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Solvability of boundary value problem at resonance for third-order functional differential equations

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Abstract. This paper is devoted to the study of boundary value problem of third-order functional differential equations. We obtain some existence results for the problem at resonance under the condition that the nonlinear terms is bounded or generally unbounded. In this paper we mainly use the topological degree theory.

Keywords. Functional boundary value problem; topological degree; Carathéodory conditions; resonance.

1. Introduction

Let X be the Banach space of $C[0, 1]$ with the sup norm $\|\cdot\|$. Denote by $\mathcal{D}$ the set of $T\colon X \longrightarrow X$ which is continuous and bounded (i.e. $T(\Omega)$ is bounded for any bounded $\Omega \subset X$).

In this paper, we consider the following third-order functional differential equations:

$$x'''(t) = f(t, x(t), (Fx)(t), x'(t), (Gx')(t), x''(t), (Hx'')(t)), \quad t \in [0, 1], \tag{1.1}$$

subject to the boundary conditions

$$x(0) = x''(0) = x''(1) = 0, \tag{1.2}$$

where $f\colon [0, 1] \times R^6 \longrightarrow R$ satisfies Carathéodory conditions, and $F, G, H \in \mathcal{D}$.

The special case of (1.1) is the following differential equation:

$$x'''(t) = g(t, x(t), x'(t), x''(t)), \quad t \in [0, 1], \tag{1.3}$$

where $g\colon [0, 1] \times R^3 \longrightarrow R$ satisfies Carathéodory conditions.

A functional boundary value problem (FBVP, for short) such as (1.1), (1.2) is called a problem at resonance, if the corresponding linear equation, say

$$x^{(''')}(t) = 0, \quad t \in [0, 1],$$

with the given linear boundary condition has nontrivial solutions. Otherwise, we call it a problem at non-resonance. So FBVP (1.1), (1.2) is a problem at resonance since linear equation $x^{(''')}(t) = 0,\ t \in [0,1]$, with boundary condition (1.2) has nontrivial solutions $x(t) = ct,\ c \in R,\ t \in [0,1]$.

Third-order functional boundary value problem at the non-resonance case were discussed in many papers in recent years, for instance, see [2, 3, 10, 11, 17] and references therein.

For the resonance case, Ma [13] combined the well-known Lyapunov–Schmidt procedure with the continuum theory for 0-epi maps to the study of existence and multiplicity of solutions for the following BVP:

$$x''' + k^2 x' + g(x, x') = p(t), \tag{1.4}$$

$$x'(0) = x'(\pi) = x(\eta) = 0, \tag{1.5}$$

where $\eta \in [0, \pi]$. For the case $k = 1$, Nagle and Pothoven [15] studied the solvability of (1.4) and (1.5) under the condition that g is bounded on one side (e.g. $g \geq 0$) and their work was motivated by Gupta [9], who studied the existence of boundary value problem of the type

$$x''' + \pi^2 x' + g(t, x, x', x'') = p(t),$$

$$x'(0) = x'(1) = x(\eta) = 0.$$

Motivated by the work [9, 13, 15], we used the topological degree theory to discuss in [5] the solvability of the following boundary value problem at resonance

$$x'''(t) = f(t, x(t), x'(t), x''(t)) + e(t), \quad t \in (0,1), \tag{1.6}$$

$$x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \quad x'(0) = 0, \quad x(1) = \beta x(\eta), \tag{1.7}$$

where f: $[0,1] \times R^3 \longrightarrow R$ is a continuous function, $e \in L^1[0,1]$, $\alpha_i \in R(1 \leq i \leq m-2)$, $\beta \geq 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ and $\eta \in (0,1)$.

In this paper, we apply the coincidence degree theory of Mawhin [14] to show sufficient conditions for the existence of solutions of FBVP (1.1), (1.2) at resonance. We obtain the existence results provided f satisfies only the sign conditions. This paper is a modified version of our papers [5, 6] in the light of papers [9, 11, 13, 15]. The problem discussed is more general than those in [13, 15]. *A priori* estimates as well as the restrictions on f are differential from [5, 6, 9]. For instance, f is only assumed to satisfy Carathéodory conditions. For recent results on nonlinear boundary value problems we refer the readers to [1, 4, 7, 8, 12, 16].

