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Regularities of multifractal measures

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Abstract. First, we prove the decomposition theorem for the regularities of multifractal Hausdorff measure and packing measure in $\mathbb{R}^d$. This decomposition theorem enables us to split a set into regular and irregular parts, so that we can analyze each separately, and recombine them without affecting density properties. Next, we give some properties related to multifractal Hausdorff and packing densities. Finally, we extend the density theorem in [6] to any measurable set.

Keywords. Multifractal measures; decomposition theorem; Hausdorff measure; packing measure; regularity.

1. Introduction and preliminaries

Suppose we want to investigate a very general set, say of finite k-dimensional measure, or a finite measure. Such a set or measure may be given as the result of a random procedure, as an abstract solution of some optimization problems or as a result of an image reconstruction algorithm. Only very little global information may be known about the set. The question of geometric measure theory is to study its regularities or irregularities. The geometric measure theorists had found several deep results that reveal a dichotomy between two classes of sets(and measures). These are on one side rectifiable or regular sets and measures, on the other side non-rectifiable or fractal sets and measures. Actually, regular and irregular sets have geometrical characterizations in terms of rectifiability. So a major concern in geometric measure theory, is finding criteria which guarantee rectifiability. Such rectifiability of sets can be expressed through their densities with respect to a given measure.

In general, regular sets are defined by density with respect to the Hausdorff measure [2–5]. Specially, in [8–10], they defined such regular sets by using the packing measure instead of the Hausdorff measure. The authors in [8, 9] proved that a subset E of $\mathbb{R}^d$ has an integer Hausdorff and packing dimension if it is strongly regular. Moreover, in [4], the authors improved the results of [8] to a generalized Hausdorff measure H^{ϕ} in a Polish space. Many researchers [1, 5–9] had formulated density theorems with respect to Hausdorff measure or packing measure in some spaces.

On the other hand, the author in [6] adopted new measures – the so-called multifractal Hausdorff measure and packing measure to analyze the multifractal structure of a given measure on the fractal set. The density theorem was also proven with respect to multifractal Hausdorff measure and packing measure.

First, we should prove the decomposition theorem for the regularities of multifractal Hausdorff measure and packing measure in $\mathbb{R}^d$. This decomposition theorem enables us to split a set into regular and irregular parts, so that we can analyze each separately, and recombine them without affecting density properties. Next, we give some properties related to multifractal Hausdorff and packing densities. Finally, we extend the density theorems in [6] to any measurable set.

Now we denote $\mathcal{P}(\mathbb{R}^d)$ the set of Borel probability measures on $\mathbb{R}^d$ and, for $E \subset \mathbb{R}^d$, put

$$\mathcal{P}_0(\mathbb{R}^d, E) = \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) | \exists a > 1, \overline{\lim}_{r\to 0} \frac{\mu(B(x, ar))}{\mu(B(x, r))} < \infty \ \forall x \in E \right\},$$

where $E \subset \operatorname{supp} \mu$, the support of a Borel probability measure μ.

Next we recall the multifractal generalizations of the centered Hausdorff measure and of the packing measure.

For $q \in \mathbb{R}$, define $\psi_q \colon [0, \infty) \to [0, \infty]$ by

$$\psi_q(x) = \begin{cases} \infty, & \text{for} \quad x = 0 \\ x^q, & \text{for} \quad 0 < x \end{cases} \quad \text{for} \quad q < 0,$$

$$\psi_q(x) = 1, \quad \text{for} \quad q = 0,$$

$$\psi_q(x) = \begin{cases} 0, & \text{for} \quad x = 0 \\ x^q, & \text{for} \quad 0 < x \end{cases} \quad \text{for} \quad q > 0.$$

