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Abel transform on PSL(2, $\mathbb{R}$) and some of its applications

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Abstract. We shall investigate the use of Abel transform on $\mathrm{PSL}_2(\mathbb{R})$ as a tool beyond K-biinvariant setup, discuss its properties and show some applications.

Keywords. Abel transform; radon transform; Helgason–Fourier transform.

1. Introduction

Let G be a noncompact connected semisimple Lie group with finite centre and let K be a fixed maximal compact subgroup. We have the following standard decompositions of G:

(i) Polar decomposition $G = KAK$
(ii) Iwasawa decomposition $G = KAN$

Here N is a nilpotent subgroup of G and A is a vector subgroup isomorphic to $\mathbb{R}^d$ for some $d \in \mathbb{N}$ (see [10] for precise statements). A function f on G is called K-biinvariant if $f(k_1 x k_2) = f(x)$ for all $x \in G$ and $k_1, k_2 \in K$. If f is K-biinvariant then clearly f can be considered as a function on $A \equiv \mathbb{R}^d$.

There is another way of relating a K-biinvariant function on G to a function on $\mathbb{R}^d$. For a suitable K-biinvariant function f, we define the Abel transform $\mathcal{A}f(a) = \mathrm{e}^{\rho(\log a)} \int_N f(an)\mathrm{d}n$. It can be verified that $\mathcal{A}(f_1 * f_2) = \mathcal{A}f_1 * \mathcal{A}f_2$, where $*$ in the left-hand side is convolution on G and $*$ in the right-hand side is convolution on $\mathbb{R}^d$. (We will always use $*$ for convolution and from the context it will be clear if it is on $\mathbb{R}^d$ or on G. We hope this will not lead to any confusion.) If f is a K-biinvariant L^1-function, then $\mathcal{A}f \in L^1(\mathbb{R}^d)$ and for $\nu \in \mathfrak{a}^* \equiv \mathbb{R}^d$, its Euclidean–Fourier transform $(\mathcal{A}f)\tilde{}(\nu) = \int_{\mathbb{R}^d} \mathcal{A}f(y)\mathrm{e}^{-i\langle \nu, y\rangle}\mathrm{d}y = \hat{f}(i\nu)$, where $\hat{f}$ is the spherical Fourier transform of f. These properties of Abel transform are crucial for reducing some questions on K-biinvariant functions on semisimple Lie groups to related questions on $\mathbb{R}^d$.

One gets the impression from the existing literature that the use of the Abel transform is mainly in the context of K-biinvariant functions, as Swayer puts it in [23] 'The Abel transform is basically a radon transform for K-biinvariant functions'. However in this article we investigate the possibility of using the Abel transform as a tool beyond the K-biinvariant set up, in particular for problems related to non K-finite functions.

To keep the exposition simple and minimize the preliminaries, in this article we will confine ourselves to the special case of PSL$(2, \mathbb{R})$. This will be sufficient to exhibit the new feature of accommodating nontrivial isotypic components of a function and tackling the problem involving the full group using the Abel transform.

All the results in this article can be extended to $\mathrm{SL}(2,\mathbb{R})$. Some of them will have ready generalization to more general groups (in particular for groups of real rank 1) of the class mentioned above. In §2 we establish the notation and put the required preliminaries. In §3 we define Abel transform for various class of functions (e.g. C_c^∞, L^1) and distributions which are not necessarily K-biinvariant and establish some basic properties of the transform. In §4 we discuss the Abel transform of complex measures and its properties. In §5 we provide two applications of Abel transform.

2. Notation and preliminaries

We will mainly use the notation of [3] with a few exceptions which we will mention here. Let G_1 be $\mathrm{SL}(2,\mathbb{R})$. Let

$$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad a_t = \begin{pmatrix} \mathrm{e}^t & 0 \\ 0 & \mathrm{e}^{-t} \end{pmatrix} \quad \text{and} \quad n_\xi = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}.$$

Then $K = \{k_\theta | \theta \in [0, 2\pi)\}$, $A = \{a_t | t \in \mathbb{R}\}$ and $N = \{n_\xi | \xi \in \mathbb{R}\}$ are three particular subgroups of G_1 of which K is a maximal compact subgroup SO(2) of G_1. It is clear from the above that A can be identified with $\mathbb{R}$. Let $G_1 = KAN$ be an Iwasawa decomposition and for $x \in G_1$, let $x = k_\theta a_t n_\xi$ be its corresponding decomposition. Then we will write $H(x)$ for t and $K(x)$ for k_θ. The Haar measure $\mathrm{d}x$ of G_1 splits according to this decomposition as $\mathrm{d}x = \mathrm{e}^{2t}\mathrm{d}k\mathrm{d}t\mathrm{d}n$ where $\mathrm{d}k = \mathrm{d}k_\theta = \frac{\mathrm{d}\theta}{2\pi}$ is the normalised Haar measure of K and $\mathrm{d}n = \mathrm{d}n_\xi = \mathrm{d}\xi$ as well as $\mathrm{d}a = \mathrm{d}a_t = \mathrm{d}t$ are both Lebesgue measures on $\mathbb{R}$.

