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Central limit theorem and almost sure central limit theorem for the product of some partial sums

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Abstract. In this paper, we give the central limit theorem and almost sure central limit theorem for products of some partial sums of independent identically distributed random variables.

Keywords. Central limit theorem; almost sure central limit theorem; products of sums.

1. Introduction

Let $(X_n)_{n\geq 1}$ be a sequence of independent identically distributed (i.i.d.) positive random variables (r.v.). Recently there have been several studies to the products of partial sums. It is well-known that the products of i.i.d. positive, square integrable random variables are asymptotically log-normal. This fact is an immediate consequence of the classical central limit theorem (CLT). This point, up to the knowledge of the author, was first argued by Arnold and Villaseñr [1], who considered the limiting properties of the sums of records. In their paper Arnold and Villaseñr obtained the following version of the CLT for a sequence of i.i.d. exponential r.v.'s $(X_n)_{n\geq 1}$ with the mean equal to one:

$$\frac{\sum_{k=1}^{n}\log S_k - n\log n + n}{\sqrt{2n}} \xrightarrow{\mathcal{L}} \Phi, \text{ as } n\to\infty,$$

where $S_k = \sum_{j=1}^{k} X_j$, $1\leq k\leq n$, and Φ is a standard normal r.v.. Rempała and Wesołowski [8] have noted that this limit behavior of a product of partial sums has a universal character and holds for any sequence of square integrable, positive i.i.d. random variables. Namely, they have proved the following.

Theorem RW. *Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. positive square integrable random variables with $\mathbb{E}X_1=\mu$, $\mathrm{Var}X_1=\sigma^2>0$ and the coefficient of variation $\gamma=\sigma/\mu$. Then*

$$\left(\frac{\prod_{k=1}^{n} S_k}{n!\mu^n}\right)^{1/(\gamma\sqrt{n})} \xrightarrow{\mathcal{L}} \mathrm{e}^{\sqrt{2}\Phi}. \tag{1.1}$$

Recently, Gonchigdanzan and Rempała [4] discussed an almost sure limit theorem for the product of the partial sums of i.i.d. positive random variables as follows.

Theorem GR. *Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. positive square integrable random variables with $\mathbb{E}X_1 = \mu > 0$, $\mathrm{Var}X_1 = \sigma^2$. Denote $\gamma = \sigma/\mu$ the coefficient of variation. Then for any real x,*

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left(\left(\frac{\prod_{k=1}^{n} S_k}{n!\mu^n}\right)^{1/(\gamma\sqrt{n})} \leq x\right) = F(x), \quad a.s. \tag{1.2}$$

where $F(\cdot)$ is the distribution function of the r.v. $\mathrm{e}^{\sqrt{2}\Phi}$.

For further discussions of the CLT, the author refers to [6,7]. Zhang and Huang [10] obtained the invariance principle of the product of sums of random variables. It is perhaps worth to notice that by the strong law of large numbers and the property of the geometric mean it follows directly that

$$\left(\frac{\prod_{k=1}^{n} S_k}{n!}\right)^{1/n} \xrightarrow{a.s.} \mu \tag{1.3}$$

if only the existence of the first moment is assumed.

Throughout the present paper let $S_{n,k} = \sum_{i=1}^{n} X_i - X_k$ for all $n \geq 1$, $1 \leq k \leq n$ and we are interested in similar results as (1.1) and (1.2).

2. Central limit theorem

Theorem 2.1. *Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. positive square integrable random variables with $\mathbb{E}X_1 = \mu$, $\mathrm{Var}X_1 = \sigma^2 > 0$ and the coefficient of variation $\gamma = \sigma/\mu$. Then*

$$\left(\frac{\prod_{k=1}^{n} S_{n,k}}{(n-1)^n\mu^n}\right)^{1/(\gamma\sqrt{n})} \xrightarrow{\mathcal{L}} \mathrm{e}^{\Phi}, \tag{2.1}$$

where Φ is a standard normal r.v.

