Proc. Indian Acad. Sci. (Math. Sci.) Vol. 118, No. 2, May 2008, pp. 295–305.

Positive solutions for higher order singular p-Laplacian boundary value problems

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MS received 18 October 2005; revised 14 September 2007

Abstract. This paper investigates $2m$-th ($m \geq 2$) order singular p-Laplacian boundary value problems, and obtains the necessary and sufficient conditions for existence of positive solutions for sublinear $2m$-th order singular p-Laplacian BVPs on closed interval.

Keywords. Positive solution; singular BVPs; sufficient and necessary conditions; p-Laplacian equations.

1. Introduction

In this paper, we are concerned with higher order singular p-Laplacian boundary value problems

$$\begin{cases} (-1)^m(\varphi_p(u^{(2m-2)}))'' = f(t,u), \quad t \in (0,1), \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad 1 \leq i \leq m-1, \end{cases} \tag{1.1}$$

where $m \geq 2$ is an integer, $1 < p < \infty$, and φ_p is the odd function which is defined by $\varphi_p(s) = |s|^{p-2}s$, for $s \neq 0$ and 0 if $s = 0$. For the nonlinear function in the equation, $f \in C((0,1) \times [0,+\infty), [0,+\infty))$ and is *quasi-homogeneous* with respect to the second variable, namely, there are constants λ, μ, N, M, $-\infty < \lambda < 0 < \mu < p-1$, $0 < N \leq 1 \leq M$, such that for all $0 < t < 1$, $u \geq 0$,

$$c^{\mu} f(t,u) \leq f(t,cu) \leq c^{\lambda} f(t,u), \quad \text{if} \quad 0 < c \leq N, \tag{1.2}$$

$$c^{\lambda} f(t,u) \leq f(t,cu) \leq c^{\mu} f(t,u), \quad \text{if} \quad c \geq M. \tag{1.3}$$

A typical *quasi-homogeneous* function is $f(t,u) = p_1(t)u^{\lambda_1} + p_2(t)u^{\lambda_2} + \cdots + p_n(t)u^{\lambda_n}$, where $0 < \lambda_i \leq p-1$, $p_i(t) \geq 0$, $i = 1, 2, \ldots, n$.

In order to explain these problems further, we consider a special example as follows:

$$\begin{cases} (\varphi_p(u''))'' = p_1(t)u^{\lambda_1} + p_2(t)u^{\lambda_2}, \quad t \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}$$

where $p_i(t) \geq 0$, $i = 1, 2$, $\lambda_1 < 0$ and $0 < \lambda_2 < p - 1$. Here it deserves to point out that when $p > 2$, λ_2 may be greater than 1, this means the nonlinear function f in (1.4), which satisfies the conditions of (1.2) and (1.3), may be superlinear in the case of p-Laplacian BVPs when $p > 2$.

When $p = 2$, $\varphi_p(s) = s$, (1.1) becomes

$$\begin{cases} u^{(4)}(t) = f(t, u(t)), & t \in (0, 1), \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{1.4}$$

The singular or nonsingular fourth-order boundary value problems (1.4) have been extensively studied by many authors [1,2,6,7,10,13–15]. Shi and Chen [10,11] gave the sufficient and necessary conditions for the existence of positive solutions to superlinear problem (1.4) by the fixed point theorem in cones when $1 < \lambda \leq \mu < +\infty$, in the case of $\varphi_p(s) = s$. Using a modified upper and lower solution method, Wei [12] obtained necessary and sufficient conditions for the existence of positive solutions to fourth-order problem (1.4) for the sublinear case. The upper and lower solution method is built in [3,5,6] to treat the singular boundary value problems for p-Laplacian and the higher order nonlinear boundary value problems. However, there are few results in the upper and lower solution methods used to treat the higher-order p-Laplacian boundary value problems [13]. In this paper, we establish a comparison theorem for $2m$-th order p-Laplacian boundary value problems, and construct a lower solution and an upper solution for the $2m$-th order p-Laplacian BVPs (1.1). By the upper and lower solution method, we obtain the necessary and sufficient conditions for existence of F_p or F_p^1 positive solutions to $2m$-th order singular p-Laplacian BVPs (1.1). By F_p positive solution of (1.1), we mean that a function $u \in C^{2m-2}[0, 1]$ and $\varphi_p(u^{(2m-2)}) \in C^2(0, 1)$ satisfying (1.1). By F_p^1 positive solution of (1.1), we mean that a positive function $u \in C^{2m-2}[0, 1]$ and $\varphi_p(u^{(2m-2)}) \in C^1[0, 1] \cap C^2(0, 1)$ satisfying (1.1). Our results extend the results of Wei [12] to the case of higher order p-Laplacian BVPs. The necessary and sufficient conditions of existence of positive solutions to (1.4) in [10–12] involve integrability conditions in terms of the function f and the Green function for the case of $p = 2$. We found that this fact is also true for the general p-Laplace BVPs, and that it is interesting that the integrability is not relying on the parameter p. In addition, our methods used in this paper can be used to investigate the higher-order p-Laplacian BVPs whose nonlinear term relies on the higher order derivates of u.

