

## Approximation of quantum Lévy processes by quantum random walks

UWE FRANZ<sup>1,2</sup> and ADAM SKALSKI<sup>3,4</sup>

<sup>1</sup>Département de Mathématiques de Besançon, Université de Franche-Comté 16, route de Gray, 25 030 Besançon cedex, France

<sup>2</sup>Current address: Graduate School of Information Sciences, Tohoku University, Sendai 980-8579, Japan

<sup>3</sup>Department of Mathematics, University of Łódź, ul. Banacha 22, 90-238 Łódź, Poland

<sup>4</sup>Current address: Department of Mathematics and Statistics, Lancaster University, Lancaster, LA1 4YF, UK

E-mail: uwe.franz@univ-fcomte.fr; a.skalski@lancaster.ac.uk

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**Abstract.** Every quantum Lévy process with a bounded stochastic generator is shown to arise as a strong limit of a family of suitably scaled quantum random walks.

**Keywords.** Quantum random walk; quantum Lévy process; discrete approximation.

### 0. Introduction

The note is concerned with investigating convergence of random walks on quantum groups to quantum Lévy processes. The theory of the latter is a natural noncommutative counterpart of the theory of classical Lévy processes on groups [Hey]. It has been initiated in [ASW] and further extensively developed by Schürmann [Sch], Franz and Schott [FSc] and Franz [Fra]. In the series of recent papers Lindsay and Skalski [LS<sub>1</sub>, LS<sub>2</sub>, Ska] introduced and investigated the corresponding notion in the topological context of compact quantum groups (or, more generally, operator space coalgebras). Recent years brought about rapid development of the theory of random walks (discrete time stochastic processes) on discrete quantum groups [Izu], [NeT], [Co] initiated by Biane [Bi<sub>1</sub>–Bi<sub>3</sub>].

In the context of quantum stochastic cocycles ([Lin] and references therein) the approximation of continuous time evolutions by random walks was first investigated by Lindsay and Parthasarathy [LiP]. They proved that under suitable assumptions scaled random walks converge weakly to \*-homomorphic quantum stochastic cocycles. Recently certain results on the strong convergence have been obtained in papers [Sin] and [Sah] (see also [Bel] for a thorough analysis from the viewpoint of the vacuum adapted cocycles). Here we apply the ideas of the latter papers to the approximation of quantum Lévy processes (continuous time processes) on a compact quantum semigroup  $A$  by quantum random walks (discrete time processes) on  $A$ .

### 1. Quantum random walks on $C^*$ -bialgebras

We start with the discussion of a notion of random walks on compact quantum semigroups. The class contains finite quantum groups, so we are in a natural way generalising the notion

of quantum random walks considered in [FGo]. Here, and in everything that follows,  $\otimes$  denotes the spatial tensor product of operator spaces (so in particular, also  $C^*$ -algebras).

DEFINITION 1

A unital  $C^*$ -algebra  $A$  is a  $C^*$ -bialgebra if it is equipped with two unital  $*$ -homomorphisms  $\Delta: A \rightarrow A \otimes A$  and  $\epsilon: A \rightarrow \mathbb{C}$  satisfying the coassociativity and counit conditions:

$$\begin{aligned} (\Delta \otimes \text{id}_A)\Delta &= (\text{id}_A \otimes \Delta)\Delta, \\ (\epsilon \otimes \text{id}_A)\Delta &= (\text{id}_A \otimes \epsilon)\Delta = \text{id}_A. \end{aligned}$$

Fix for the rest of the note a  $C^*$ -bialgebra  $A$ .

DEFINITION 2

Let  $B$  be a unital  $C^*$ -algebra. A family of unital  $*$ -homomorphisms  $J_n: A \rightarrow B^{\otimes n}$  ( $n \in \mathbb{N}_0$ ) is called a quantum random walk on  $A$  with values in  $B$  if

$$J_0 = \epsilon, \quad J_n = (J_{n-1} \otimes J_1)\Delta, \quad n \in \mathbb{N}.$$

If  $(J_n)_{n \in \mathbb{N}}$  is a quantum random walk in the above sense, the family  $(\tilde{J}_n)_{n \in \mathbb{N}}$  given by  $\tilde{J}_n = (\text{id}_A \otimes J_n)\Delta: A \rightarrow A \otimes B^{\otimes n}$  is a quantum random walk in the sense of [LiP]. For any state  $\phi$  on  $B$  the family  $(\kappa_n)_{n \in \mathbb{N}_0}$  of states on  $A$  defined by

$$\kappa_n = \phi^{\otimes n} \circ J_n, \quad n \in \mathbb{N}_0,$$

is a (discrete) convolution semigroup of states on  $A$ .