Proof. Let $Y_i = (X_i - \mu)/\sigma$, $i = 1, 2, \ldots$. Then

$$\frac{1}{\gamma\sqrt{n}} \sum_{k=1}^{n} \left(\frac{S_{n,k}}{(n-1)\mu} - 1\right) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \left(\frac{\sum_{i\neq k, i\leq n}(X_i - \mu)}{(n-1)\sigma}\right) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} Y_k. \tag{2.2}$$

Therefore from the classical central limit theorem and $\mathbb{E}Y_i = 0$, $\mathrm{Var}(Y_i) = 1$ for all $i = 1, 2, \ldots$, we know that

$$\frac{1}{\gamma\sqrt{n}} \sum_{k=1}^{n} \left(\frac{S_{n,k}}{(n-1)\mu} - 1\right) \xrightarrow{\mathcal{L}} \Phi. \tag{2.3}$$

Furthermore, let $C_{n,k} = S_{n,k}/((n-1)\mu)$, $k = 1, 2, \ldots$. By the strong law of large numbers it follows that for any $\delta > 0$, $\exists\, R$ such that

$$\mathbb{P}\left(\sup_{n\geq R, 1\leq k\leq n} |C_{n,k} - 1| \geq \delta\right) < \delta.$$

Taking $\delta < 1/2$, for any $x \in \mathbb{R}$, we have

$$\begin{aligned}
&\mathbb{P}\left(\frac{1}{\gamma\sqrt{n}}\sum_{k=1}^{n}\log(C_{n,k}) \le x\right) \\
&= \mathbb{P}\left(\frac{1}{\gamma\sqrt{n}}\sum_{k=1}^{n}\log(C_{n,k}) \le x, \sup_{n\ge R, 1\le k\le n}|C_{n,k}-1| \ge \delta\right) \\
&\quad + \mathbb{P}\left(\frac{1}{\gamma\sqrt{n}}\sum_{k=1}^{n}\log(C_{n,k}) \le x, \sup_{n\ge R, 1\le k\le n}|C_{n,k}-1| < \delta\right) \\
&:= A_n + B_n
\end{aligned}$$

and

$$A_n \le \delta. \tag{2.4}$$

Next we will control the term B_n. By the following logarithm:

$$\log(1+x) = x + \frac{x^2}{(1+\theta x)^2},$$

where $\theta \in (0,1)$ depends on $x \in (-1,1)$, we have

$$\begin{aligned}
B_n &= \mathbb{P}\left(\frac{1}{\gamma\sqrt{n}}\sum_{k=1}^{n}\log(C_{n,k}) \le x, \sup_{n\ge R, 1\le k\le n}|C_{n,k}-1| < \delta\right) \\
&= \mathbb{P}\left\{\frac{1}{\gamma\sqrt{n}}\sum_{k=1}^{n}(C_{n,k}-1) + \frac{1}{\gamma\sqrt{n}}\sum_{k=1}^{n}\frac{(C_{n,k}-1)^2}{(1+\theta_k(C_{n,k}-1))^2} \le x,\right. \\
&\qquad \left.\sup_{n\ge R, 1\le k\le n}|C_{n,k}-1| < \delta\right\} \\
&= \mathbb{P}\left\{\frac{1}{\gamma\sqrt{n}}\sum_{k=1}^{n}(C_{n,k}-1) + \left[\frac{1}{\gamma\sqrt{n}}\sum_{k=1}^{n}\frac{(C_{n,k}-1)^2}{(1+\theta_k(C_{n,k}-1))^2}\right]\right. \\
&\qquad \left.\times I\left(\sup_{n\ge R, 1\le k\le n}|C_{n,k}-1| < \delta\right) \le x\right\} \\
&\qquad - \mathbb{P}\left\{\frac{1}{\gamma\sqrt{n}}\sum_{k=1}^{n}(C_{n,k}-1) \le x, \sup_{n\ge R, 1\le k\le n}|C_{n,k}-1| \ge \delta\right\} \\
&:= D_n + F_n,
\end{aligned}$$