The paper is organized as follows. Section 2 briefly introduces some notations and an abstract existence result. Section 3 is devoted to the existence results of FBVP (1.1), (1.2) for bounded nonlinearity f. Section 4 gives the existence results of FBVP (1.1), (1.2) for the case that the nonlinearity f is unbounded.

2. Preliminary

First we present some preliminaries needed to understand the continuation theorem of Mawhin [14].

Let Y, Z be real Banach spaces and L: $\operatorname{dom} L \subset Y \longrightarrow Z$ a linear operator which is a Fredholm map of index zero (that is, $\operatorname{Im} L$, the image of L, $\operatorname{Ker} L$, the kernel of L, is finite dimensional with the same dimension as $Z/\operatorname{Im} L$) and P: $Y \longrightarrow Y$, Q: $Z \longrightarrow Z$ continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$ and $Y = \operatorname{Ker} L \oplus \operatorname{Ker} P$, $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. It follows that $L|_{\operatorname{dom} L \cap \operatorname{Ker} P}$: $\operatorname{dom} L \cap \operatorname{Ker} P \longrightarrow \operatorname{Im} L$ is invertible and we denote it as K_P of the inverse of that map. Let Ω be an open bounded subset of Y such that $\operatorname{dom} L \cap \Omega \neq \emptyset$, the map N: $Y \longrightarrow Z$ is said to be L-compact on $\bar{\Omega}$ if the map $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N$: $\bar{\Omega} \longrightarrow Y$ is compact.

Theorem A (Mawhin continuation theorem (Theorem IV.13 of [14])). *Let $\Omega \subset Y$ be an open bounded set and L be a Fredholm operator of index zero and let N be L-compact on $\bar{\Omega}$. Assume that*

(i) *for each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*
(ii) $QNx \neq 0$ *for every* $x \in \operatorname{Ker} L \cap \partial\Omega$;
(iii) $\deg(JQN, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, *where Q: $Z \longrightarrow Z$ is a projection as above with* $\operatorname{Im} L = \operatorname{Ker} Q$ *and J:* $\operatorname{Im} Q \longrightarrow \operatorname{Ker} L$ *is a linear isomorphism.*

Then the operator equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \bar{\Omega}$.

Now we shall recall a definition and a notation.

DEFINITION 1

We say that a function h: $[0, 1] \times R^6 \longrightarrow R$ satisfies Carathéodory conditions, if

(i) the map $y \mapsto h(t, y)$ is continuous for a.e. $t \in [0, 1]$;
(ii) the map $t \mapsto h(t, y)$ is measurable for all $y \in R^6$, and
(iii) for each $c > 0$ there exists $\psi_c \in L^1[0, 1]$, such that $|y| \leq c$ implies $|h(t, y)| \leq \psi_c(t)$ for a.e. $t \in [0, 1]$.

Throughout this paper, we assume that the following condition (H) hold:

(H) Both functions f: $[0, 1] \times R^6 \longrightarrow R$ and g: $[0, 1] \times R^3 \longrightarrow R$ satisfy Carathéodory conditions.

Notation 1. For constants $r_1, r_2, L_1, L_2 \in R$, $r_1 \leq r_2$, $L_1 \leq 0 \leq L_2$, operators $F, G, H \in \mathcal{D}$, we denote

$$\rho(F, \Omega) = \sup\{\|Fx\|: x \in \Omega\},$$

$$(r_1, r_2)_X = \{x \in X: r_1 \leq x(t) \leq r_2, \quad \text{for } t \in [0, 1]\},$$

$$(r_1, r_2; F, G)_4 = \{(x, u, w, \sigma) \in R^4: r_1 \leq x \leq r_2,$$
$$|u| \leq \rho(F, (r_1, r_2)_X), |w| \leq \rho(G, (r_1, r_2)_X)\},$$

$$(r_1, r_2; F, G)_6 = \{(x, u, v, w, \tau, \sigma) \in R^6: r_1 \leq x \leq r_2,$$
$$|u| \leq \rho(F, (r_1, r_2)_X), r_1 \leq v \leq r_2, |w| \leq \rho(G, (r_1, r_2)_X)\},$$