2. Main result

We first need to establish some notations. Fix a Hilbert space  $k$  and denote by  $\hat{k}$  the Hilbert space  $\mathbb{C} \oplus k$  (sometimes written as  $\mathbb{C}\Omega \oplus k$ ). We use the Dirac notation, so that for example  $|k\rangle$  denotes the space of all linear maps from  $\mathbb{C}$  to  $k$ . The symmetric Fock space over  $L^2(\mathbb{R}_+; k)$  is denoted by  $\mathcal{F}$  and its exponential vectors by  $\epsilon(f)$ , whenever  $f \in L^2(\mathbb{R}_+; k)$ . The usual shift semigroup of endomorphisms on  $B(\mathcal{F})$  will be written  $\{\sigma_s: s \geq 0\}$  and by  $\mathbb{P}(\mathcal{F})$  is meant the space of bounded adapted operator-valued processes on  $\mathcal{F}$ . By a Fock space quantum Lévy process  $l \in \mathbb{P}(A; \mathcal{F})$  is understood a  $*$ -homomorphic map  $l: A \rightarrow \mathbb{P}(\mathcal{F})$ , such that

$$l_0(a) = \epsilon(a)l_{\mathcal{F}}, \quad l_{s+t}(a) = (l_s \otimes (\sigma_s \circ l_t))\Delta(a)$$

( $a \in A, s, t \geq 0$ ). It is said to be Markov regular if its Markov convolution semigroup of states  $\{P_t = \langle \epsilon(0), l_t(\cdot)\epsilon(0) \rangle: t \geq 0\}$  is norm continuous. For more information on quantum Lévy processes on  $C^*$ -bialgebras we refer to [LS<sub>2</sub>].

The proof of the main theorem is based on the following lemma.

Lemma 3. Assume that  $v: A \rightarrow B(k)$  is a unital representation, and  $\tilde{\xi} \in k$  is nonzero. Let  $\gamma: A \rightarrow \mathbb{C}$  and  $\delta: A \rightarrow |k\rangle$  be given by

$$\begin{aligned} \gamma(a) &= \langle \tilde{\xi}, (v(a) - \epsilon(a))\tilde{\xi} \rangle, \\ \delta(a) &= |(v(a) - \epsilon(a))\tilde{\xi} \rangle, \end{aligned}$$

( $a \in \mathbf{A}$ ). Define the map  $\varphi: \mathbf{A} \rightarrow B(\hat{\mathbf{k}})$  by

$$\varphi = \begin{bmatrix} \gamma & \delta^\dagger \\ \delta & v(\cdot) - \epsilon(\cdot)I_{\mathbf{k}} \end{bmatrix}. \quad (1)$$

Put  $\lambda = \|\tilde{\xi}\|^2$ . For each  $h \in (0, \lambda^{-1}]$  there exists a unital \*-homomorphism  $\beta^{(h)}: \mathbf{A} \rightarrow B(\hat{\mathbf{k}})$  such that

$$\beta^{(h)} = \begin{bmatrix} \beta_1^{(h)} & \beta_2^{(h)} \\ \beta_3^{(h)} & \beta_4^{(h)} \end{bmatrix}, \quad (2)$$