We also recall that G_1 has Cartan decomposition $G_1 = K\overline{A^+}K$, $x = k_1 a_t k_2$ where $k_1, k_2 \in K, t \geq 0$. The Haar measure of G_1 splits according to this decomposition as $\mathrm{d}x = \mathrm{d}k_1 \sinh 2t \mathrm{d}t \mathrm{d}k_2$. Let $\sigma(x) = \sigma(k_1 a_t k_2) = |t|$. In fact $\sigma(x) = d(xK, o)$, where $o = eK$ is the 'origin' of the symmetric space G_1/K and d is the distance function on G_1/K.

Let $\hat{K} = \{\chi_n | n \in \mathbb{Z}\}$ be the set of continuous characters of K, where $\chi_n(k_\theta) = \mathrm{e}^{in\theta}$. Instead of χ_n, by abuse of language, we will call the integers n as K-types. A complex-valued function f on G_1 is said to be of left (respectively right) K-type n if $f(kx) = \chi_n(k) f(x)$ (respectively $f(xk) = \chi_n(k) f(x)$) for all $k \in K$ and $x \in G_1$. A function is called spherical of type (m, n) if its left K-type is m and right K-type is n. For a suitable function f, the (m, n)-th isotypical component of f is denoted by $f_{m,n}$ and is given by

$$f_{m,n}(x) = \int_{K\times K} \overline{\chi_m(k_1)}\ \overline{\chi_n(k_2)} f(k_1 x k_2) \mathrm{d}k_1 \mathrm{d}k_2. \tag{2.1}$$

It can be verified that $f_{m,n}$ is itself a function of type (m, n) and $f_{m,n} \equiv 0$ when m and n are of opposite parity. The function f can be decomposed as $f = \sum_{m,n\in\mathbb{Z}} f_{m,n}$. In fact when $f \in C^\infty(G_1)$ this is an absolutely convergent series in the C^∞-topology. When $f \in L^p(G_1)$, $p \in [1, \infty)$, the equality is in the sense of distribution. A function f is called K-finite if there are only finitely many components in the decomposition of f as above.

Let G be the group $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$, where I is the 2×2 identity matrix. The group of all Möbius transformations preserving the upper half plane $\mathbb{H} = \{z \in \mathbb{C}\colon \mathrm{Im}\, z > 0\}$ is isomorphic to G. Any suitable function space $\mathfrak{F}$, (e.g. $L^p(G)$, $C_c^\infty(G)$) can be identified with $\mathfrak{F}(G_1)_{\text{even}}$, where $\mathfrak{F}(G_1)_{\text{even}}$ is the subspace of even functions in $\mathfrak{F}(G_1)$. Note that for the decomposition $f = \sum_{m,n} f_{m,n}$ of any $f \in \mathfrak{F}(G_1)_{\text{even}}$, $f_{m,n} \not\equiv 0$ only when $m, n \in 2\mathbb{Z}$. From the realization above of the function spaces of $G = \mathrm{PSL}(2, \mathbb{R})$, it is clear that only those representations of $G_1 = \mathrm{SL}(2, \mathbb{R})$ which are relevant for even functions are required to study G. Let us denote by $\mathfrak{F}(G)_{m,n}$ the subspace of $\mathfrak{F}(G)$ consisting of spherical functions of type (m, n). In particular $\mathfrak{F}(G)_{0,0}$ is the space of K-biinvariant functions which is also denoted by $\mathfrak{F}(G//K)$. We denote by $\mathfrak{F}(G/K)_n$ the space of functions in $\mathfrak{F}(G)$ whose right type is n.

Let $\mathfrak{a}$ be the Lie algebra of A. Let $\mathfrak{a}^*$ be the real dual of $\mathfrak{a}$ and $\mathfrak{a}^*_{\mathbb{C}}$ be the complexification of $\mathfrak{a}^*$. Then $\mathfrak{a}^*$ and $\mathfrak{a}^*_{\mathbb{C}}$ can be identified with $\mathbb{R}$ and $\mathbb{C}$ respectively via ρ, the half-sum of the positive roots, i.e. $\rho = 1$ under this identification.

