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On split Lie algebras with symmetric root systems

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Abstract. We develop techniques of connections of roots for split Lie algebras with symmetric root systems. We show that any of such algebras L is of the form $L = \mathcal{U} + \sum_j I_j$ with $\mathcal{U}$ a subspace of the abelian Lie algebra H and any I_j a well described ideal of L, satisfying $[I_j, I_k] = 0$ if $j \neq k$. Under certain conditions, the simplicity of L is characterized and it is shown that L is the direct sum of the family of its minimal ideals, each one being a simple split Lie algebra with a symmetric root system and having all its nonzero roots connected.

Keywords. Infinite dimensional Lie algebras; split Lie algebras; roots.

1. Introduction and previous definitions

Throughout this paper, Lie algebras L are considered of arbitrary dimension and over an arbitrary field $\mathbb{K}$. It is worth to mention that, unless otherwise stated, there is not any restriction on dim L_α, the products $[L_\alpha, L_{-\alpha}]$, or $\{k \in \mathbb{K}\colon k\alpha \in \Lambda$, for a fixed $\alpha \in \Lambda\}$.

In the study of different classes of split Lie algebras has appeared several notions of 'connections of roots' that have had an important role in the classification of the algebras under consideration. These notions are related to the particular characteristics of each class. Let us see some examples:

An L^*-algebra is defined (see [2–4]) as a complex involutive Hilbert–Lie algebra for which the inner product $(\cdot|\cdot)$ satisfies the H^*-identities

$$([x, y]|z) = (y|[x^*, z]) = (x|[z, y^*]).$$

See also [1] for a definition in the real case.

Schue proved in [4] the existence of a dense split Lie algebra, with respect to a Cartan subalgebra H, for any separable semisimple L^*-algebra, and in [3] the following notion of 'connections of roots' in order to classify them: Given two nonzero roots α, β, a chain from α to β is a finite set of nonzero roots $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n = \beta$ such that $(h_{\alpha_i}|h_{\alpha_{i+1}}) \neq 0$ for $i = 1, \ldots, n-1$, where any h_{α_i} is the unique autoadjoint element in H with $\|h_{\alpha_i}\| = 1$ and $\alpha(h) = (h|h_{\alpha_i})$ for every $h \in H$. This notion depends on the Hilbert space structure of an L^*-algebra. In the study of semisimple locally finite split Lie algebras over a field of characteristic zero $\mathbb{K}$ [5], Stumme introduces the next definition (Definition III.21 of [5]): A subset M of nonzero roots is called irreducible if for every two roots $\alpha, \beta \in M$ there exists a family of roots $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n = \beta$ such that $\alpha_j(\check{\alpha}_{j+1}) \neq 0$ for $j = 1, \ldots, n-1$, where $\check{\alpha}_{j+1}$ is the coroot associated to α_{j+1} (it is said that α and β are 'connected'). This concept holds on the existence of the coroot for any nonzero root. This fact is ensured for any of these algebras as a consequence of any nonzero root α of a semisimple locally finite split Lie algebra $\mathcal{L}$ over a field of characteristic zero $\mathbb{K}$ which is integrable, and implies that $\alpha([x_\alpha, x_{-\alpha}]) \neq 0$ for $0 \neq x_{\pm\alpha} \in \mathcal{L}_{\pm\alpha}$, and $\dim \mathcal{L}_{\pm\alpha} = 1$. The split Lie algebras of these examples are particular cases of Lie algebras having a symmetric root system.

To extend the above notions, in this paper we introduce the concept of connections of roots for the wide class of split Lie algebras, over arbitrary fields, having a symmetric root system, and begin the study of this class of algebras.

DEFINITION 1.1

A *splitting Cartan subalgebra* H of a Lie algebra L is defined as a maximal abelian subalgebra of L satisfying that the adjoint mappings $\mathrm{ad}(h)$ for $h \in H$ are simultaneously diagonalizable. If L contains a splitting Cartan subalgebra H, then L is called a *split Lie algebra*. This means that we have a *root decomposition* $L = H + \left(\sum_{\alpha\in\Lambda} L_\alpha\right)$ where $L_\alpha = \{v_\alpha \in L\colon [h, v_\alpha] = \alpha(h)v_\alpha$ for any $h \in H\}$ for a linear functional $\alpha \in H^*$ and $\Lambda := \{\alpha \in H^* \backslash \{0\}\colon L_\alpha \neq 0\}$ is the corresponding *root system*. The subspaces L_α for $\alpha \in H^*$ are called *root spaces* of L (with respect to H) and the elements $\alpha \in \Lambda \cup \{0\}$ are called *roots* of L with respect to H.