2. Comparison theorem

In this section, we mainly give the comparison theorem for $2m$-th order p-Laplacian boundary value problems.

Denote

$$F_p = \{u \in C^{2m-2}[0, 1] \quad \text{and} \quad \varphi_p(u^{(2m-2)}) \in C^2(0, 1)\},$$

and

$$F_1 = \left\{ u \in F_p \quad \text{and} \quad \int_0^1 t(1-t)(\varphi_p(u^{(2m-2)}(t)))'' \mathrm{d}t < \infty \right\}.$$

Lemma 1 (*Comparison theorem*). *Suppose* $x, y \in F_1$ *such that*

$$(-1)^m[\varphi_p(y^{(2m-2)}(t))]'' - (-1)^m[\varphi_p(x^{(2m-2)}(t))]'' \geq 0, \quad 0 < t < 1.$$

If

$$(-1)^i y^{(2i)}(0) \geq (-1)^i x^{(2i)}(0),$$

$$(-1)^i y^{(2i)}(1) \geq (-1)^i x^{(2i)}(1), \quad i = 0, 1, \ldots, m-1,$$

then $y(t) \geq x(t)$, $0 \leq t \leq 1$.

Proof. Let

$$(-1)^m(\varphi_p(x^{(2m-2)}(t)))'' = p_1(t), \quad 0 < t < 1, \tag{2.1}$$

$$(-1)^i x^{(2i)}(0) = a_i^0, \quad (-1)^i x^{(2i)}(1) = a_i^1, \quad i = 1, 2, \ldots, m-1 \tag{2.2}$$

and

$$(-1)^m(\varphi_p(y^{(2m-2)}(t)))'' = p_2(t), \quad 0 < t < 1, \tag{2.3}$$

$$(-1)^i y^{(2i)}(0) = b_i^0, \quad (-1)^i y^{(2i)}(1) = b_i^1, \quad i = 1, 2, \ldots, m-1. \tag{2.4}$$

Then $p_1(t) \leq p_2(t)$, $0 < t < 1$ and

$$a_i^0 \leq b_i^0 \quad \text{and} \quad a_i^1 \leq b_i^1, \quad i = 0, 1, \ldots, m-1. \tag{2.5}$$

From (2.1) and (2.2), for $0 \leq t \leq 1$,

$$\begin{aligned}(-1)^{m-1}\varphi_p(x^{(2m-2)}(t)) &= (-1)^{m-1}(1-t)\varphi_p(x^{(2m-2)}(0)) \\ &\quad + (-1)^{m-1}t\varphi_p(x^{(2m-2)}(1)) + \int_0^1 G(t,s)p_1(s)\mathrm{d}s,\end{aligned} \tag{2.6}$$

where $G(t, s)$ is the Green function of

$$\begin{cases} -u'' = 0, & t \in (0, 1), \\ u(0) = u(1) = 0, \end{cases}$$

and

$$G(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

In view of φ_p being an odd function, (2.6) can be written as

$$\begin{aligned}&\varphi_p((-1)^{m-1}x^{(2m-2)}(t)) \\ &\quad = (1-t)\varphi_p(a_{m-1}^0) + t\varphi_p(a_{m-1}^1) + \int_0^1 G(t,s)p_1(s)\mathrm{d}s.\end{aligned} \tag{2.7}$$