We define e_n on K by $e_n(k_\theta) = \mathrm{e}^{in\theta}$. For $\lambda \in \mathfrak{a}^*_{\mathbb{C}} = \mathbb{C}$, let $(\pi_\lambda, \mathcal{H})$ be the principal series representation of G where $\mathcal{H}$ is a subspace of $L^2(K)$ generated by the orthonormal basis elements e_n with $n \in 2\mathbb{Z}$. The representation π_λ is normalized so that it is unitary if and only if $\lambda \in i\mathfrak{a}^* = i\mathbb{R}$. In fact (eq. (4.1) of [3]):

$$(\pi_\lambda(x)e_n)(k) = \mathrm{e}^{-(\lambda+1)H(x^{-1}k^{-1})} e_{-n}(K(x^{-1}k^{-1})). \tag{2.2}$$

For every $k \in 2\mathbb{Z} + 1$, the set of odd integers, there is a discrete series representation π_k which occurs as a subrepresentation of $\pi_{|k|}$. For $m, n \in 2\mathbb{Z}$ and $k \in 2\mathbb{Z} + 1$, let $\Phi^{m,n}_\lambda(x) = \langle \pi_\lambda(x)e_m, e_n\rangle$ and $\Psi^{m,n}_k(x) = \langle \pi_k(x)e^k_m, e^k_n\rangle_k$, be the matrix coefficients of the principal series and discrete series representations respectively, where $\{e^k_n\}$ are the renormalised basis and $\langle\ ,\ \rangle_k$ is the renormalised inner product of π_k (see p. 20 of [3]). In particular, $\Phi^{0,0}_\lambda$ is clearly the elementary spherical function, which we also denote by ϕ_λ.

From (2.2) and the fact that $\pi_\lambda(x)^* = \pi_{-\bar{\lambda}}(x^{-1})$ (see [3]) we have for $\lambda \in i\mathbb{R}$:

$$\Phi^{m,n}_\lambda(xy) = \int_K \mathrm{e}^{-(\lambda+1)H(y^{-1}k^{-1})} e_{-m}(K(y^{-1}k^{-1}))\mathrm{e}^{(\lambda-1)H(xk^{-1})} e_n(K(xk^{-1}))\mathrm{d}k. \tag{2.3}$$

For a function $f \in L^1(G)$, let $\hat{f}(\lambda)$ and $\hat{f}(k)$ denote its (operator-valued) principal and discrete Fourier transforms respectively. Precisely,

$$\hat{f}(\lambda) = \int_G f(x)\pi_\lambda(x^{-1})\mathrm{d}x \quad \text{and} \quad \hat{f}(k) = \int_G f(x)\pi_k(x^{-1})\mathrm{d}x.$$

The (m, n)-th matrix entries of $\hat{f}(\lambda)$ and $\hat{f}(k)$ are denoted by $\hat{f}(\lambda)_{m,n}$ and $\hat{f}(k)_{m,n}$ respectively. Thus $\hat{f}(\lambda)_{m,n} = \langle \hat{f}(\lambda)e_m, e_n\rangle = \int_G f(x)\Phi^{m,n}_\lambda(x^{-1})\mathrm{d}x$ and $\hat{f}(k)_{m,n} = \int_G f(x)\Psi^{m,n}_k(x^{-1})\mathrm{d}x$. As $\int_G f(x)\Phi^{m,n}_\lambda(x^{-1})\mathrm{d}x = \int_G f_{m,n}(x)\Phi^{m,n}_{\sigma,\lambda}(x^{-1})\mathrm{d}x$, clearly, $\hat{f}(\lambda)_{m,n} = \hat{f}_{m,n}(\lambda)$. Similarly $\hat{f}(k)_{m,n} = \hat{f}_{m,n}(k)$. Henceforth we will not distinguish between $\hat{f}(\lambda)_{m,n}$ (respectively $\hat{f}(k)_{m,n}$) and $\hat{f}_{m,n}(\lambda)$ (respectively $\hat{f}_{m,n}(k)$).