DEFINITION 1.2

A root system Λ is called *symmetric* if it satisfies that $\alpha \in \Lambda$ implies $-\alpha \in \Lambda$.

It is clear that the root space associated to the zero root is contained in H and, by Jacobi identity, that if $\alpha + \beta$ is a root then $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$ and if $\alpha + \beta$ is not a root then $[L_\alpha, L_\beta] = 0$.

DEFINITION 1.3

A subset Λ_0 of Λ is called a *root subsystem* (relative to H) if it is symmetric and $\alpha, \beta \in \Lambda_0$, $\alpha + \beta \in \Lambda$ implies $\alpha + \beta \in \Lambda_0$.

Let Λ_0 be a root subsystem of Λ. We define $H_{\Lambda_0} := \mathrm{span}_{\mathbb{K}}\{[L_\alpha, L_{-\alpha}]\colon \alpha \in \Lambda_0\}$ and $V_{\Lambda_0} := \sum_{\alpha\in\Lambda_0} L_\alpha$. It is straightforward to verify $L_{\Lambda_0} := H_{\Lambda_0} + V_{\Lambda_0}$ is a Lie subalgebra of L. We will say that L_{Λ_0} is the Lie subalgebra *associated* to the root subsystem Λ_0.

2. Connections of roots: Decompositions

Unless otherwise stated, in the following L denotes a split Lie algebra with a symmetric root system, and $L = H + \left(\sum_{\alpha\in\Lambda} L_\alpha\right)$ the corresponding root decomposition.

DEFINITION 2.1

Let α and β be two nonzero roots, we shall say that α and β are *connected* if there exist $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that

$$\{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \cdots + \alpha_{n-1} + \alpha_n\}$$

is a family of nonzero roots, $\alpha_1 = \alpha$ and $\alpha_1 + \cdots + \alpha_{n-1} + \alpha_n \in \pm\beta$. We shall also say that $\{\alpha_1, \ldots, \alpha_n\}$ is a *connection* from α to β.

We denote by

$$\Lambda_\alpha := \{\beta \in \Lambda\colon \alpha \text{ and } \beta \text{ are connected}\}.$$

Let us observe that $\{\alpha\}$ is a connection from α to itself and to $-\alpha$. Therefore $\pm\alpha \in \Lambda_\alpha$. The next result shows that the connection relation is of equivalence.

PROPOSITION 2.1

The relation $\sim$ in Λ, defined by $\alpha \sim \beta$ if and only if $\beta \in \Lambda_\alpha$, is of equivalence.

Proof. $\{\alpha\}$ is a connection from α to itself and therefore $\alpha \sim \alpha$.

If $\alpha \sim \beta$ and $\{\alpha_1, \ldots, \alpha_n\}$ is a connection from α to β, then $\{\alpha_1 + \cdots + \alpha_n, -\alpha_n, -\alpha_{n-1}, \ldots, -\alpha_2\}$ is a connection from β to α in case $\alpha_1 + \cdots + \alpha_n = \beta$, and $\{-\alpha_1 - \cdots - \alpha_n, \alpha_n, \alpha_{n-1}, \ldots, \alpha_2\}$ in case $\alpha_1 + \cdots + \alpha_n = -\beta$. Therefore $\beta \sim \alpha$.

Finally, suppose $\alpha \sim \beta$ and $\beta \sim \gamma$, and write $\{\alpha_1, \ldots, \alpha_n\}$ for a connection from α to β and $\{\beta_1, \ldots, \beta_m\}$ for a connection from β to γ. If $m > 1$, then $\{\alpha_1, \ldots, \alpha_n, \beta_2, \ldots, \beta_m\}$ is a connection from α to γ in case $\alpha_1 + \cdots + \alpha_n = \beta$, and $\{\alpha_1, \ldots, \alpha_n, -\beta_2, \ldots, -\beta_m\}$ in case $\alpha_1 + \cdots + \alpha_n = -\beta$. If $m = 1$, then $\gamma \in \pm\beta$ and so $\{\alpha_1, \ldots, \alpha_n\}$ is a connection from α to γ. Therefore $\alpha \sim \gamma$ and $\sim$ is of equivalence. □