For the same reason in (2.7), from (2.3) and (2.4), for $0 \le t \le 1$,

$$\varphi_p((-1)^{m-1}y^{(2m-2)}(t)) = (1-t)\varphi_p(b_{m-1}^0) + t\varphi_p(b_{m-1}^1) + \int_0^1 G(t,s)p_2(s)\mathrm{d}s. \tag{2.8}$$

By the monotone of the function φ_p and (2.5), we have

$$\varphi_p((-1)^{m-1}x^{(2m-2)}(t)) \le \varphi_p((-1)^{m-1}y^{(2m-2)}(t)), \quad 0 \le t \le 1. \tag{2.9}$$

Then

$$(-1)^{m-1}x^{(2m-2)}(t) \le (-1)^{m-1}y^{(2m-2)}(t), \quad 0 \le t \le 1. \tag{2.10}$$

In view of (2.5), by recursion, it can be deduced that $y(t) \ge x(t)$. □

3. Main results

First, we define the upper and lower solutions for fourth order singular p-Laplacian boundary value problems (1.1).

DEFINITION 1

Letting $\alpha \in F_p$, α is said to be a lower solution for $2m$-th order singular p-Laplacian BVPs (1.1) if

$$\begin{cases} (-1)^m(\varphi_p(\alpha^{(2m-2)}(t)))'' \le f(t,\alpha(x)), & x \in (0,1), \\ (-1)^i\alpha^{(2i)}(0) \le 0, \quad (-1)^i\alpha^{(2i)}(1) \le 0, & i = 0,1,\ldots,m-1. \end{cases} \tag{3.1}$$

DEFINITION 2

Letting $\beta \in F_p$, β is said to be an upper solution for $2m$-th order singular p-Laplacian BVPs (1.1) if

$$\begin{cases} (-1)^m(\varphi_p(\beta^{(2m-2)}(t)))'' \ge f(t,\beta(x)), & x \in (0,1), \\ (-1)^i\beta^{(2i)}(0) \ge 0, \quad (-1)^i\beta^{(2i)}(1) \ge 0, & i = 0,1,\ldots,m-1. \end{cases} \tag{3.2}$$

Our main results are the following theorems.

Theorem 1. *The $2m$-th order singular p-Laplacian BVPs* (1.1) *have a F_p positive solution $u \in F_p$, if and only if,*

$$\int_0^1 t(1-t)f(t,t(1-t))\mathrm{d}t < \infty. \tag{3.3}$$

Denote

$$F_p^1 = \{u \in C^{2m-2}[0,1], \quad \varphi_p(u^{(2m-2)}) \in C^1[0,1] \cap C^2(0,1)\}.$$

Theorem 2. *The* $2m$*-th order singular* p*-Laplacian BVPs* (1.1) *have a* F_p^1 *positive solution* $u \in F_p^1$, *if and only if,*

$$\int_0^1 f(t, t(1-t))\mathrm{d}t < \infty. \tag{3.4}$$

Proof of Theorem 1. We prove the necessity first. Let $u \in F_p$ be a F_p positive solution of (1.1). Then $(-1)^m \varphi_p(u^{(2m-2)}(t))'' > 0$, $0 < t < 1$. In view of φ_p being an odd function, $\varphi_p(u^{(2m-2)}(0)) = \varphi_p(u^{(2m-2)}(1)) = 0$. One can easily see that, $(-1)^{m-1}\varphi_p(u^{(2m-2)}(t)) \geq 0$ for $0 \leq t \leq 1$. Then by the property of function φ_p, we have $(-1)^{m-1}u^{(2m-2)}(t) \geq 0$ for $0 \leq t \leq 1$. In addition, considering $(-1)^i u^{(2i)}(0) = (-1)^i u^{(2i)}(1) = 0$, $i = 0, 1, \ldots, m-2$, we have $(-1)^i u^{(2i)}(t) \geq 0$, $t \in [0, 1]$, $i = 0, 1, \ldots, m-2$. This means $(-1)^i u^{(2i)}$ is concave on $[0, 1]$, $i = 0, 1, \ldots, m-2$. It follows from $u(0) = u(1) = 0$ that $u'(0) > 0$ and $u'(1) < 0$. Consequently, there must be a positive number k such that $u(t) \geq kt(1-t)$. Let $c \geq \max\{M, 1/(kN)\}$. Then, for $0 < t < 1$, $t(1-t)/(cu(t)) < N$, we get