Let $\mathcal{S}_1 = \{\lambda \in \mathbb{C}\colon |\mathrm{Re}\ \lambda| \le 1\}$ be the Helgason–Johnson strip. Using (2.2) we see that for $\lambda \in \mathbb{C}$,

$$|\Phi^{m,n}_\lambda(x)| \le \int_K \mathrm{e}^{-(\mathrm{Re}\ \lambda+1)H(x^{-1}k^{-1})}\mathrm{d}k = \Phi^{0,0}_{\mathrm{Re}\ \lambda} = \phi_{\mathrm{Re}\ \lambda},$$

where Re λ stands for the *real part* of λ. It is well-known [14] that $|\phi_\lambda(x)| \le 1$ for $x \in G$ and $\lambda \in \mathcal{S}_1$. Hence $\Phi^{m,n}_\lambda(x)$ is also uniformly bounded by 1 on $\mathcal{S}_1$. We also have the following well-known estimate (see (3.2) of [3]):

$$\mathrm{e}^{-\sigma(x)} \le \Xi(x) \le C(1 + \sigma(x))\mathrm{e}^{-\sigma(x)} \text{ where } \Xi = \Phi^{0,0}_0 = \phi_0. \tag{2.4}$$

The Plancherel measure [17]. The Plancherel measure on the unitary principal series representations (parametrized by $i\mathbb{R}$) is $\mathrm{d}\mu(\lambda) = \mu(\lambda)\mathrm{d}\lambda$ where

$$\mu(i\xi) = \left(\frac{\xi}{4\pi}\right)\tanh\left(\frac{\xi\pi}{2}\right) \text{ for } \xi \in \mathbb{R}. \tag{2.5}$$

The Plancherel measure on the discrete series is given by $\mu(\pi_k) = \frac{|k|}{2\pi}$, for $k \in \mathbb{Z}^*$.

Throughout this paper for a function $g \in L^1(\mathbb{R})$ we denote its Euclidean–Fourier transform at ν i.e. $\int_{\mathbb{R}} g(y)\mathrm{e}^{-i\nu y}\mathrm{d}y$ by $\tilde{g}(\nu)$.

3. Abel transform and Fourier transform for spherical functions of arbitrary type

For a function f on G we define its Abel transform as $\mathcal{A}f(t) = \mathrm{e}^t \int_N f(a_t n)\mathrm{d}n$, whenever the integral makes sense.

PROPOSITION 3.1

Suppose f_1, f_2 are two functions in $C_c^\infty(G)$ of types (m, n) and (r, s) respectively. Then $\mathcal{A}(f_1 * f_2) = \delta_{n,r}(\mathcal{A}f_1 * \mathcal{A}f_2)$, where $\delta_{n,r}$ is the Kronecker delta.

We omit the proof as it is quite straightforward and is an exercise in the use of Haar measure expressed with respect to the Iwasawa decomposition $G = KAN$.

The dual of the Abel transform. For every fixed $m, n \in \mathbb{Z}$ of the same parity, let the (m, n)-th dual $\mathcal{A}^*_{m,n}$: $C(A) \longrightarrow C(G)_{m,n}$ of the Abel transform $\mathcal{A}$ be defined as follows: For $f \in C(A)$,

$$\mathcal{A}^*_{m,n}(f)(x) = \int_K \mathrm{e}^{-H(xk^{-1})} f(\mathrm{e}^{H(xk^{-1})}) e_{-m}(K(xk^{-1})) e_n(k^{-1})\mathrm{d}k.$$

It can be verified that $\mathcal{A}^*_{m,n} f$ is a function of type $(-m, -n)$. The dual of Abel transform is also called *back projection operator* by some authors. The following is a easy consequence of Fubini's theorem and the Iwasawa decomposition.

PROPOSITION 3.2

Let $h \in C_c^\infty(G)_{m,n}$ and $f \in C(A)$. Then

$$\int_A \mathcal{A}(h)(a) f(a)\mathrm{d}a = \int_G h(g)\mathcal{A}^*_{m,n}(f)(g)\mathrm{d}g.$$

We take $f(a_t) = \mathrm{e}^{-\lambda t} = \mathrm{e}^{-\lambda \log(a_t)}$ in the proposition above where $\lambda \in i\mathbb{R}$. Then

$$\begin{aligned}\mathcal{A}^*_{m,n}(f)(x) &= \int_K \mathrm{e}^{-H(xk^{-1})}\mathrm{e}^{-\lambda H(xk^{-1})} e_{-m}(K(xk^{-1})) e_n(k^{-1})\mathrm{d}k \\ &= \Phi_\lambda^{m,n}(x^{-1}).\end{aligned}$$

This shows that for $f \in C_c^\infty(G)_{m,n}$ and $\nu \in \mathbb{R}$,

$$\widetilde{\mathcal{A}f}(\nu) = \hat{f}_{m,n}(i\nu) \tag{3.6}$$

or equivalently for $\lambda \in i\mathbb{R}$,

$$\widetilde{\mathcal{A}f}(-i\lambda) = \hat{f}_{m,n}(\lambda). \tag{3.7}$$

Equations (3.6) and (3.7) are the generalizations of the *slice theorem* for K-biinvariant functions. Note that equations (3.6), (3.7) are valid for any $\lambda, \nu \in \mathbb{C}$ for which both sides make sense.