PROPOSITION 2.2

Let α be a nonzero root. Then the following assertions hold:

1. *Λ_α is a root subsystem.*
2. *If γ is a nonzero root such that $\gamma \notin \Lambda_\alpha$, then $[L_\beta, L_\gamma] = 0$ and $\gamma([L_\beta, L_{-\beta}]) = 0$ for any $\beta \in \Lambda_\alpha$.*

Proof.
(1) If $\beta \in \Lambda_\alpha$ then there exists a connection $\{\alpha_1, \ldots, \alpha_n\}$ from α to β. It is clear that $\{\alpha_1, \ldots, \alpha_n\}$ also connects α to $-\beta$ and therefore $-\beta \in \Lambda_\alpha$. If $\beta, \gamma \in \Lambda_\alpha$ and $\beta + \gamma \in \Lambda$, then there exists a connection $\{\alpha_1, \ldots, \alpha_n\}$ from α to β. Hence, $\{\alpha_1, \ldots, \alpha_n, \gamma\}$ is a connection from α to $\beta + \gamma$ in case $\alpha_1 + \cdots + \alpha_n = \beta$ and $\{\alpha_1, \ldots, \alpha_n, -\gamma\}$ in case $\alpha_1 + \cdots + \alpha_n = -\beta$. So $\beta + \gamma \in \Lambda_\alpha$.
(2) Let us suppose that there exists $\beta \in \Lambda_\alpha$ such that $[L_\beta, L_\gamma] \neq 0$ with $\gamma \notin \Lambda_\alpha$. Then $\beta + \gamma \in \Lambda$ and we have as in (1) that α is connected to $\beta + \gamma$. Since Λ_α is a root subsystem, $\gamma \in \Lambda_\alpha$, a contradiction. Therefore $[L_\beta, L_\gamma] = 0$ for any $\beta \in \Lambda_\alpha$ and $\gamma \notin \Lambda_\alpha$. As $-\beta \in \Lambda_\alpha$ for any $\beta \in \Lambda_\alpha$, we also have that $[L_{-\beta}, L_\gamma] = 0$. By applying Jacobi identity to $[[L_\beta, L_{-\beta}], L_\gamma]$ we conclude that $\gamma([L_\beta, L_{-\beta}]) = 0$ for any $\beta \in \Lambda_\alpha$. □

Theorem 2.1. *The following assertions hold*:

(1) *For any $\alpha \in \Lambda$, the Lie subalgebra*

$$L_{\Lambda_\alpha} = H_{\Lambda_\alpha} + V_{\Lambda_\alpha}$$

of L associated to the root subsystem Λ_α is an ideal of L.

(2) *If L is simple, then there exists a connection from α to β for any $\alpha, \beta \in \Lambda$.*

Proof.
(1) We have by Proposition 2.2(2) that $[L_\beta, L_\gamma] = 0$ and that $[[L_\beta, L_{-\beta}], L_\gamma] = 0$ for any $\beta \in \Lambda_\alpha$ and $\gamma \notin \Lambda_\alpha$. Hence,

$$[L_{\Lambda_\alpha}, L] = \left[\sum_{\beta\in\Lambda_\alpha} [L_\beta, L_{-\beta}] + \sum_{\beta\in\Lambda_\alpha} L_\beta, H + \left(\sum_{\gamma\in\Lambda_\alpha} L_\gamma\right) + \left(\sum_{\gamma\notin\Lambda_\alpha} L_\gamma\right)\right] \subset L_{\Lambda_\alpha}.$$

(2) The simplicity of L implies $L_{\Lambda_\alpha} = L$ and therefore $\Lambda_\alpha = \Lambda$. □

Theorem 2.2. *For a vector space complement* $\mathcal{U}$ *of* $\text{span}_{\mathbb{K}}\{[L_\alpha, L_{-\alpha}]: \alpha \in \Lambda\}$ *in* H, *we have*

$$L = \mathcal{U} + \sum_{[\alpha]\in\Lambda/\sim} I_{[\alpha]},$$

where any $I_{[\alpha]}$ *is one of the ideals* L_{Λ_α} *of* L *described in Theorem* 2.1(1), *satisfying* $[I_{[\alpha]}, I_{[\beta]}] = 0$ *if* $[\alpha] \neq [\beta]$.