$$\begin{aligned} f(t, t(1-t)) &\leq c^{\mu} f(t, t(1-t)u(t)/(cu(t))) \\ &\leq c^{\mu-\lambda}k^{-\lambda} f(t, u(t)) = (-1)^m c^{\mu-\lambda}k^{-\lambda}(\varphi_p(u^{(2m-2)}(t)))''. \end{aligned} \tag{3.5}$$

On the other hand, the boundary value conditions in (1.1) imply $\varphi_p(u^{(2m-2)}(0)) = \varphi_p(u^{(2m-2)}(1)) = 0$. Then

$$\begin{aligned} (-1)^{m-1}\varphi_p\left(u^{(2m-2)}\left(\frac{1}{2}\right)\right) &= \int_0^1 G\left(\frac{1}{2}, s\right)\left[(-1)^m(\varphi_p(u^{(2m-2)}(s)))''\right]\mathrm{d}s \\ &= \frac{1}{2}\int_0^t s[(-1)^m(\varphi_p(u^{(2m-2)}(s)))'']\mathrm{d}s \\ &\quad + \frac{1}{2}\int_t^1 (1-s)[(-1)^m(\varphi_p(u^{(2m-2)}(s)))'']\mathrm{d}s \\ &\geq \frac{1}{2}\int_0^1 s(1-s)[(-1)^m(\varphi_p(u^{(2m-2)}(s)))'']\mathrm{d}s, \end{aligned} \tag{3.6}$$

we obtain (3.3) by combining (3.5) and (3.6). The proof of the necessity is completed.

To prove the sufficiency, denote

$$H(t, s) := \int_0^1 \cdots \int_0^1 G(t, s_{m-2}) \cdots G(s_1, s)\mathrm{d}s_1 \cdots \mathrm{d}s_{m-2}.$$

Then $H(t, s)$ is the Green function of $(2m-2)$th order boundary value problems

$$\begin{cases} (-1)^{m-1}u^{(2m-2)}(t) = 0, & 0 < t < 1, \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, & i = 0, 1, \ldots, m-2. \end{cases}$$

Denote

$$H_1(t,s) = \begin{cases} \int_0^1 \cdots \int_0^1 G(t, s_{m-3}) \cdots G(s_1, s) \mathrm{d}s_1 \cdots \mathrm{d}s_{m-3}, & m > 2, \\ 1, & m = 2. \end{cases}$$

We know $\varphi_p^{-1}(u) = \varphi_q(u)$, where $q = \frac{p}{p-1}$. Denote

$$F(t) = \int_0^1 H_1(t,s)\varphi_p^{-1}\left(\int_0^1 G(\tau,s) f(s, s(1-s))\mathrm{d}s\right)\mathrm{d}\tau.$$

Let

$$h(t) = \int_0^1 H(t,\tau)\varphi_p^{-1}\left[\int_0^1 G(\tau,s) f(s, s(1-s))\mathrm{d}s\right]\mathrm{d}\tau. \tag{3.7}$$

Then by the condition (3.3) in Theorem 1, it is true that $h \in F_p$, and

$$a_1 t(1-t) \le h(t) \le a_2 t(1-t). \tag{3.8}$$

Here

$$a_1 = \int_0^1 s(1-s)F(s)\mathrm{d}s, \tag{3.9}$$

$$a_2 = \int_0^1 F(s)\mathrm{d}s. \tag{3.10}$$

By a modified method which was used in [12], let $\alpha(t) = k_1 h(t)$, $\beta(t) = k_2 h(t)$. Choose a $c = \max\left\{M, \left(\frac{a_2}{a_1}\right)^{\left(\frac{\mu}{\mu-\lambda}\right)}\right\}$. Then