$$(r_1, r_2, L_1, L_2; F, G, H)_4 = \{(x, u, w, \sigma) \in R^4: r_1 \leq x \leq r_2,$$
$$|u| \leq \rho(F, (r_1, r_2)_X), |w| \leq \rho(G, (r_1, r_2)_X),$$
$$|\sigma| \leq \rho(H, (L_1, L_2)_X)\},$$

$$(r_1, r_2, L_1, L_2; F, G, H)_5 = \{(x, u, v, w, \sigma) \in R^5: r_1 \le x \le r_2,$$
$$|u| \le \rho(F, (r_1, r_2)_X), r_1 \le v \le r_2, |w| \le \rho(G, (r_1, r_2)_X),$$
$$|\sigma| \le \rho(H, (L_1, L_2)_X)\}.$$

3. Existence results for bounded nonlinearity *f*

Theorem 1. *Assume that*
(A1) *there exist* $r_1, r_2 \in R$ *and* $\varphi \in L^1([0, 1])$ *such that* $r_1 \le 0 \le r_2$ *and*

$$f(t, x, u, r_1, w, 0, \sigma) \le 0 \le f(t, x, u, r_2, w, 0, \sigma)$$

for a.e. $t \in [0, 1]$ *and for each* $(x, u, w, \sigma) \in (r_1, r_2; F, G)_4$, *and*

$$|f(t, x, u, v, w, \tau, \sigma)| \le \varphi(t)$$

for a.e. $t \in [0, 1]$ *and for each* $(x, u, v, w, \tau, \sigma) \in (r_1, r_2; F, G)_6$. *Then FBVP* (1.1), (1.2) *has a solution u such that*

$$r_1 \le u(t) \le r_2, \quad r_1 \le u'(t) \le r_2, \quad |u''(t)| \le \int_0^1 \varphi(s)\mathrm{d}s, \quad \textit{for } t \in [0, 1]. \tag{3.1}$$

To prove Theorem 1, we define the auxiliary functions $f_n: [0, 1] \times R^6 \longrightarrow R$ for each $n \in N$ as follows:

$$f_n(t, x, u, v, w, \tau, \sigma)$$
$$= \begin{cases} f(t, \bar{x}, \bar{u}, r_2, \bar{w}, 0, \sigma) + \dfrac{v - r_2 - \frac{1}{n}}{v - r_2 + 1}, & v > r_2 + \dfrac{1}{n}; \\ f(t, \bar{x}, \bar{u}, r_2, \bar{w}, \tau, \sigma) + p_n(x, u, r_2, w, \tau, \sigma), & r_2 < v \le r_2 + \dfrac{1}{n}; \\ f(t, \bar{x}, \bar{u}, v, \bar{w}, \tau, \sigma), & r_1 \le v \le r_2; \\ f(t, \bar{x}, \bar{u}, r_1, \bar{w}, \tau, \sigma) - p_n(x, u, r_1, w, \tau, \sigma), & r_1 - \dfrac{1}{n} \le v < r_1; \\ f(t, \bar{x}, \bar{u}, r_1, \bar{w}, 0, \sigma) + \dfrac{v - r_1 + \frac{1}{n}}{r_1 - v + 1}, & v < r_1 - \dfrac{1}{n}, \end{cases} \tag{3.2}$$

where

$$p_n(x, u, r_i, w, \tau, \sigma) = [f(t, \bar{x}, \bar{u}, r_i, \bar{w}, 0, \sigma) - f(t, \bar{x}, \bar{u}, r_i, \bar{w}, \tau, \sigma)](v - r_i)n,$$
$$i = 1, 2;$$

$$\bar{x} = \begin{cases} r_2, & x > r_2; \\ x, & r_1 \le x \le x_2; \\ r_1, & x < r_1, \end{cases}$$

$$\bar{u} = \begin{cases} u, & |u| \le \rho(F, (r_1, r_2)_X); \\ \rho(F, (r_1, r_2)_X)\operatorname{sign} u, & |u| > \rho(F, (r_1, r_2)_X), \end{cases}$$

$$\bar{w} = \begin{cases} w, & |w| \le \rho(G, (r_1, r_2)_X); \\ \rho(G, (r_1, r_2)_X)\operatorname{sign} w, & |w| > \rho(G, (r_1, r_2)_X). \end{cases}$$