Proof. By Proposition 2.1, we can consider the quotient set $\Lambda/\sim := \{[\alpha]: \alpha \in \Lambda\}$. Let us denote by $I_{[\alpha]} := L_{\Lambda_\alpha}$. $I_{[\alpha]}$ is well-defined and by Theorem 2.1(1) is an ideal of L. Therefore

$$L = \mathcal{U} + \sum_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}.$$

By applying Proposition 2.2(2) we also obtain $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$. □

Let us denote by $\mathcal{Z}(L)$ the center of L.

COROLLARY 2.1

If $\mathcal{Z}(L) = 0$ *and* $[L, L] = L$, *then* L *is the direct sum of the ideals given in Theorem* 2.2,

$$L = \bigoplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}.$$

Proof. From $[L, L] = L$ it is clear that $L = \sum_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$. The direct character of the sum now follows from the facts $[I_{[\alpha]}, I_{[\beta]}] = 0$, if $[\alpha] \neq [\beta]$, and $\mathcal{Z}(L) = 0$. □

3. The simple components

The study of the structure of these algebras has been reduced to consider those with all their roots connected. It is a natural question if these algebras are simple. Under certain conditions we give an affirmative answer. From now on $\text{char}(\mathbb{K}) = 0$.

Lemma 3.1. *Let* L *be a split Lie algebra. For any* $\alpha, \beta \in \Lambda$ *with* $\alpha \neq k\beta$, $k \in \mathbb{K}$, *there exists* $h_{\alpha,\beta} \in H$ *such that* $\alpha(h_{\alpha,\beta}) \neq 0$ *and* $\beta(h_{\alpha,\beta}) = 0$.

Proof. As $\alpha \neq 0$, there exists $h \in H - \{0\}$ such that $\alpha(h) \neq 0$. If $\beta(h) = 0$ we take $h_{\alpha,\beta} := h$. Suppose therefore $\beta(h) \neq 0$, let us write $k = \frac{\alpha(h)}{\beta(h)}$. As $\alpha \neq k\beta$, there exists $h' \in H$ such that $\alpha(h') \neq k\beta(h')$. We can take $h_{\alpha,\beta} := \beta(h')h - \beta(h)h'$. □

Lemma 3.2. *Let* L *be a split Lie algebra. If* I *is an ideal of* L *and* $x = h_0 + \sum_{j=1}^{m} e_{\beta_j} \in I$, *with* $h_0 \in H$, $e_{\beta_j} \in L_{\beta_j}$ *and* $\beta_j \neq \beta_k$ *if* $j \neq k$. *Then any* $e_{\beta_j} \in I$.

Proof. Let us fix β_1. If $e_{\beta_1} = 0$ then $e_{\beta_1} \in I$. Suppose then $e_{\beta_1} \neq 0$. For any $\beta_{k_r} \neq p\beta_1$, $p \in \mathbb{K}$ and $k_r \in \{2, \ldots, m\}$, Lemma 3.1 gives us $h_{\beta_1,\beta_{k_r}} \in H$ satisfying $\beta_1(h_{\beta_1,\beta_{k_r}}) \neq 0$ and $\beta_{k_r}(h_{\beta_1,\beta_{k_r}}) = 0$. From here,

$$[[\ldots[[x, h_{\beta_1,\beta_{k_2}}], h_{\beta_1,\beta_{k_3}}], \ldots], h_{\beta_1,\beta_{k_s}}] = p_1 e_{\beta_1} + \sum_{t=1}^{u} p_{k_t} e_{k_t\beta_1} \in I, \qquad (1)$$

$p_1, k_t \in \mathbb{K} - \{0\}$, $k_t \neq 1$ and $p_{k_t} \in \mathbb{K}$.

If any $p_{k_t} = 0$, $t = 1, \dots, u$, then $p_1 e_{\beta_1} \in I$ and so $e_{\beta_1} \in I$. Let us suppose some $p_{k_t} \neq 0$ and write (1) as

$$p_1 e_{\beta_1} + \sum_{t=1}^{v} p_{k_t} e_{k_t \beta_1} \in I, \tag{2}$$

with $p_1, k_t, p_{k_t} \in \mathbb{K} - \{0\}$, $k_t \neq 1$, $v \leq u$.