$$c^{\mu-\lambda} \ge \left(\frac{a_2}{a_1}\right)^{\mu}. \tag{3.11}$$

Moreover, k_1 and k_2 can be chosen such that

$$\frac{k_1 a_2}{c} \le N \quad \text{and} \quad k_2 a_1 \ge M. \tag{3.12}$$

In view of $p - 1 > \mu$, by (3.11), we can choose k_1, k_2 such that $k_2 \ge k_1$ and

$$k_1 \le \min\left\{ \left(a_2^{\mu} c^{\lambda-\mu}\right)^{\frac{1}{p-1-\mu}}, \quad \left(\frac{a_1^{\mu}}{M^{\mu-\lambda}}\right)^{\frac{1}{p-1-\mu}}, \quad \left(a_1^{\lambda} N^{\mu-\lambda}\right)^{\frac{1}{p-1-\lambda}} \right\}, \tag{3.13}$$

$$k_2 \ge \max\left\{ \left(a_1^{\mu}\right)^{\frac{1}{p-1-\lambda}}, \quad \left(a_2^{\lambda} M^{\mu-\lambda}\right)^{\frac{1}{p-1-\lambda}}, \quad \left(\frac{a_2^{\mu}}{N^{\mu-\lambda}}\right)^{\frac{1}{p-1-\mu}} \right\}. \tag{3.14}$$

Then for $t \in (0, 1)$,

$$\frac{k_1 h(t)}{ct(1-t)} \le \frac{k_1 a_2}{c} \le N.$$

Thus, by (3.12) and (3.13),

$$\begin{aligned} f(t, \alpha(t)) &= f(t, k_1 h(t)) \ge c^{\lambda} f\left(t, \frac{k_1 h(t)}{c}\right) \ge c^{\lambda} \left(\frac{k_1 a_2}{c}\right)^{\mu} f(t, t(1-t)) \\ &= c^{\lambda-\mu} k_1^{\mu} a_2^{\mu} f(t, t(1-t)) \ge k_1^{p-1} f(t, t(1-t)). \end{aligned} \tag{3.15}$$

Therefore,

$$(-1)^m (\varphi_p(\alpha^{(2m-2)}(t)))'' = k_1^{p-1} f(t, t(1-t)) \le f(t, \alpha(t)), \tag{3.16}$$

and

$$(-1)^i \alpha^{(2i)}(0) = (-1)^i \alpha^{(2i)}(1) = 0, \quad i = 0, 1, \ldots, m-1.$$

With the same method, by (3.12) and (3.14), for $t \in (0, 1)$,

$$\frac{k_2 h(t)}{t(1-t)} \ge k_2 a_1 \ge M.$$

$$\begin{aligned} f(t, \beta(t)) &= f(t, k_2 h(t)) = f\left(t, \frac{k_2 h(t)}{t(1-t)} t(1-t)\right) \\ &\le (k_2 a_1)^{\mu} f(t, t(1-t)) \le k_2^{p-1} f(t, t(1-t)). \end{aligned} \tag{3.17}$$

Hence,

$$(-1)^m (\varphi_p(\beta^{(2m-2)}(t)))'' = k_2^{p-1} f(t, t(1-t)) \ge f(t, \beta(t)) \tag{3.18}$$

and

$$(-1)^i \beta^{(2i)}(0) = (-1)^i \beta^{(2i)}(1) = 0, \quad i = 0, 1, \ldots, m-1.$$

Then $\alpha(t)$ and $\beta(t)$ are the lower solution and the upper solution for the problems (1.1).

Denote

$$S_p = \{u \in F_p, \quad \alpha(t) \le u(t) \le \beta(t), \quad 0 \le t \le 1\}.$$

Now we arrive at a position to prove that there exists a F_p positive solution u in S_p.

Let

$$g(t) = \begin{cases} f(t, \beta(t)), & \beta(t) \le u(t), \\ f(t, u(t)), & \alpha(t) \le u(t) \le \beta(t), \\ f(t, \alpha(t)), & u(t) \le \alpha(t). \end{cases} \tag{3.19}$$

Consider the problems

$$\begin{cases} (\varphi_p(u^{(2m-2)}))'' = g(x,u), & x \in (0,1) \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{3.20}$$