In the following, we consider the differential equation

$$x'''(t) = \lambda f_n(t, x(t), (Fx)(t), x'(t), (Gx')(t), x''(t), (Hx'')(t)),$$
$$\lambda \in [0, 1], \quad t \in [0, 1]. \tag{$3.3_\lambda)_n$}$$

In order to use Mawhin continuation theorem, we denote $Y = C^2[0, 1]$, $Z = L^1[0, 1]$ be the Banach space with the usual norms, $AC^2([0, 1]) = \{x\colon [0, 1] \longrightarrow R | x''$ are absolutely continuous on $[0, 1]\}$, and L is the linear operator from $\operatorname{dom} L \subset Y$ to Z with

$$\operatorname{dom} L = \{x\colon x \in AC^2[0, 1], x(0) = x''(0) = x''(1) = 0\}$$

and $Lx = x'''$, $x \in \operatorname{dom} L$. For each $n \in N$, define the operator $N\colon Y \longrightarrow Z$ by setting

$$Nx = f_n(t, x(t), (Fx)(t), x'(t), (Gx')(t), x''(t), (Hx'')(t)), \quad t \in [0, 1].$$

Then FBVP $(3.3_\lambda)_n$, (1.2) can be written as $Lx = \lambda Nx$, $\lambda \in [0, 1]$.

We show Theorem 1 via the following lemmas.

Lemma 1 (A priori *estimates*). *Assume that* (A1) *holds and FBVP* $(3.3_\lambda)_n$, (1.2) *have a solution u for some* $\lambda \in (0, 1]$ *and* $n \in N$. *Then the following estimate is fulfilled*:

$$r_1 - \frac{1}{n} \le u(t) \le r_2 + \frac{1}{n}, \quad r_1 - \frac{1}{n} \le u'(t) \le r_2 + \frac{1}{n},$$
$$|u''(t)| \le \int_0^1 \varphi(s)\mathrm{d}s, \quad t \in [0, 1]. \tag{3.4}$$

Proof. Suppose by contradiction the estimates $r_1 - \frac{1}{n} \le u'(t) \le r_2 + \frac{1}{n}$ is not true. Then there exists $t \in [0, 1]$, such that $u'(t) > r_2 + \frac{1}{n}$ or $u'(t) < r_1 - \frac{1}{n}$. Without loss the generality, we suppose that the first case hold. let

$$\max\{u'(t)\colon t \in [0, 1]\} := u'(t_0)\left(> r_2 + \frac{1}{n}\right), \quad \text{for a } t_0 \in [0, 1].$$

Then $u''(t_0) = 0$. It is clear that if $t_0 \in (0, 1)$ comes from boundary conditions (1.2), $t_0 \in \{0, 1\}$. We select $t_1, t_2 \in (t_0, 1)$, $t_1 < t_2$, such that $u'(t_1) \ge u'(t_2) > r_2 + \frac{1}{n}$. Thus

$$\int_{t_1}^{t_2} u'''(s)\mathrm{d}s = u''(t_2) - u''(t_1) \le 0. \tag{3.5}$$

On the other hand, by (3.2) and (A1), we get

$$\begin{aligned}
&\int_{t_1}^{t_2} u'''(s)\mathrm{d}s \\
&= \lambda \int_{t_1}^{t_2} f_n(s, u(s), (Fu)(s), u'(s), (Gu')(s), u''(s), (Hu'')(s))\mathrm{d}s \\
&= \lambda \int_{t_1}^{t_2} \left[f_n(s, \bar{u}(s), \overline{(Fu)}(s), r_2, \overline{(Gu')}(s), 0, (Hu'')(s)) + \frac{u'(s) - r_2 - \frac{1}{n}}{u'(s) - r_2 + 1} \right] \mathrm{d}s \\
&> 0
\end{aligned}$$

which contradicts (3.5). Similarly, we can verify that the estimates $r_1 - \frac{1}{n} \leq u'(t)$ holds. Moreover, since $u(0) = 0$, the estimates $r_1 - \frac{1}{n} \leq u(t) \leq r_2 + \frac{1}{n}$ is easily obtained by integration.