$\beta_1^{(h)}: \mathbf{A} \rightarrow \mathbb{C}$ ,  $\beta_3^{(h)} = (\beta_2^{(h)})^\dagger: \mathbf{A} \rightarrow \mathbf{k}$ ,  $\beta_4^{(h)}: \mathbf{A} \rightarrow B(\mathbf{k})$  and for some constant  $M > 0$ ,

$$\|\beta_1^{(h)} - (\epsilon + h\gamma)\|_{cb} \leq Mh^2,$$

$$\|\beta_3^{(h)} - \sqrt{h}\delta\|_{cb} \leq Mh^{\frac{3}{2}},$$

$$\|\beta_4^{(h)} - v\|_{cb} \leq Mh.$$

*Proof.* Let  $\xi = \tilde{\xi}\|\tilde{\xi}\|^{-1}$  and denote by  $v_\xi$  the functional given by

$$v_\xi(a) = \langle \xi, v(a)\xi \rangle, \quad a \in \mathbf{A}.$$

To construct the required \*-homomorphism let for  $a \in \mathbf{A}$

$$\beta_1^{(h)}(a) = (\epsilon + h\gamma)(a) = (1 - \lambda h)\epsilon(a) + \lambda h v_\xi(a),$$

$$\beta_3^{(h)}(a) = |\sqrt{\lambda h}(v(a)\xi - \sqrt{1 - \lambda h}\epsilon(a)\xi) + (\sqrt{1 - \lambda h} - 1)v_\xi(a)\xi|,$$

$$\begin{aligned} \beta_4^{(h)}(a) &= (\lambda h\epsilon(a) + (2 - 2\sqrt{1 - \lambda h} - \lambda h)v_\xi(a))|\xi\rangle\langle\xi| \\ &\quad + (\sqrt{1 - \lambda h} - 1)|v(a)\xi\rangle\langle\xi| + (\sqrt{1 - \lambda h} - 1)|\xi\rangle\langle v(a^*)\xi| + v(a), \end{aligned}$$

where Dirac notation has been used again. Further let  $\beta_2^{(h)} = (\beta_3^{(h)})^\dagger$  and define  $\beta^{(h)}$  as the matrix (2).

It may be checked that  $\beta^{(h)}$  satisfies all requirements of the lemma.  $\square$

Some remarks are in place. In fact  $\beta^{(h)}$  in the proof above has been constructed via the GNS construction for the state  $(1 - \lambda h)\epsilon + \lambda h v_\xi$ . The GNS triple may be realised by  $(\epsilon \oplus v, \mathbb{C}\Omega \oplus \mathbf{k}, \Omega_h)$ , where  $\Omega_h = \sqrt{1 - \lambda h}\Omega \oplus \sqrt{\lambda h}\xi$ . Defining  $\tilde{\beta} = \epsilon \oplus v$ ,

$$\tilde{\beta}_1^{(h)}(a) = P_{\mathbb{C}\Omega_h}\tilde{\beta}^{(h)}(a)P_{\mathbb{C}\Omega_h},$$

$$\tilde{\beta}_2^{(h)}(a) = P_{\mathbb{C}\Omega_h}\tilde{\beta}^{(h)}(a)P_{(\mathbb{C}\Omega_h)^\perp},$$

$$\tilde{\beta}_3^{(h)}(a) = P_{(\mathbb{C}\Omega_h)^\perp}\tilde{\beta}^{(h)}(a)P_{\mathbb{C}\Omega_h},$$

$$\tilde{\beta}_4^{(h)}(a) = P_{(\mathbb{C}\Omega_h)^\perp}\tilde{\beta}^{(h)}(a)P_{(\mathbb{C}\Omega_h)^\perp},$$

( $a \in \mathbf{A}$ ), it remains to ‘rotate’ the GNS space to  $\hat{\mathbf{k}}$  so that the decomposition  $\mathbb{C}\Omega_h \oplus (\mathbb{C}\Omega_h)^\perp$  corresponds to  $\mathbb{C}\Omega \oplus \mathbf{k}$ . This is achieved by applying the unitary  $U_h: \mathbb{C}\Omega_h \oplus \mathbf{k} \rightarrow \mathbb{C}\Omega \oplus \mathbf{k}$  given by

$$U_h(\alpha\Omega_h \oplus \alpha'\Sigma_h \oplus \eta) = \alpha\Omega \oplus \alpha'\xi \oplus \eta,$$

where  $\Sigma_h = -\sqrt{\lambda h}\Omega_h \oplus \sqrt{1-\lambda h}\xi$  and  $\eta \in (\mathbb{C}\Omega_h)^\perp \cap (\mathbb{C}\Sigma_h)^\perp$ . It remains to check that the maps given by ( $a \in \mathbf{A}$ ),