Let $h \in H$ be such that $\beta_1(h) \neq 0$. Then

$$\left[p_1 e_{\beta_1} + \sum_{t=1}^{v} p_{k_t} e_{k_t \beta_1}, h \right] = p_1 \beta_1(h) e_{\beta_1} + \sum_{t=1}^{v} p_{k_t} k_t \beta_1(h) e_{k_t \beta_1} \in I,$$

and so

$$p_1 e_{\beta_1} + \sum_{t=1}^{v} p_{k_t} k_t e_{k_t \beta_1} \in I, \ k_t \neq 1. \tag{3}$$

From (2) and (3) it follows easily that

$$q_1 e_{\beta_1} + \sum_{t=1}^{w} q_{k_t} e_{q_t \beta_1} \in I \tag{4}$$

with $q_1, q_{k_t} \in \mathbb{K} - \{0\}$, $q_t \in \{k_t \colon t = 1, \dots, v\}$ and $w < v$. Following this process, (multiply (4) with h and compare the result with (4) taking into account $q_t \neq 1$, and so on), we obtain $e_{\beta_1} \in I$. The same argument holds for any β_j, $j \neq 1$. □

DEFINITION 3.1

We say that a split Lie algebra L is *root-multiplicative* if $\alpha, \beta, \alpha + \beta \in \Lambda$ implies $[L_\alpha, L_\beta] \neq 0$.

As examples of root-multiplicative split Lie algebras we have the separable semisimple L^*-algebras and the semisimple locally finite split Lie algebras over a field of characteristic zero. Indeed, as we can take a locally finite split subalgebra dense in any separable semisimple L^*-algebra [2–4], it is enough to consider a semisimple locally finite split Lie algebra $\mathcal{L}$, (over a field of characteristic zero $\mathbb{K}$), but it is well-known that in any such algebras, if $\alpha, \beta, \alpha+\beta \in \Lambda$ then $[\mathcal{L}_\alpha, \mathcal{L}_\beta] = \mathcal{L}_{\alpha+\beta}$ (see Proposition I.7 (v) and Theorem III.19 of [5]), and so $\mathcal{L}$ is root-multiplicative.

Theorem 3.1. *Let L be root-multiplicative, with $\mathcal{Z}(L) = 0$, $[L, L] = L$ and satisfying* $\dim L_\alpha = 1$ *for any $\alpha \in \Lambda$. Then L is simple if and only if it has all its nonzero roots connected.*

Proof. The first implication is Theorem 2.1(2). Let us prove the converse: Let I be a nonzero ideal of L. We can find $0 \neq x \in I$ such that $x = h_0 + \sum_{j=1}^{m} e_{\beta_j} \in I$, with $h_0 \in H$, $e_{\beta_j} \in L_{\beta_j}$, $\beta_j \neq \beta_k$ if $j \neq k$ and satisfying some $e_{\beta_j} \neq 0$. Indeed, if $e_{\beta_j} = 0$ for any j, then $I \subset H$. As $\left[I, \sum_{\alpha\in\Lambda} L_\alpha\right] \subset H$ implies $\alpha(I) = 0$ for any $\alpha \in \Lambda$ and so $\left[I, \sum_{\alpha\in\Lambda} L_\alpha\right] = 0$, we get $I \subset \mathcal{Z}(L) = 0$, a contradiction. Let us choose such an $x \in I$, and fix any β_{j_0}, $j_0 \in \{1, \dots, m\}$ such that $e_{\beta_{j_0}} \neq 0$. By applying Lemma 3.2, $e_{\beta_{j_0}} \in I$ and, as $\dim L_{\beta_{j_0}} = 1$, we conclude $L_{\beta_{j_0}} \subset I$. We obtain that $L_{-\beta_{j_0}} \subset I$