Now we show

$$|f_n(t, u(t), (Fu)(t), u'(t), (Gu')(t), u''(t), (Hu'')(t))| \leq \varphi(t), \quad \text{for a.e. } t \in [0, 1] \tag{3.6}$$

is fulfilled. From the first and second estimates in (3.4) and definition of f_n in (3.2), we show (3.6) in three cases:

Case (i): $r_2 < u' \leq r_2 + \frac{1}{n}$;
Case (ii): $r_1 \leq u' \leq r_2$;
Case (iii): $r_1 - \frac{1}{n} \leq u' < r_1$.

We only show Case (i), and the others can be proved similarly. If $r_2 < u' \leq r_2 + \frac{1}{n}$, then

$$\begin{aligned}
&|f_n(t, u(t), (Fu)(t), u'(t), (Gu')(t), u''(t), (Hu'')(t))| \\
&= |f(t, \bar{u}(t), \overline{(Fu)}(t), r_2, \overline{(Gu')}(t), u''(t), (Hu'')(t)) \\
&\quad + [f(t, \bar{u}(t), \overline{(Fu)}(t), r_2, \overline{(Gu')}(t), 0, (Hu'')(t)) \\
&\quad - f(t, \bar{u}(t), \overline{(Fu)}(t), r_2, \overline{(Gu')}(t), u''(t), (Hu'')(t))](u'(t) - r_2)n| \\
&\leq |f(t, \bar{u}(t), \overline{(Fu)}(t), r_2, \overline{(Gu')}(t), u''(t), (Hu'')(t))|[1 - (u'(t) - r_2)n] \\
&\quad + |f(t, r_2, \overline{(Fu)(t)}, r_2, \overline{(Gu')}(t), 0, (Hu'')(t))|(u'(t) - r_2)n \\
&\leq \varphi(t)[1 - (u'(t) - r_2)n] + \varphi(t)(u'(t) - r_2)n = \varphi(t).
\end{aligned}$$

Integrating $(3.3_\lambda)_n$ from 0 to t, we obtain the estimate $|u''(t)| \leq \int_0^1 \varphi(s)\mathrm{d}s$ in (3.4) by inequality (3.6) and $u''(0) = 0$.

Lemma 2. *L is a Fredholm map of index* 0 *and N is L-compact on $\bar{\Omega}$ for any open bounded set $\Omega \subset Y$.*

Proof. Firstly, we show that L is a Fredholm map of index 0. It is clear that

$$\operatorname{Ker} L = \{x \in \operatorname{dom} L\colon x = ct,\, c \in R,\, t \in [0, 1]\}.$$

Now we show that

$$\operatorname{Im} L = \left\{ y \in Z\colon \int_0^1 y(s)\mathrm{d}s = 0 \right\}. \tag{3.7}$$

The problem

$$x''' = y \tag{3.8}$$

has a solution $x(t)$ satisfying boundary conditions (1.2) if and only if

$$\int_0^1 y(s)\mathrm{d}s = 0. \tag{3.9}$$

In fact, if (3.8) has solution $x(t)$ satisfying boundary conditions (1.2), then we have

$$x(t) = x'(0)t + \int_0^t \int_0^s \int_0^\tau y(v)\mathrm{d}v\mathrm{d}\tau\mathrm{d}s.$$

According to $x''(1) = 0$, we obtain $\int_0^1 y(s)\mathrm{d}s = 0$.

On the other hand, if (3.9) holds, setting

$$x(t) = ct + \int_0^t \int_0^s \int_0^\tau y(v)\mathrm{d}v\mathrm{d}\tau\mathrm{d}s,$$

where c is an arbitrary constant, then $x(t)$ is a solution of (3.8) and satisfies (1.2). Hence (3.7) is valid. Clearly, $\operatorname{Im} L$ is closed in Z and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 1$. Thus L is a Fredholm operator of index zero.

Now we consider the continuous projection

$$P\colon Y \longrightarrow Y, \quad x \longmapsto x(0)t$$

$$Q\colon Z \longrightarrow Z, \quad y \longmapsto \int_0^1 y(s)\mathrm{d}s.$$

Then the generalized inverse (to L) $K_P\colon \operatorname{Im} L \longrightarrow \operatorname{dom} L \cap \operatorname{Ker} P$ can be written as

$$K_P(y) = \int_0^t \int_0^s \int_0^\tau y(v)\mathrm{d}v\mathrm{d}\tau\mathrm{d}s.$$