For $\alpha(t) \le u(t) \le \beta(t)$, $0 \le t \le 1$, let $c_1 = \max\{M, k_2 a_2/N\}$. We have, for $0 < t < 1$,

$$\begin{aligned} 0 \le f(t,u(t)) &\le c_1^{\mu} f\left(t, \frac{u(t)}{c_1 t(1-t)} t(1-t)\right) \\ &\le c_1^{\mu} \left(\frac{u(t)}{c_1 t(1-t)}\right)^{\lambda} f(t, t(1-t)) \\ &\le c_1^{\mu} \left(\frac{k_2 a_2}{c_1}\right)^{\lambda} f(t, t(1-t)). \end{aligned} \tag{3.21}$$

Denote

$$B = \{u \in C^{2m-2}[0,1] | u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad 1 \le i \le m-1\}.$$

For every $u \in B$, we define the norm $\|u\| = \sum_{i=0}^{m-1} |u^{(2i)}|_0$, where $|\cdot|_0$ is the usual sup-norm for continuous functions on [0, 1]. It is clear that B equipped with norm $\|\cdot\|$ is a Banach space.

Next, we define the operator $T\colon B \to B$ by

$$Tu(t) = \int_0^1 H(t,\tau)\varphi_p^{-1}\left[\int_0^1 G(\tau,s) g(s,u(s)) \mathrm{d}s\right] \mathrm{d}\tau, \quad u \in B. \tag{3.22}$$

Equations (3.3), (3.21) and the monotone of the function φ_p assert that the operator T is well defined on B. It is easy to prove that, the existence of the positive solution for the integral equation $u = Tu$ implies the existence of F_p positive solution for $2m$-th order singular p-Laplacian boundary value problems (1.1).

Denote

$$Ku(t) := \varphi_p^{-1}\left[\int_0^1 G(t,s) g(s,u(s)) \mathrm{d}s\right], \quad 0 \le t \le 1, \quad u \in B. \tag{3.23}$$

From the condition (3.3), by the inequality (3.21) and the continuous property of function f, we know that the operator T is continuous. By the definition of g, it is easy to see that $T(B)$ is bounded. Let $\sup_{t\in[0,1]} |Ku(t)| \le C$, where C is a constant which does not rely on $u \in B$.

Next we prove that the operator T is a pre-compact operator on B. Because Green function $G(t,s)$ is continuous on $[0,1] \times [0,1]$, it is obvious that $G(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$. Then this property of $G(t,s)$ asserts that $H(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < t_1 < t_2 < 1$, and $|t_1 - t_2| < \delta$, $|H(t_1,s) - H(t_2,s)| < \varepsilon$. Let $u \in B$,

$$|Tu(t_1) - Tu(t_2)| = \left|\int_0^1 [H(t_1,s) - H(t_2,s)] Ku(s) \mathrm{d}s\right| < \varepsilon C. \tag{3.24}$$

This means that $T(B)$ is a relative compact set. Then by Schauder fixed point theorem, we know that there exists at least one fixed point $u \in B$ of T.

We need to show that the fixed point u satisfies $\alpha(t) \leq u(t) \leq \beta(t)$, $t \in [0, 1]$. In view of (3.14), we can choose $c_1 > 0$ such that

$$M \leq c_1 \leq \left(\frac{k_2^{p-1-\lambda}}{a_2^\lambda}\right)^{\frac{1}{\mu-\lambda}} \quad \text{and} \quad \frac{a_2k_2}{c_1} \leq N.$$

Then for $\alpha(t) \leq u(t) \leq \beta(t)$, $0 \leq t \leq 1$,

$$\begin{aligned} 0 \leq f(t, u(t)) &\leq c_1^\mu f\left(t, \frac{u(t)}{c_1 t(1-t)} t(1-t)\right) \\ &\leq c_1^\mu \left(\frac{a_2k_2}{c_1}\right)^\lambda f(t, t(1-t)) \leq k_2^{p-1} f(t, t(1-t)). \end{aligned} \tag{3.25}$$

From (3.13), we can choose a $c_2 > 0$ such that

$$\max\{1/N, M/a_1k_1\} \leq c_2 \leq \left(\frac{a_1^\lambda}{k_1^{p-1-\lambda}}\right)^{\frac{1}{\mu-\lambda}}.$$