$$\beta_1^{(h)}(a) := U_h \bar{\beta}_1^{(h)}(a) U_h^*,$$

$$\beta_2^{(h)}(a) := U_h \bar{\beta}_2^{(h)}(a) U_h^*,$$

$$\beta_3^{(h)}(a) := U_h \bar{\beta}_3^{(h)}(a) U_h^*,$$

$$\beta_4^{(h)}(a) := U_h \bar{\beta}_4^{(h)}(a) U_h^*,$$

reduce indeed to the ones given by formulas in the proof above. This can be done via straightforward (though very tedious) calculations. Note the fact that  $\beta^{(h)}$  is a unital  $*$ -homomorphism follows immediately from the analogous property of  $\epsilon \oplus \nu$ . We suggest the reader that it is worth to analyse carefully what happens to each part of the above construction as  $h$  tends to 0. Note also that the construction of  $\beta^{(h)}$  with all the properties formulated in the lemma becomes trivial if  $\tilde{\xi} = 0$ .

We are now ready to formulate and prove the main theorem of the paper.

**Theorem 4.** *Let  $l \in \mathbb{P}(\mathbf{A}; \mathcal{F})$  be a Markov regular Fock space quantum Lévy process on  $\mathbf{A}$ . There exists a family of quantum random walks  $(J_n^{(h)})_{n \in \mathbb{N}_0}$  on  $\mathbf{A}$  with values in  $B(\hat{\mathbf{k}})$  and a family of injective embeddings  $\iota_n^{(h)}: B(\hat{\mathbf{k}})^{\otimes n} \hookrightarrow B(\mathcal{F})$ , given by discretization of the Fock space, (indexed by a parameter  $h \in (0, \mu]$  for some  $\mu > 0$ ) such that for each  $a \in \mathbf{A}, t \geq 0, \zeta \in \mathcal{F}$ ,*

$$(\iota_{\lfloor t/h \rfloor}^{(h)} \circ J_{\lfloor t/h \rfloor}^{(h)}(a))(\zeta) \xrightarrow{h \rightarrow 0^+} l_t(a)\zeta.$$

*Proof.* Theorem 6.2 of [LS<sub>2</sub>] implies that the cocycle  $l$  is stochastically generated by a map  $\varphi: \mathbf{A} \rightarrow B(\hat{\mathbf{k}})$  given by the formula (1) for some vector  $\tilde{\xi} \in \mathbf{k}$  and representation  $\nu: \mathbf{A} \rightarrow B(\mathbf{k})$ . We may assume that the vector  $\tilde{\xi}$  is nonzero; otherwise the approximation method described below still works, and there is no need to restrict the range of  $h > 0$  in any way (see the Remark before the theorem).

Let  $\lambda = \|\tilde{\xi}\|^2$ . Let, for each  $h \in (0, \lambda^{-1}]$ ,  $\beta^{(h)}: \mathbf{A} \rightarrow B(\hat{\mathbf{k}})$  be a  $*$ -homomorphism satisfying all the properties described in Lemma 3. Define the approximating random walk by the formulas

$$J_0^{(h)} = \epsilon, \quad J_1^{(h)} = \beta^{(h)},$$

$$J_{n+1}^{(h)}(a) = (J_n^{(h)} \otimes J_1^{(h)})\Delta, \quad n \in \mathbb{N}.$$

The embeddings  $l_n^{(h)}$  are given by the standard discretization procedure for the Fock space [Att], [Sah]. Precisely speaking, take any  $T^{(1)}, \dots, T^{(n)} \in B(\hat{k})$ ,

$$T^{(i)} = \begin{bmatrix} T_1^{(i)} & T_2^{(i)} \\ T_3^{(i)} & T_4^{(i)} \end{bmatrix},$$

and write  $T = T^{(1)} \otimes \dots \otimes T^{(n)}$ . Then

$$l_n^{(h)}(T)\varepsilon(f) = \bigotimes_{i=1}^k N_i^{(h)}(T^{(i)})\varepsilon(f_{[(i-1)h, ih]}) \otimes \varepsilon(f_{[kh]}),$$

where

$$N_i^{(h)} = \sum_{l=1}^4 N_{T_l^{(i)}}^l[(i-1)h, ih],$$

and the operators  $N_{T_l^{(i)}}^l$  are discretized versions of time ( $N^1$ ), annihilation ( $N^2$ ), creation ( $N^3$ ) and preservation ( $N^4$ ) integral, defined as in [Sah].