Thus,

$$QN\colon Y \longrightarrow Z, \quad x \longmapsto \int_0^1 f_n(s, u(s), (Fu)(s), u'(s), (Gu')(s), u''(s), (Hu'')(s))\mathrm{d}s, \tag{3.10}$$

$K_P(I-Q)N\colon Y \longrightarrow Y,$

$$x \longmapsto \int_0^t \int_0^s \int_0^\tau f_n(v, u(v), (Fu)(v), u'(v), (Gu')(v), u''(v), (Hu'')(v))\mathrm{d}v\mathrm{d}\tau\mathrm{d}s$$
$$- \frac{t^3}{6}\int_0^1 f_n(s, u(s), (Fu)(s), u'(s), (Gu')(s), u''(s), (Hu'')(s))\mathrm{d}s.$$

Since $F, G, H \in \mathcal{D}$ and from (3.2), (A1), we obtain

$$|f_n(t, x, u, v, w, \tau, \sigma)| \le \varphi(t), \quad \text{for a.e. } t \in [0, 1] \text{ and each } (x, u, v, w, \tau, \sigma) \in R^6.$$

By the Lebesgue dominated convergence theorem, we show QN and $K_P(I-Q)N$ are continuous. Furthermore, $QN(\bar{\Omega})$ and $K_P(I-Q)N(\bar{\Omega})$ are relatively compact for any open bounded set $\Omega \subset Y$. Hence N is L-compact on $\bar{\Omega}$ for any open bounded set $\Omega \subset Y$.

Lemma 3. *Assume that* (A1) *holds, then for each* $n \in N$, *FBVP* $(3.3_\lambda)_n$, (1.2) *has a solution u satisfying* (3.4).

Proof. We only need to show that all conditions of Theorem A are satisfied. For $n \in N$, let P, Q and K_P be as in the proof of Lemma 2 and let

$$\Omega = \left\{ x \in Y\colon r_1 - \frac{2}{n} \le x'(t) \le r_2 + \frac{2}{n}, \quad |x''(t)| \le \int_0^1 \varphi(s)\mathrm{d}s + 1, \quad \text{for } t \in [0, 1] \right\}.$$

From Lemma 2, N is L-compact on $\bar{\Omega}$ and then Lemma 1 implies that condition (i) of Theorem A is fulfilled.

To check condition (ii), we assume that $x \in \operatorname{Ker} L \cap \partial\Omega$, then $x' = r_1 - \frac{2}{n}$ or $x' = r_2 + \frac{2}{n}$, so $x = \left(r_1 - \frac{2}{n}\right)t$ or $x = \left(r_2 + \frac{2}{n}\right)t$ by $x(0) = 0$. Due to (3.10), (3.2) and (A1), we have

$$\begin{aligned} QN\left(\left(r_1 - \frac{2}{n}\right)t\right) &= \int_0^1 f_n\left(s, \left(r_1 - \frac{2}{n}\right)s, \left(F\left(\left(r_1 - \frac{2}{n}\right)s\right)\right)(s),\right. \\ &\qquad \left. r_1 - \frac{2}{n}, \left(G\left(r_1 - \frac{2}{n}\right)\right)(s), 0, (H(0))(s)\right)\mathrm{d}s \\ &= \int_0^1 \left[f\left(s, r_1, \overline{\left(F\left(\left(r_1 - \frac{2}{n}\right)s\right)\right)}(s), r_1,\right.\right. \\ &\qquad \left.\left. \overline{\left(G\left(r_1 - \frac{2}{n}\right)\right)}(s), 0, (H(0))(s)\right) - \frac{1}{n+2}\right]\mathrm{d}s \\ &\le -\frac{1}{n+2} < 0, \end{aligned} \tag{3.11}$$

$$\begin{aligned} QN\left(\left(r_2 + \frac{2}{n}\right)t\right) &= \int_0^1 f_n\left(s, \left(r_2 + \frac{2}{n}\right)s, \left(F\left(\left(r_2 + \frac{2}{n}\right)s\right)\right)(s),\right. \\ &\qquad \left. r_2 + \frac{2}{n}, \left(G\left(r_2 + \frac{2}{n}\right)\right)(s), 0, (H(0))(s)\right)\mathrm{d}s \end{aligned}$$

$$= \int_0^1 \left[f\left(s, \overline{\left(r_2 + \frac{2}{n}\right) s}, \overline{\left(F\left(\left(r_2 + \frac{2}{n}\right) s\right)\right)}(s), r_2, \right.\right.$$

$$\left.\left. \overline{\left(G\left(r_2 + \frac{2}{n}\right)\right)}(s), 0, (H(0))(s) \right) + \frac{1}{n+2} \right] \mathrm{d}s$$

$$\geq \frac{1}{n+2} > 0. \tag{3.12}$$

This condition (ii) of Theorem A is satisfied.