For $\mu \in \mathcal{P}(\mathbb{R}^d)$, $E \subset \mathbb{R}^d$, $q, t \in \mathbb{R}$ and $\delta > 0$, we denote

$$\bar{H}^{q,t}_{\mu,\delta}(E) = \inf \left\{ \sum_i \psi_q(\mu(B(x_i, r_i)))(2r_i)^t | \{B(x_i, r_i\}_i \text{ is a centered, } \delta\text{-covering of } E \right\}, \quad E \neq \phi,$$

$$\bar{H}^{q,t}_{\mu,\delta}(\phi) = 0,$$

$$\bar{H}^{q,t}_{\mu}(E) = \sup_{\delta > 0} \bar{H}^{q,t}_{\mu,\delta}(E)$$

and

$$H^{q,t}_{\mu}(E) = \sup_{F \subset E} \bar{H}^{q,t}_{\mu}(F).$$

We also make the dual definitions:

$$\bar{P}^{q,t}_{\mu,\delta}(E) = \sup \left\{ \sum_i \psi_q(\mu(B(x_i, r_i)))(2r_i)^t \mid \{B(x_i, r_i)\}_i \text{ is a centered, } \delta\text{-packing of } E \right\}, \quad E \neq \phi,$$

$$\bar{p}^{q,t}_{\mu,\delta}(\phi) = 0,$$

$$\bar{p}^{q,t}_{\mu}(E) = \inf_{\delta>0} \bar{p}^{q,t}_{\mu,\delta}(E)$$

and

$$p^{q,t}_{\mu}(E) = \inf_{E\subset\cup_i E_i} \sum_i \bar{p}^{q,t}_{\mu}(E_i).$$

The measure $H^{q,t}_{\mu}$ is a multifractal generalisation of the centered Hausdorff measure, whereas $p^{q,t}_{\mu}$ is a multifractal generalisation of the packing measure. There are several basic properties for the relations of multifracal measures in [6] including the fact that $H^{q,t}_{\mu}$, $p^{q,t}_{\mu}$ are Borel metric outer measures. He used those measures to analyze multifractal structures of regular and irregular sets. So, such measures are in the limelight of mutifractal analysts.

Now we give definitions of densities for multifractal measures.

For $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$,

$$\overline{\text{den}}^{q,t}_{\mu}(x,\nu) = \overline{\lim_{r\to 0}} \frac{\nu(B(x,r))}{\mu(B(x,r))^q (2r)^t},$$

$$\underline{\text{den}}^{q,t}_{\mu}(x,\nu) = \underline{\lim_{r\to 0}} \frac{\nu(B(x,r))}{\mu(B(x,r))^q (2r)^t}.$$

In particular, we define

$$\bar{D}^{q,t}_{\mu}(x,E) = \overline{\text{den}}^{q,t}_{\mu}(x,\nu),\ \underline{D}^{q,t}_{\mu}(x,E) = \underline{\text{den}}^{q,t}_{\mu}(x,\nu) \text{ if } \nu = H^{q,t}_{\mu}|_E$$

and

$$\bar{\Delta}^{q,t}_{\mu}(x,E) = \overline{\text{den}}^{q,t}_{\mu}(x,\nu),\ \underline{\Delta}^{q,t}_{\mu}(x,E) = \underline{\text{den}}^{q,t}_{\mu}(x,\nu) \text{ if } \nu = p^{q,t}_{\mu}|_E.$$

If $\bar{D}^{q,t}_{\mu}(x,E) = \underline{D}^{q,t}_{\mu}(x,E)$, we write $D^{q,t}_{\mu}(x,E)$ for the common value. Similarly, we write $\Delta^{q,t}_{\mu}(x,E)$. In particular, when $0 < H^{q,t}_{\mu}(E) < \infty$, a point x in E is called $H^{q,t}_{\mu}$*-regular point* of E if $\bar{D}^{q,t}_{\mu}(x,E) = \underline{D}^{q,t}_{\mu}(x,E)$ otherwise x is a $H^{q,t}_{\mu}$*-irregular point*. When $0 < H^{q,t}_{\mu}(E) < \infty$, E is said to be $H^{q,t}_{\mu}$*-regular* if $H^{q,t}_{\mu}$-a.a of its points are $H^{q,t}_{\mu}$-regular and $H^{q,t}_{\mu}$*-irregular* if $H^{q,t}_{\mu}$-a.a of its points are $H^{q,t}_{\mu}$-irregular. Similarly we can define regularities for $p^{q,t}_{\mu}$.

