γ joining z_1 with z_2 such that for all $z \in \gamma$,

$$\min_{j=1,2} \ell(\gamma[z_j, z]) \leq c \operatorname{dist}(z, \partial D).$$

Let z_1' be the first intersection point of γ from z_1 to z_2 with $\partial \mathbb{B}_t^n$, and let z_2' be the intersection point of the ray starting from p_1 and passing through z_2 with $\partial \mathbb{B}_t^n$. We use $\mathbb{B}_t^2$ to denote the disk determined by z_1', and z_2' and p_1 in $\mathbb{B}_t^n$. Then z_1' and z_2' divide $\partial \mathbb{B}_t^2$ into two subarcs. Denote by α, the subarc with the shorter arclength. We set $\beta = \gamma[z_1, z_1'] \cup \alpha \cup [z_2', z_2]$. Now we divide our discussion into two subcases.

Subcase I. Let $z \in [z_2', z_2]$.

In this subcase, it is clear that

$$\min_{j=1,2} \ell(\beta[z_j, z]) \leq \ell(\beta[z_2, z]) \leq (\pi + 1) \operatorname{dist}(z, \partial G). \tag{3.11}$$

Subcase II. Assume that $z \in \gamma[z_1, z_1']$.

If $\min_{j=1,2} \ell(\gamma[z_j, z]) = \ell(\gamma[z_1, z])$, then

$$\begin{aligned}\min_{j=1,2} \ell(\beta[z_j, z]) &\le \ell(\gamma[z_1, z])\\ &\le c\,\mathrm{dist}(z, \partial D)\\ &\le 3c\,\mathrm{dist}(z, \partial G), \end{aligned} \tag{3.12}$$

because $\mathrm{dist}(z, \partial D) \le 3\,\mathrm{dist}(z, \partial G)$.

If $\min_{j=1,2} \ell(\gamma[z_j, z]) = \ell(\gamma[z_2, z])$, then we know that

$$\ell(\beta[z_2, z_1']) \le (\pi + 1)t.$$

If $\ell(\gamma[z_1', z]) < \frac{t}{2}$, then it yields that

$$\begin{aligned}\min_{j=1,2} \ell(\beta[z_j, z]) &\le (\pi + 1)t + \ell(\gamma[z_1', z])\\ &\le (\pi + (3/2))t\\ &\le (2\pi + 3)\,\mathrm{dist}(z, \partial G), \end{aligned} \tag{3.13}$$

because $\mathrm{dist}(z, \partial G) \ge \frac{t}{2}$.

If $\ell(\gamma[z_1', z]) \ge \frac{t}{2}$, then we have that

$$\begin{aligned}\min_{j=1,2} \ell(\beta[z_j, z]) &\le (\pi + 1)t + \ell(\gamma[z_1', z])\\ &\le (2\pi + 3)\ \ell(\gamma[z_1', z])\\ &\le (2\pi + 3)c\,\mathrm{dist}(z, \partial D)\\ &\le 3(2\pi + 3)c\,\mathrm{dist}(z, \partial G), \end{aligned} \tag{3.14}$$

because $\mathrm{dist}(z, \partial D) \le 3\,\mathrm{dist}(z, \partial G)$.

Thus, eqs (3.6)–(3.14) show that the proof for the case that P contains a single point is complete. ■

4. Proof of theorem 1.7

If D is a simply connected sub domain of $\mathbb{R}^2$, then the hyperbolic density at $z \in D$ is defined by

$$\rho_D(z) = \frac{2|g'(z)|}{1 - |g(z)|^2},$$

where g is a conformal mapping of D onto the unit disk $\mathbb{B}(0, 1)$. Then for any pair of points z_1 and z_2 in D, the *hyperbolic distance* is defined by

$$h_D(z_1, z_2) = \inf_\alpha \int_\alpha \rho_D(z)|\mathrm{d}z|,$$

where the infimum is taken over all rectifiable arcs α joining z_1 and z_2 in D. Moreover, for any domain D in $\mathbb{R}^2$, as in [4] (see also [2]), we define a metric $j_D(z_1, z_2)$ for any z_1, $z_2 \in D$ by

$$j_D(z_1, z_2) = \log\left(1 + \frac{|z_1 - z_2|}{\mathrm{dist}(z_1, \partial D)}\right)\left(1 + \frac{|z_1 - z_2|}{\mathrm{dist}(z_2, \partial D)}\right).$$

Many basic properties of this metric may be found from [4]. For any $z_1,\ z_2 \in D$, the λ-*distance* between them is defined by

$$\lambda_D(z_1, z_2) = \inf\{\ell(\alpha)\colon \alpha \subset D \text{ is a rectifiable arc joining } z_1 \text{ and } z_2\}.$$

A point w in the boundary ∂D of D is said to be *rectifiably accessible* if there is a half open rectifiable arc α in D ending in w. Let $\partial_r D$ denote the subset of ∂D which consists of all the rectifiably accessible points, that is

$$\partial_r D = \{w \in \partial D\colon w \ \text{ is rectifiably accessible}\}.$$

As in [2] (see also [1, 6]), we define the λ-Apollonian metric a'_D as follows:

$$a'_D(z_1, z_2) = \sup_{w_1, w_2 \in \partial_r D} \log(|z_1, z_2, w_1, w_2|_{\lambda_D}),$$

where

$$|z_1, z_2, w_1, w_2|_{\lambda_D} = \frac{\lambda_D(z_1, w_1)\lambda_D(z_2, w_2)}{\lambda_D(z_1, w_2)\lambda_D(z_2, w_1)}.$$

Proof of Theorem 1.7. The proof of statement (1) follows from Theorem 1.4.

For the proof of statement (2), we choose

$$\mathbb{B}^n_t = \mathbb{B}^n(p, t)\left(0 < t < \frac{1}{3}\operatorname{dist}(p, \partial D)\right) \quad \text{and} \quad \operatorname{dist}(p, \partial D) = s,$$

and take $z_1, z_2 \in \partial \mathbb{B}^n_t$ with $|z_1 - z_2| = 2t$. Then we easily know that

$$j_G(z_1, z_2) = 2\log 3.$$

Also, on one hand we have

$$\begin{aligned} a'_G(z_1, z_2) &\le \log\left(1 + \frac{|z_1 - z_2|}{\operatorname{dist}(z_1, \partial D)}\right)\left(1 + \frac{|z_1 - z_2|}{\operatorname{dist}(z_2, \partial D)}\right) \\ &= \log\left(1 + \frac{2t}{s - t}\right)\left(1 + \frac{2t}{s - t}\right) \end{aligned}$$

and, on the other hand

$$\lim_{t \longrightarrow 0} \frac{2\log 3}{\log\left(1 + \frac{2t}{s-t}\right)\left(1 + \frac{2t}{s-t}\right)} = \infty.$$

This completes the proof.

Acknowledgement

The work of the first author was supported by INSA-JRD TATA fellowship, the second author by National Board for Higher Mathematics, India, and the third author was supported by National Board for Higher Mathematics, India, Commission on Development and Exchanges, and by grants NCET (No. 04-0783), NSF of China (No. 10771059).

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