Let $J\colon \operatorname{Im} Q \longrightarrow \operatorname{Ker} L$ be a linear isomorphism given by $J(c) = ct, \forall c \in R, t \in [0, 1]$. From inequalities (3.11) and (3.12), we obtain that

$$\deg(JQN, \Omega \cap \operatorname{Ker} L, 0) \neq 0.$$

Then condition (iii) of Theorem A also holds. So the assentation of our Lemma 3 follows from Theorem A and Lemma 1.

The proof of Theorem 1 is now an easy consequence of the above lemmas.

Proof of Theorem 1. For $n \in N$, we consider the sequence of FBVP $\{(3.3_\lambda)_n, (1.2)\}$. By lemma 3, we get an appropriate sequence of solutions u_n for which (3.4) (with $u = u_n$). Then by (3.2), (3.4), one has

$$|u_n'''(t)| = |f_n(t, u(t), (Fu)(t), u'(t), (Gu')(t), u''(t), (Hu'')(t))| \leq \varphi(t)$$

for a.e. $t \in [0, 1]$ and each $n \in N$. By the Ascoli–Arzela theorem, there exists a subsequence $\{u_{k_n}\}$ of $\{u_n\}$ converging in $C^2([0, 1])$ to some u. The function u satisfies (3.1) and, hence it is a solution of FBVP (1.1), (1.2) from (3.2).

COROLLARY 1

Assume that there exist $r_1, r_2 \in R$, $\varphi \in L^1([0, 1])$ *such that* $r_1 \leq 0 \leq r_2$ *and*

$$g(t, x, r_1, 0) \leq 0 \leq g(t, x, r_2, 0), \quad \textit{for a.e. } t \in [0, 1] \textit{ and each } x \in [r_1, r_2]$$

$$|g(t, x, y, z)| \leq \varphi(t), \quad \textit{for a.e. } t \in [0, 1] \textit{ and each}$$

$$(x, y, z) \in [r_1, r_2] \times [r_1, r_2] \times R.$$

Then FBVP (1.3), (1.2) *has a solution u satisfying* (3.1).

4. Existence results for generally unbounded nonlinearity *f*

Theorem 2. *Assume that*
(A2) *there exist* $r_1, r_2, L_1, L_2 \in R$ *and* $\mu, \nu \in \{-1, 1\}$ *such that* $r_1 \leq 0 \leq r_2, L_1 \leq 0 \leq L_2$, *and*

$$f(t, x, u, r_1, w, 0, \sigma) \leq 0 \leq f(t, x, u, r_2, w, 0, \sigma)$$

for a.e. $t \in [0, 1]$ *and for each* $(x, u, w, \sigma) \in (r_1, r_2, L_1, L_2; F, G, H)_4$, *and*

$$\nu f(t, x, u, v, w, L_1, \sigma) \leq 0 \leq \mu f(t, x, u, v, w, L_2, \sigma)$$

for a.e. $t \in [0, 1]$ *and for each* $(x, u, v, w, \sigma) \in (r_1, r_2, L_1, L_2; F, G)_5$. *Then FBVP* (1.1), (1.2) *has a solution u such that*

$$r_1 \le u(t) \le r_2, \quad r_1 \le u'(t) \le r_2, \quad L_1 \le u''(t) \le L_2, \quad \textit{for } t \in [0, 1]. \tag{4.1}$$