Then for $u(t) \geq \alpha(t)$, $0 \leq t \leq 1$,

$$\begin{aligned} f(t, u(t)) &\geq \left(\frac{1}{c_2}\right)^\mu f\left(t, \frac{c_2 u(t)}{t(1-t)} t(1-t)\right) \\ &\geq \left(\frac{1}{c_2}\right)^\mu (c_2k_1a_1)^\lambda f(t, t(1-t)) \geq k_1^{p-1} f(t, t(1-t)). \end{aligned} \tag{3.26}$$

From (3.25) and (3.26),

$$k_1^{p-1} f(t, t(1-t)) \leq g(t, u(t)) \leq k_2^{p-1} f(t, t(1-t)). \tag{3.27}$$

It is true that, for $u \in B$,

$$\begin{aligned} &(-1)^m (\varphi_p(\beta^{(2m-2)}(t)))'' - (-1)^m (\varphi_p(u^{(2m-2)}(t)))'' \\ &\quad \geq k_2^{p-1} f(t, t(1-t)) - g(t, u(t)) \geq 0. \end{aligned}$$

Then by Lemma 1, we have $\beta(t) \geq u(t)$. By the same method, it is also true that $\alpha(t) \leq u(t)$.

Hence the solution $u(t)$ is the positive solution of the boundary value problems (1.1). □

Proof of Theorem 2. We prove the sufficiency first. Since (3.4) implies (3.3), Theorem 1 provides a positive solution $u \in F_p$ of $2m$-th order singular p-Laplacian BVPs (1.1), and

$$\begin{aligned} u(t) &= \int_0^1 G(t, s)(-u''(s))\mathrm{d}s \\ &\leq \left(\int_0^t s(1-t)\mathrm{d}s + \int_t^1 t(1-s)\mathrm{d}s\right) |u''|_0 = \frac{1}{2} t(1-t)|u''|_0. \end{aligned} \tag{3.28}$$

To prove that $(\varphi_p(u^{(2m-2)}(t)))' \in C[0,1]$, choose a positive number $c \geq \max\{M, |u''|_0/(2N)\}$. Then from (1.1), we have

$$\int_0^1 |(\varphi_p(u^{(2m-2)}(t)))''| \mathrm{d}s = \int_0^1 f(s, u(s)) \mathrm{d}s \leq c^\mu \int_0^1 f\left(s, \frac{u(s)}{c}\right) \mathrm{d}s$$

$$\leq c^\mu \left(\frac{|u''|_0}{2c}\right)^\lambda \int_0^1 f(s, s(1-s)) \mathrm{d}s.$$

Thus $(\varphi_p(u^{(2m-2)}(t)))''$ is an absolute integrable over $[0,1]$ from (3.4), and hence, $(\varphi_p(u^{(2m-2)}(t)))' \in C[0,1]$.

To prove the necessity, let there be a positive solution $u \in F_p^1$ of (1.1). The same reasoning at the beginning of the proof of Theorem 1 asserts that $u(t) \geq k_1 t(1-t)$ for all $t \in [0,1]$, for some constant $k_1 > 0$. Let $c \geq \max\{M, 1/(k_1 N)\}$. Then, from (1.2) and (1.3),

$$f(t, t(1-t)) \leq c^\mu f\left(t, \frac{t(1-t)u(t)}{cu(t)}\right) \leq c^{\mu-\lambda} k_1^{-\lambda} f(t, u(t)),$$

and hence,

$$\int_0^1 f(t, t(1-t)) \mathrm{d}t \leq c^{\mu-\lambda} k_1^{-\lambda} \int_0^1 f(t, u(t)) \mathrm{d}t$$

$$= c^{\mu-\lambda} k_1^{-\lambda} \int_0^1 [\varphi_p(u''(t))]'' \mathrm{d}t$$

$$= c^{\mu-\lambda} k_1^{-\lambda} ([\varphi_p(u''(1))]' - [\varphi_p(u''(0))]') < +\infty.$$

Thus, (3.4) holds and the proof of Theorem 2 is completed. □

Acknowledgement

This research was supported by Liuhui Center of Applied Mathematics at Nankai University and Tianjin University, under grant No. NNSFC10671141. The authors express their gratitude to the referee for careful reading of their manuscript and helpful comments.

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