Now we will prove the decomposition theorem for the regularities of multifractal Hausdorff measure and packing measure in $\mathbb{R}^d$. This decomposition theorem says that the set of regular points with respect to $H^{q,t}_{\mu}(p^{q,t}_{\mu})$ is regular and the set of irregular points with respect to $H^{q,t}_{\mu}(p^{q,t}_{\mu})$ is irregular.

2. Main results

To achieve our purpose, we list some useful lemmas from [6].

Lemma 2.1. *If E is $H^{q,t}_{\mu}$-measurable, $H^{q,t}_{\mu}(E) < \infty$ and $\mu \in \mathcal{P}_0(\mathbb{R}^d, E)$, then*

$$H^{q,t}_{\mu}(E) \inf_{x\in E} \overline{\text{den}}^{q,t}_{\mu}(x,\nu) \le \nu(E).$$

Lemma 2.2. *If E is $H_\mu^{q,t}$-measurable and $H_\mu^{q,t}(E) < \infty$, then*

$$\nu(E) \le H_\mu^{q,t}(E) \sup_{x\in E} \overline{\text{den}}_\mu^{q,t}(x,\nu).$$

Lemma 2.3. *If E is $p_\mu^{q,t}$-measurable and $p_\mu^{q,t}(E) < \infty$, then*

$$p_\mu^{q,t}(E) \inf_{x\in E} \underline{\text{den}}_\mu^{q,t}(x,\nu) \le \nu(E) \le p_\mu^{q,t}(E) \sup_{x\in E} \underline{\text{den}}_\mu^{q,t}(x,\nu).$$

PROPOSITION 2.4

Let E be a $H_\mu^{q,t}$-measurable set with $H_\mu^{q,t}(E) < \infty$ and μ in $\mathcal{P}_0(\mathbb{R}^d, E)$. If $\nu_E(F) = \nu(F\cap E)$, then $\underline{\text{den}}_\mu^{q,t}(x,\nu) = \underline{\text{den}}_\mu^{q,t}(x,\nu_E)$ and $\overline{\text{den}}_\mu^{q,t}(x,\nu) = \overline{\text{den}}_\mu^{q,t}(x,\nu_E)$ for $H_\mu^{q,t}$-a.a on E.

Proof. Clearly $\underline{\text{den}}_\mu^{q,t}(x,\nu) \ge \underline{\text{den}}_\mu^{q,t}(x,\nu_E)$ and $\overline{\text{den}}_\mu^{q,t}(x,\nu) \ge \overline{\text{den}}_\mu^{q,t}(x,\nu_E)$. Define $\lambda(A) = \nu(A\backslash E)$ for all Borel set A. Then $\nu(A) = \nu(A\cap(E^c\cup E)) = \nu(A\backslash E) + \nu(A\cap E) = \lambda(A) + \nu_E(A)$.

Thus $\underline{\text{den}}_\nu^{q,t}(x,\nu) \le \underline{\text{den}}_\mu^{q,t}(x,\nu_E) + \overline{\text{den}}_\mu^{q,t}(x,\lambda)$ and $\overline{\text{den}}_\mu^{q,t}(x,\nu) \le \overline{\text{den}}_\mu^{q,t}(x,\nu_E) + \overline{\text{den}}_\mu^{q,t}(x,\lambda)$. Let $E_k = \left\{x \in E | \overline{\text{den}}_\mu^{q,t}(x,\lambda) \ge \frac{1}{k}\right\}$. Then, by Lemma 2.1, $\frac{1}{k} H_\mu^{q,t}(E_k) \le \lambda(E_k) = \nu(E_k\backslash E) = 0$ for all k. So $\overline{\text{den}}_\lambda^{q,t}(x,E) = 0$ for $H_\mu^{q,t}$-a.a on E. Hence $\underline{\text{den}}_\mu^{q,t}(x,\nu) = \underline{\text{den}}_\mu^{q,t}(x,\nu_E)$ and $\overline{\text{den}}_\mu^{q,t}(x,\nu) = \overline{\text{den}}_\mu^{q,t}(x,\nu_E)$ for $H_\mu^{q,t}$-a.a on E.