where $\theta_k, k = 1, \ldots, n$ are $(0,1)$-valued random variables and $F_n \le \delta$. To estimate the term D_n, by the following elementary inequality: for $|x| < 1/2$ and any $\theta \in (0,1)$ it follows that $x^2/(1+\theta x)^2 \le 4x^2$. Then we have

$$\left[\frac{1}{\gamma\sqrt{n}}\sum_{k=1}^{n}\frac{(C_{n,k}-1)^2}{(1+\theta_k(C_{n,k}-1))^2}\right] I\left(\sup_{n\ge R, 1\le k\le n}|C_{n,k}-1|<\delta\right)$$

$$\le \frac{4}{\gamma\sqrt{n}}\sum_{k=1}^{n}(C_{n,k}-1)^2 \xrightarrow{\mathbb{P}} 0, \tag{2.5}$$

as $n\to\infty$. Relation (2.5) is a consequence of the Markov inequality, since for any $r>0$,

$$\mathbb{P}\left(\frac{4}{\gamma\sqrt{n}}\sum_{k=1}^{n}(C_{n,k}-1)^2 \ge r\right) \le \frac{4}{r\gamma\sqrt{n}}\sum_{k=1}^{n}\mathrm{Var}(C_{n,k}-1)$$

$$= \frac{4}{r\gamma\sqrt{n}}\frac{n\gamma^2}{n-1} \to 0.$$

Therefore $D_n \longrightarrow \Phi(x)$. For any $x\in\mathbb{R}$, we have

$$\mathbb{P}\left(\log\left(\frac{\prod_{k=1}^{n}S_{n,k}}{(n-1)^n\mu^n}\right)^{1/(\gamma\sqrt{n})} \le x\right) = \mathbb{P}\left(\frac{1}{\gamma\sqrt{n}}\sum_{k=1}^{n}\log C_{n,k} \le x\right)$$

$$= A_n + D_n + F_n$$

which implies our result. □

3. Almost sure central limit theorem

In this section we will consider the almost sure central limit theorem as (1.2). Starting with Brosamler [3] and Schatte [9], in the past decade several authors investigated the a.s. central limit theorem and related 'logarithmic' limit theorems for partial sums of independent random variables. The simplest form of the a.s. central limit theorem [3], [9], [5] states that if $X_1, X_2, \ldots$ are i.i.d. random variables with mean 0, variance 1 and partial sums $S_n = \sum_{i=1}^{n} X_i$, then

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{k=1}^{N}\frac{1}{k}I\left(\frac{S_k}{\sqrt{k}}\le x\right) = \Phi(x) \ \text{ a.s. } \forall x, \tag{3.1}$$

where I denotes indicator function. Berkes and Csáki [2] extended this theory and showed that not only the central limit theorem, but every weak limit theorem for independent random variables, subject to minor technical conditions, has an analogous almost sure version. However under our model we only need the simplest version of (3.1).

Theorem 3.1. *Let $(X_n)_{n\ge 1}$ be a sequence of i.i.d. positive square integrable random variables with $\mathbb{E}X_1=\mu>0$, $\mathrm{Var}X_1=\sigma^2$. Denote $\gamma=\sigma/\mu$ the coefficient of variation. Then for any real x,*

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}I\left(\left(\frac{\prod_{k=1}^{n}S_{n,k}}{(n-1)^n\mu^n}\right)^{1/(\gamma\sqrt{n})}\le x\right) = F(x), \ \ a.s. \tag{3.2}$$

where $F(\cdot)$ is the distribution function of the r.v. e^{Φ}.