Proof. We define the auxiliary functions $\bar{f}_{\mu\nu}$: $[0, 1] \times R^6 \longrightarrow R$ as follows:

$$\bar{f}_{\mu\nu}(t, x, u, v, w, \tau, \sigma) = \begin{cases} f(t, x, u, v, w, L_2, \bar{\sigma}) + \mu \dfrac{\tau - L_2}{\tau - L_2 + 1}, & \tau > L_2; \\ f(t, x, u, v, w, \tau, \bar{\sigma}), & L_1 \le \tau \le L_2; \\ f(t, x, u, v, w, L_1, \bar{\sigma}) + \nu \dfrac{\tau - L_1}{L_1 - \tau + 1}, & \tau < L_1, \end{cases} \tag{4.2}$$

where

$$\bar{\sigma} = \begin{cases} \sigma, & |\sigma| \le \rho(H, (L_1, L_2)_X); \\ \rho(H, (L_1, L_2)_X) \operatorname{sign} \sigma, & |\sigma| > \rho(H, (L_1, L_2)_X). \end{cases}$$

Then $\bar{f}_{\mu\nu}$ fulfills assumption (A1) with

$$\begin{aligned} \varphi(t) = 1 + \sup\{|f(t, x, u, v, w, \tau, \sigma)|: (x, u, v, w, \tau, \sigma) \in R^6, \\ r_1 \le x \le r_2, |u| \le \rho(F, (r_1, r_2)_X), r_1 \le v \le r_2, \\ |w| \le \rho(G, (r_1, r_2)_X), L_1 \le \tau \le L_2, |\sigma| \le \rho(H, (L_1, L_2)_X)\}. \end{aligned}$$

We consider the differential equation

$$x'''(t) = \bar{f}_{\mu\nu}(t, x(t), (Fx)(t), x'(t), (Gx')(t), x''(t), (Hx'')(t)), \quad t \in [0, 1]. \tag{4.3}$$

Theorem 1 implies that FBVP (4.3), (1.2) has a solution u such that inequalities in (3.1) hold. Now we show that solution u also satisfies inequalities in (4.1).

We show that the estimates $L_1 \le u''(t) \le L_2$ is fulfilled. Otherwise, we set

$$\max\{u''(t): t \in [0, 1]\} := u''(t_0) > L_2 \ge 0, \quad \text{for } t_0 \in [0, 1].$$

Then from boundary conditions (1.2), $t_0 \in (0, 1)$. So there exists $\delta > 0$, such that $L_2 < u''(t) \le u''(t_0)$ for each t belonging to the interval with the end points t_0 and $t_0 + \mu\delta$, and

$$\int_{t_0}^{t_0+\mu\delta} u'''(s)\mathrm{d}s = u''(t_0 + \mu\delta) - u''(t_0) \le 0. \tag{4.4}$$

On the other hand, by (4.2) and (A2), we get

$$\begin{aligned}
&\int_{t_0}^{t_0+\mu\delta} u'''(s)\mathrm{d}s \\
&\quad = \int_{t_0}^{t_0+\mu\delta} \bar{f}_{\mu\nu}(s, u(s), (Fu)(s), u'(s), (Gu')(s), u''(s), (Hu'')(s))\mathrm{d}s \\
&\quad = \mu \int_{t_0}^{t_0+\mu\delta} \Bigg[\mu f(s, u(s), (Fu)(s), u'(s), (Gu')(s)\mathrm{d}s, L_2, \overline{(Hu'')(s)}) \\
&\qquad + \frac{u''(s) - L_2}{u''(s) - L_2 + 1}\Bigg]\mathrm{d}s \\
&\quad > 0
\end{aligned}$$

which contradicts (4.4). Hence $u''(t) \leq L_2$ on [0, 1]. Similarly, we can verify that the estimate $L_1 \leq u''(t)$ on [0, 1] holds. By (3.1) and (4.2), we can show that u is a solution of FBVP (1.1), (1.2) and satisfies (4.1).

COROLLARY 2

Assume that there exist $r_1, r_2, L_1, L_2 \in R$ *and* $\mu, \nu \in \{-1, 1\}$ *such that* $r_1 \leq 0 \leq r_2$, $L_1 \leq 0 \leq L_2$, *and*

$$g(t, x, r_1, 0) \leq 0 \leq g(t, x, r_2, 0), \quad \textit{for a.e. } t \in [0, 1], \ \textit{and each } x \in [r_1, r_2],$$

$g(t, x, y, L_i)$ *do not change their signs for a.e.* $t \in [0, 1]$, *and each* $(x, y) \in [r_1, r_2] \times [r_1, r_2]$, $i = 1, 2$. *Then FBVP* (1.3), (1.2) *has a solution u satisfying* (4.1).

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