COROLLARY 2.5

(1) *Let E be $H_\mu^{q,t}$-measurable with $H_\mu^{q,t}(E) < \infty$ and μ be in $\mathcal{P}_0(\mathbb{R}^d, E)$. If $F \subset E$, then $\underline{D}_\mu^{q,t}(x,E) = \underline{D}_\mu^{q,t}(x,F)$ and $\bar{D}_\mu^{q,t}(x,E) = \bar{D}_\mu^{q,t}(x,F)$ for $H_\mu^{q,t}$-a.a on F.*
(2) *Let E be a $p_\mu^{q,t}$-measurable set with $p_\mu^{q,t}(E) < \infty$ and μ be in $\mathcal{P}_0(\mathbb{R}^d, E)$. If $F \subset E$, then $\underline{\Delta}_\mu^{q,t}(x,E) = \underline{\Delta}_\mu^{q,t}(x,F)$ and $\bar{\Delta}_\mu^{q,t}(x,E) = \bar{\Delta}_\mu^{q,t}(x,F)$ for $p_\mu^{q,t}$-a.a on F.*

Proof. We can easily obtain this Corollary by letting $\nu = H_\mu^{q,t}|_E$ $(p_\mu^{q,t}|_E)$ in Proposition 2.4.

PROPOSITION 2.6

Let E be a $p_\mu^{q,t}$-measurable set with $p_\mu^{q,t}(E) < \infty$, $\mu \in \mathcal{P}(\mathbb{R}^d)$ and let $F = \{x \in E | \bar{\Delta}_\mu^{q,t}(x,E) < \infty\}$. If G is a Borel subset of F such that $H_\mu^{q,t}(G) = 0$, then $p_\mu^{q,t}(G) = 0$.

Proof. Letting $\nu = p_\mu^{q,t}|_E$ in Lemma 2.2, we can easily get this Proposition.

Now we prove a decomposition theorem of Besicovitch type for multifractal Hausdorff measures and packing measures.

Theorem 2.7. *Let $E \subset \mathbb{R}^d$ and $q, t \in \mathbb{R}$. Also, let $\mu \in \mathcal{P}_0(\mathbb{R}^d, E)$.*

(1) *If $H_\mu^{q,t}(E) < \infty$, then the following two statements hold.*

 (1.1) *The set $\{x \in E | x$ is a $H_\mu^{q,t}$-regular point$\}$ is $H_\mu^{q,t}$-regular.*
 (1.2) *The set $\{x \in E | x$ is a $H_\mu^{q,t}$-irregular point$\}$ is $H_\mu^{q,t}$-irregular.*

(2) *If $p_\mu^{q,t}(E) < \infty$, then the following two statements hold.*

(2.1) *The set $\{x \in E | x$ is a $p_\mu^{q,t}$-regular point$\}$ is $p_\mu^{q,t}$-regular.*
(2.2) *The set $\{x \in E | x$ is a $p_\mu^{q,t}$-irregular point$\}$ is $p_\mu^{q,t}$-irregular.*

Proof. The proof of (1) is similar to that of (2).

For (2), put $F = \{x \in E | \underline{\Delta}_\mu^{q,t}(x, E) = 1 = \bar{\Delta}_\mu^{q,t}(x, E)\}$. Since $F \subset E$ and $p_\mu^{q,t}(E) < \infty$, we can show that $\underline{\Delta}_\mu^{q,t}(x, F) = 1$ for $p_\mu^{q,t}$-a.a on F (see [6]). So we only have to prove that $\bar{\Delta}_\mu^{q,t}(x, F) = 1$ for $p_\mu^{q,t}$-a.a on F.