The idea of the proof is to pull the situation back to the realm of standard Markov stochastic cocycles and apply a slightly improved version of the main theorem of [Sah]. To this end assume that  $\mathbf{A}$  is faithfully and nondegenerately represented on a Hilbert space  $\mathfrak{h}$ . Define

$$\tilde{l}_t = (\text{id}_{\mathbf{A}} \otimes l_t)\Delta: \mathbf{A} \rightarrow \mathbf{A} \otimes B(\mathcal{F}),$$

$$\tilde{J}_n^{(h)} = (\text{id}_{\mathbf{A}} \otimes (l_n^{(h)} \circ J_n^{(h)}))\Delta: \mathbf{A} \rightarrow \mathbf{A} \otimes B(\mathcal{F}).$$

Lemma 4.1 and Proposition 3.3 of [LS<sub>2</sub>] imply that  $\tilde{l}$  is stochastically generated by an operator  $\phi = (\text{id}_{\mathbf{A}} \otimes \varphi)\Delta: \mathbf{A} \rightarrow \mathbf{A} \otimes B(\hat{k})$ . It may also be shown that  $\tilde{J}_n^{(h)}$  coincides with the map  $p_{nh}^{(h)}: \mathbf{A} \rightarrow \mathbf{A} \otimes B(\mathcal{F})$  constructed as in [Sah] via the \*-homomorphisms  $\tilde{\beta}^{(h)} = (\text{id}_{\mathbf{A}} \otimes \beta^{(h)})\Delta: \mathbf{A} \rightarrow \mathbf{A} \otimes B(\hat{k})$ . It is easy to note that the conditions of Lemma 3 imply that if

$$\Phi = \begin{bmatrix} \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_4 \end{bmatrix}, \quad \tilde{\beta}^{(h)} = \begin{bmatrix} \tilde{\beta}_1^{(h)} & \tilde{\beta}_2^{(h)} \\ \tilde{\beta}_3^{(h)} & \tilde{\beta}_4^{(h)} \end{bmatrix},$$

then

$$\|\tilde{\beta}_1^{(h)} - \text{id}_{\mathbf{A}} - h\Phi_1\|_{\text{cb}} \leq Mh^2,$$

$$\|\tilde{\beta}_2^{(h)} - \sqrt{h}\Phi_2\|_{\text{cb}} \leq Mh^{\frac{3}{2}},$$

$$\|\tilde{\beta}_3^{(h)} - \sqrt{h}\Phi_3\|_{\text{cb}} \leq Mh^{\frac{3}{2}},$$

$$\|\tilde{\beta}_4^{(h)} - \text{id}_{\mathbf{A} \otimes B(\hat{k})} - \Phi_4\|_{\text{cb}} \leq Mh.$$

Now one may check that this is sufficient for all the assumptions of the principal theorem of [Sah] to be satisfied, and we deduce the following statement: for each  $a \in \mathbf{A}$ ,  $v \in \mathfrak{h}$  and  $\zeta \in \mathcal{F}$ ,

$$\tilde{J}_{[\frac{t}{h}]}^{(h)}(a)(v \otimes \zeta) \xrightarrow{h \rightarrow 0^+} \tilde{l}_t(a)(v \otimes \zeta).$$

The careful analysis of the estimates used in the proof of the theorem mentioned above shows that in fact one can obtain a stronger result, which is of use for what follows. Define for each  $\zeta \in \mathcal{F}$ ,  $t \geq 0$ ,  $n \in \mathbb{N}$  the maps  $\tilde{l}_{t,\zeta}: \mathbf{A} \rightarrow B(\mathfrak{h}; \mathfrak{h} \otimes \mathcal{F})$  and  $\tilde{J}_{n,\zeta}^{(h)}: \mathbf{A} \rightarrow B(\mathfrak{h}; \mathfrak{h} \otimes \mathcal{F})$  by the formulas

$$\begin{aligned} (\tilde{l}_{t,\zeta}(a))(v) &= \tilde{l}_t(a)(v \otimes \zeta), \\ (\tilde{J}_{n,\zeta}^{(h)}(a))(v) &= (\tilde{J}_n^{(h)}(a))(v \otimes \zeta), \end{aligned}$$