Proof. Let $Y_i = (X_i - \mu)/\sigma$, $i = 1, 2, \ldots$. Then $\mathbb{E}Y_i = 0$ and $\mathrm{Var}(Y_i) = 1$ for all $i = 1, 2, \ldots$ and from (2.2), for any real x, we have

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left(\frac{1}{\gamma\sqrt{n}} \sum_{k=1}^{n} \left(\frac{S_{n,k}}{(n-1)\mu} - 1\right) \le x\right)$$

$$= \lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} Y_k \le x\right) = \Phi(x) \quad \text{a.s.} \tag{3.3}$$

Note that in order to prove (3.2) it is sufficient to show that for any real x,

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left(\frac{1}{\gamma\sqrt{n}} \sum_{k=1}^{n} \log \frac{S_{n,k}}{(n-1)\mu} \le x\right) = \Phi(x), \quad \text{a.s.} \tag{3.4}$$

To this end let, as before, $C_{n,k} = S_{n,k}/((n-1)\mu)$ and note that by the law of the iterated logarithm we have for $n \to \infty$,

$$\max_{1\le k\le n} |C_{n,k} - 1| = O\left(\left(\frac{\log\log n}{n}\right)^{1/2}\right) \quad \text{a.s.}$$

Since for $|x| < 1$ we have $\log(1+x) = x + R(x)$ with $\lim_{x\to 0} R(x)/x^2 = 1/2$,

$$\left|\sum_{k=1}^{n} \log C_{n,k} - \sum_{k=1}^{n} (C_{n,k} - 1)\right| \ll \sum_{k=1}^{n} (C_{n,k} - 1)^2$$

$$\ll \sum_{k=1}^{n} \frac{\log\log n}{n} \ll \log\log n \log n \quad \text{a.s.},$$

where '$\ll$' denote the inequality '$\le$' up to some universal constant. Hence for almost every ω and any $\varepsilon > 0$ there exists $n_0 = n_0(\omega, \varepsilon, x)$ such that for $n \ge n_0$,

$$I\left(\frac{1}{\gamma\sqrt{n}} \sum_{k=1}^{n} \left(\frac{S_{n,k}}{(n-1)\mu} - 1\right) \le x - \varepsilon\right)$$

$$\le I\left(\frac{1}{\gamma\sqrt{n}} \sum_{k=1}^{n} \log \frac{S_{n,k}}{(n-1)\mu} \le x\right)$$

$$\le I\left(\frac{1}{\gamma\sqrt{n}} \sum_{k=1}^{n} \left(\frac{S_{n,k}}{(n-1)\mu} - 1\right) \le x + \varepsilon\right).$$

Thus (3.3) implies (3.4).

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References

[1] Arnold B C and Villaseñr J A, The asymptotic distribution of sums of records, *Extremes* **1(3)** (1998) 351–363

[2] Berkes I and Csáki E, A universal result in almost sure central limit theory, *Stoch. Proc. Appl.* **94** (2001) 105–134

[3] Brosamler G, An almost everywhere central limit theorem, *Math. Proc. Cambridge Philos. Soc.* **104** (1988) 561–574

[4] Gonchigdanzan K and Rempała G A, A note on the almost sure limit theorem for the product of partial sums, *Appl. Math. Lett.* **19** (2006) 191–196

[5] Lacey M and Philipp W A, note on the almost everywhere central limit theorem, *Statist. Probab. Lett.* **9** (1990) 201–205

[6] Lu X W and Qi Y C, A note on asymptotic distribution of products of sums, *Statist. Probab. Lett.* **68(4)** (2004) 407–413

[7] Qi Y C, Limit distributions for products of sums, *Statist. Probab. Lett.* **62(1)** (2003) 93–100

[8] Rempała G A and Wesołowski J, Asymptotics for products of sums and U-statistics, *Electron. Comm. Probab.* **7** (2002) 47–54

[9] Schatte P, On strong versions of the central limit theorem, *Math. Nachr.* **137** (1988) 249–256

[10] Zhang L X and Huang W, A note on the invariance principle of the product of sums of random variables, arXiv:math.PR/0610515 v1, 17 Oct. 2006