By Corollary 2.5, $\bar{\Delta}_\mu^{q,t}(x, F) = \bar{\Delta}_\mu^{q,t}(x, E)$ for $H_\mu^{q,t}$-a.a on F. So $\bar{\Delta}_\mu^{q,t}(x, F) = 1$ for $H_\mu^{q,t}$-a.a on F. By Proposition 3.2, we have $\bar{\Delta}_\mu^{q,t}(x, F) = 1$ for $p_\mu^{q,t}$-a.a on F. Next, we must show that it is not $p_\mu^{q,t}$-a.a on $E \backslash F$ that $\underline{\Delta}_\mu^{q,t}(x, E\backslash F) = \bar{\Delta}_\mu^{q,t}(x, E\backslash F) = 1$. Now again by Corollary 2.5 and Proposition 3.2, $\Delta_\mu^{q,t}(x, E\backslash F) = \Delta_\mu^{q,t}(x, E\backslash F)$ for $p_\mu^{q,t}$-a.a on $E\backslash F$. Hence $p_\mu^{q,t}\{x \in E\backslash F \mid \bar{\Delta}_\mu^{q,t}(x, E\backslash F) = 1\} = 0$.

3. Applications

For convenience, we adapt the following definition.

DEFINITION 3.1

Let $(X, \mathcal{E}, \mu)$ be a measure space and $E, F \in \mathcal{E}$. We will say that E is a subset of F for μ-almost-all and write $E \subset F$ for μ-a.a., if $\mu(F\backslash E) = 0$.

PROPOSITION 3.2

Let E be a $p_\mu^{q,t}$-measurable set with $p_\mu^{q,t}(E) < \infty$, $\mu \in \mathcal{P}_0(\mathbb{R}^d, E)$ and A a measurable subset of E. Then $H_\mu^{q,t}(A) = p_\mu^{q,t}(A)$ if and only if $A \subset \{x \in E | \Delta_\mu^{q,t}(x, E) = 1\}$ for $p_\mu^{q,t}$-a.a.

Proof. Without loss of generality, we may assume that $p_\mu^{q,t}(A) > 0$.

First, we suppose that $H_\mu^{q,t}(A) = p_\mu^{q,t}(A)$. Then $\mu \in \mathcal{P}_0(\mathbb{R}^d, A)$. It follows from [6] that $\Delta_\mu^{q,t}(x, A) = 1$ for $p_\mu^{q,t}$-a.a on A. By Corollary 2.5, $\Delta_\mu^{q,t}(x, E) = \Delta_\mu^{q,t}(x, A)$ for $p_\mu^{q,t}$-a.a on A.

Secondly, we suppose that $\Delta_\mu^{q,t}(x, E) = 1$ for $p_\mu^{q,t}$-a.a. on A. Then we easily see that $\Delta_\mu^{q,t}(x, E) = 1$ for $H_\mu^{q,t}$-a.a. on A. By Corollary 2.5, $\Delta_\mu^{q,t}(x, A) = \Delta_\mu^{q,t}(x, E)$ for $H_\mu^{q,t}$-a.a. on A. We have $\Delta_\mu^{q,t}(x, A) = 1$ for $H_\mu^{q,t}$-a.a. on A. We can replace $H_\mu^{q,t}$ by $p_\mu^{q,t}$ from Proposition 3.2. So we get $\Delta_\mu^{q,t}(x, A) = 1$ for $p_\mu^{q,t}$-a.a on A. Finally, it follows from [6] that $H_\mu^{q,t}(A) = p_\mu^{q,t}(A)$.

We remark here that if $p_\mu^{q,t}(A) > 0$ for a subset A of p-irregular set, then $p_\mu^{q,t}(A) > H_\mu^{q,t}(A)$.