( $a \in \mathbf{A}$ ,  $v \in \mathfrak{h}$ ). It is easy to see that in our context both  $\tilde{l}_{t,\zeta}$  and  $\tilde{J}_{n,\zeta}^{(h)}$  take indeed values in the operator space  $\mathbf{A} \otimes |\mathcal{F}$ ; in general, von Neumann algebraic framework of [Sah] would take values in the von Neumann module  $\mathbf{A}'' \overline{\otimes} |\mathcal{F}$ . As all the estimates in [Sah] are independent of  $v \in \mathfrak{h}$ , it may be deduced in fact that for each  $\zeta \in \mathcal{F}$ ,  $t \geq 0$ ,  $a \in \mathbf{A}$ ,

$$\tilde{J}_{[\frac{t}{h},\zeta]}^{(h)}(a) \xrightarrow{h \rightarrow 0^+} \tilde{l}_{t,\zeta}(a). \tag{3}$$

Simple argument [LS<sub>2</sub>], [Ska] also shows that for each  $a \in \mathbf{A}$ ,  $\zeta \in \mathcal{F}$ ,  $t \geq 0$ ,  $n \in \mathbb{N}$ ,

$$\begin{aligned} l_t(a)\zeta &= (\epsilon \otimes \text{id}_{|\mathcal{F}}) \circ \tilde{l}_{t,\zeta}(a), \\ (l_n^{(h)} \circ J_n^{(h)}(a))\zeta &= (\epsilon \otimes \text{id}_{|\mathcal{F}}) \circ \tilde{J}_{n,\zeta}^{(h)}(a). \end{aligned}$$

In conjunction with (3) we obtain ( $a \in \mathbf{A}$ ,  $\zeta \in \mathcal{F}$ ),

$$\begin{aligned} \|l_t(a)\zeta - (l_{[\frac{t}{h}] }^{(h)} \circ J_{[\frac{t}{h}]}^{(h)}(a))\zeta\| &= \|(\epsilon \otimes \text{id}_{|\mathcal{F}})(\tilde{l}_{t,\zeta}(a) - \tilde{J}_{[\frac{t}{h},\zeta]}^{(h)}(a))\| \\ &\leq \|\tilde{l}_{t,\zeta}(a) - \tilde{J}_{[\frac{t}{h},\zeta]}^{(h)}(a)\| \xrightarrow{h \rightarrow 0^+} 0. \end{aligned}$$

This ends the proof. □

The main theorem above could be obtained without appealing at all to the theory of standard quantum stochastic cocycles, essentially by rewriting the proof of Sahu [Sah] replacing everywhere the composition by the convolution operation. This is possible only in the context of completely bounded operators; consequently, the original proof of [Sah] would have to be formulated solely in the language of the ‘column’ operators (an element of a reasoning of that type may be seen in the proof above).

Markov-regular Fock space quantum Lévy processes may be thought of as compound Poisson processes [Fra], [LS<sub>2</sub>]. It is therefore easy to describe conceptually how our approximations are built: the quantum random walk constructed above, after embedding in the algebra of Fock space operators, corresponds to taking random jumps, governed by the generating measure of the original compound Poisson process scaled by  $h$ , at discrete times  $h, 2h, \dots$ . It is then clear that the limit as  $h \rightarrow 0^+$  yields the original process.

The case of Lévy processes with unbounded generators is classically resolved via treating separately the part of the process responsible for ‘big’ jumps and the continuous/‘small’ jumps part (for the extensive bibliography of the subject and applications for numerical simulations of stochastic processes, we refer to [KIP]); it is not clear how to apply this procedure in the noncommutative framework.

*Note added in proof.* The main result of this paper has now been extended by J M Lindsay and the second named author to the context of quantum Lévy processes on multiplier  $C^*$ -bialgebras, i.e. locally compact quantum semigroups [LS<sub>3</sub>].

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