as follows: Since $\beta_{j_0} \neq 0$ and $[L, L] = L$, there exists $[e_\gamma, e_{-\gamma}] \neq 0$, $e_{\pm\gamma} \in L_{\pm\gamma}$, $\gamma \in \Lambda$, such that $\beta_{j_0}([e_\gamma, e_{-\gamma}]) \neq 0$. If $\gamma \in \pm\beta_{j_0}$, as $0 \neq [e_{\beta_{j_0}}, e_{-\beta_{j_0}}] \in I$ then $e_{-\beta_{j_0}} = -\beta_{j_0}([e_{\beta_{j_0}}, e_{-\beta_{j_0}}])^{-1}[[e_{\beta_{j_0}}, e_{-\beta_{j_0}}], e_{-\beta_{j_0}}] \in I$ and so $L_{-\beta_{j_0}} \in I$. If $\gamma \notin \pm\beta_{j_0}$, as β_{j_0} and γ are connected, the root-multiplicativity of L and the assumption $\dim L_\alpha = 1$ for any $\alpha \in \Lambda$, give us a connection $\{\alpha_1, \ldots, \alpha_r\}$ from β_{j_0} to γ such that $\alpha_1 = \beta_{j_0}, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_r \in \pm\gamma \in \Lambda$, with $[L_{\alpha_1}, L_{\alpha_2}] = L_{\alpha_1+\alpha_2}$, $[[L_{\alpha_1}, L_{\alpha_2}], L_{\alpha_3}] = L_{\alpha_1+\alpha_2+\alpha_3}, \ldots, [[\ldots[[L_{\alpha_1}, L_{\alpha_2}], L_{\alpha_3}], \ldots], L_{\alpha_r}] = L_{\epsilon\gamma}$, where $\epsilon \in \pm 1$, and deduce that either $L_\gamma \subset I$ or $L_{-\gamma} \subset I$. In both cases $[L_\gamma, L_{-\gamma}] \subset I$ and so $[e_\gamma, e_{-\gamma}] \in I$. As given any $e_{-\beta_{j_0}} \in L_{-\beta_{j_0}}$, we have $e_{-\beta_{j_0}} = -\beta_{j_0}([e_\gamma, e_{-\gamma}])^{-1}[[e_\gamma, e_{-\gamma}], e_{-\beta_{j_0}}] \in I$, we conclude $L_{-\beta_{j_0}} \subset I$ and so $[L_{\beta_{j_0}}, L_{-\beta_{j_0}}] \subset I$. Given any $\delta \in \Lambda$, $\delta \neq \pm\beta_{j_0}$, arguing as above we have some $L_{\epsilon\delta} \subset I$, $\epsilon \in \pm 1$, and from here $L_{-\epsilon\delta} \subset I$. As a consequence $[L_\delta, L_{-\delta}] \subset I$. We have proved $\sum_{\alpha\in\Lambda}[L_\alpha, L_{-\alpha}] + \sum_{\beta\in\Lambda} L_\beta \subset I$. Since the condition $[L, L] = L$ implies $\sum_{\alpha\in\Lambda}[L_\alpha, L_{-\alpha}] + \sum_{\beta\in\Lambda} L_\beta = L$, $I = L$ and we conclude that L is simple. □

Theorem 3.2. *Let L be root-multiplicative, with $\mathcal{Z}(L) = 0$, $[L, L] = L$ and satisfying* $\dim L_\alpha = 1$ *for any $\alpha \in \Lambda$. Then L is the direct sum of the family of its minimal ideals, each one being a simple split Lie algebra with a symmetric root system and having all its nonzero roots connected.*

Proof. By Corollary 2.1 and Theorem 3.1 we just have to prove that any of the simple Lie algebras (see Corollary 2.1), $I_{[\alpha_0]} = \sum_{\alpha\in\Lambda_{\alpha_0}}[L_\alpha, L_{-\alpha}] + \sum_{\alpha\in\Lambda_{\alpha_0}} L_\alpha$ is split (with respect to its MASA $H_{[\alpha_0]} := \sum_{\alpha\in\Lambda_{\alpha_0}}[L_\alpha, L_{-\alpha}]$). The abelian maximal character of $H_{[\alpha_0]}$ follows that of $H = \sum_{\beta\in\Lambda}[L_\beta, L_{-\beta}]$ and that $[I_{[\alpha_0]}, I_{[\beta_0]}] = 0$ when $[\alpha_0] \neq [\beta_0]$. It is straightforward to verify that the set of nonzero roots is $\{\alpha|_{H_{[\alpha_0]}} : \alpha \in [\alpha_0]\}$, the fact $\alpha|_{H_{[\alpha_0]}} \neq 0$ being a consequence of $\alpha \neq 0$, $H = \sum_{\beta\in\Lambda}[L_\beta, L_{-\beta}]$ and Proposition 2.2(2). Finally, taking into account that Proposition 2.2(2) gives us that $\alpha|_{H^0_{[\alpha_0]}} = \beta|_{H^0_{[\alpha_0]}}$ implies $\alpha = \beta$, the root spaces associated to nonzero roots are $(I_{[\alpha_0]})_{\alpha|_{H^0_{[\alpha_0]}}} = L_\alpha$. We have showed that $I_{[\alpha_0]}$ is a split Lie algebra which clearly has a symmetric root system. □

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