PROPOSITION 3.3

Let E be a $p_\mu^{q,t}$-measurable set with $p_\mu^{q,t}(E) < \infty$ and $\mu \in \mathcal{P}_0(\mathbb{R}^d, E)$. Then $\{x \in E | \Delta_\mu^{q,t}(x, E) = 1\} \subset \{x \in E | D_\mu^{q,t}(x, E) = 1\}$ for $p_\mu^{q,t}$-a.a.

Proof. Let $A = \{x \in E | \Delta_\mu^{q,t}(x, E) = 1\}$. We may assume that $p_\mu^{q,t}(A) > 0$. We can obtain that A is a p-regular set by using Theorem 2.7. In other words, $\Delta_\mu^{q,t}(x, A) = 1$ for $p_\mu^{q,t}$-a.a on A. So, by Corollary 2.16 [6], $D_\mu^{q,t}(x, A) = 1$ for $p_\mu^{q,t}$-a.a on A and hence $D_\mu^{q,t}(x, A) = 1$ for $H_\mu^{q,t}$-a.a on A. By Corollary 2.5, $D_\mu^{q,t}(x, A) = D_\mu^{q,t}(x, E)$ for $H_\mu^{q,t}$-a.a on A. Then $D_\mu^{q,t}(x, E) = 1$ for $H_\mu^{q,t}$-a.a on A. By Proposition 3.2, $D_\mu^{q,t}(x, E) = 1$ for $p_\mu^{q,t}$-a.a on A.

PROPOSITION 3.4

Let E be a $p_\mu^{q,t}$-measurable set with $p_\mu^{q,t}(E) < \infty$ and $\mu \in \mathcal{P}_0(\mathbb{R}^d, E)$. If $\bar{\Delta}_\mu^{q,t}(x, E) < \infty$ on E, then $\{x \in E | D_\mu^{q,t}(x, E) = 1\} \subset \{x \in E | \Delta_\mu^{q,t}(x, E) = 1\}$ for $p_\mu^{q,t}$-a.a.

Proof. Put $B = \{x \in E | D_\mu^{q,t}(x, E) = 1\}$. We may assume that $p_\mu^{q,t}(B) > 0$. Then B is $H_\mu^{q,t}$-regular by Theorem 2.7. In other words, $D_\mu^{q,t}(x, B) = 1$ for $H_\mu^{q,t}$-a.a on B. Since $\bar{\Delta}_\mu^{q,t}(x, E) < \infty$, by Proposition 3.2, $D_\mu^{q,t}(x, B) = 1$ for $p_\mu^{q,t}$-a.a on B. By Corollary 2.16 [6], $\Delta_\mu^{q,t}(x, B) = 1$ for $p_\mu^{q,t}$-a.a on B. By Corollary 2.5, $\Delta_\mu^{q,t}(x, B) = \Delta_\mu^{q,t}(x, E)$ for $p_\mu^{q,t}$-a.a on B. Then $\Delta_\mu^{q,t}(x, E) = 1$ for $p_\mu^{q,t}$-a.a on B.

We can prove the next theorem from the above Propositions as a generalization of the density theorem of Olsen [6].

Theorem 3.5. *Let E be a $p_\mu^{q,t}$-measurable set with $p_\mu^{q,t}(E) < \infty$ and $\mu \in \mathcal{P}_0(\mathbb{R}^d, E)$. If $\bar{\Delta}_\mu^{q,t}(x, E) < \infty$ on E, then the following are equivalent.*

(1) $H_\mu^{q,t}(A) = p_\mu^{q,t}(A)$ *for a measurable subset A of E.*
(2) $A \subset \{x \in E | D_\mu^{q,t}(x, E) = 1\}$ *for $p_\mu^{q,t}(E)$-a.a.*
(3) $A \subset \{x \in E | \Delta_\mu^{q,t}(x, E) = 1\}$ *for $p_\mu^{q,t}(E)$-a.a.*

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