In general, for a function f which is both left and right K-finite, we have

$$\widetilde{\mathcal{A}f}(-i\lambda) = \sum_{m,n} \hat{f}_{m,n}(\lambda).$$

Note that the sum on the right-hand side is a finite sum.

Transmutation. We consider a standard basis $\{X, E, F\}$ of $\mathfrak{sl}(2, \mathbb{C})$, where

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad F = \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}.$$

We recall that the universal enveloping algebra $\mathcal{U}$ of G is generated by $\{X, E, F\}$. The left and right invariant vector fields L_Y, R_Y respectively associated with an element $Y \in \mathfrak{g}$ are given by

$$L_Y f(x) = f(x; Y) = (\mathrm{d}/\mathrm{d}t) f(x \cdot \exp tY))|_{t=0}$$

and

$$R_Y f(x) = f(Y; x) = (\mathrm{d}/\mathrm{d}t) f(\exp tY \cdot x))|_{t=0}.$$

Through this identification we have an isomorphism of $\mathcal{U}$ with the algebra of left invariant differential operator and an anti isomorphism with the algebra of right invariant differential operators. For instance, let $g = X_1X_2 + X_3X_4 \in \mathcal{U}$, where $X_i \in \{X, E, F\}$. Then g gives rise to a left invariant differential operator as

$$f(x; X_1X_2 + X_3X_4) = \mathrm{d}/\mathrm{d}s\,\mathrm{d}/\mathrm{d}t f(x \exp tX_1 \exp sX_2)|_{t=0}|_{s=0}$$
$$+ \mathrm{d}/\mathrm{d}s\,\mathrm{d}/\mathrm{d}t f(x \exp tX_3 \exp sX_4)|_{t=0}|_{s=0}.$$

Similarly g also gives rise to a right invariant differential operator.

The infinitesimal action of the principal series representation π_λ on $\mathfrak{sl}(2, \mathbb{C})$ is given by (see eq. (4.6) of [3]):

$$\pi_\lambda(X)e_n = ine_n,\ \pi_\lambda(E)e_n = (n + \lambda + 1)e_{n+2} \text{ and } \pi_\lambda(F)e_n$$
$$= (n - \lambda - 1)e_{n-2}.$$

From this it is clear that for $X_i, X_j \in \{X, E, F\}$,

$$\Phi_\lambda^{m,n}(X_i; x; X_j) = P_{m,n}(\lambda)\Phi_\lambda^{r,s}(x),$$

where P is a polynomial and $m, n, r, s \in 2\mathbb{Z}$. In general, for $g_1, g_2 \in \mathcal{U}$, $\Phi_\lambda^{m,n}(g_1; x; g_2)$ is a finite sum of the form

$$P_1(\lambda, m, n)\Phi_\lambda^{r_1,s_1}(x) + P_2(\lambda, m, n)\Phi_\lambda^{r_2,s_2}(x) + \cdots + P_l(\lambda, m, n)\Phi_\lambda^{r_l,s_l}(x).$$

With this preparation we can state and prove the following *transmutation* property, satisfied by the Abel transform.

PROPOSITION 3.3

Let g_1, g_2 be two elements of $\mathcal{U}$ interpreted as right and left invariant differential operators on G respectively. For $f \in C_c^\infty(G)_{m,n}$, let $f'(x) = f(g_1; x; g_2)$. Then $\mathcal{A}f' = \mathcal{D}\mathcal{A}f$, where $\mathcal{D}$ is a constant coefficient differential operator on $\mathbb{R}$.

Proof. Notice that the Paley–Wiener theorem on $\mathbb{R}$ and on PSL$(2, \mathbb{R})$ (see p. 47 of [3]) imply that for f as above $\mathcal{A}f \in C_c^\infty(\mathbb{R})$.

We start with the particular case when $g_1 = X_1, g_2 = X_2$ are elements of the basis $\{X, E, F\}$ of $\mathfrak{sl}(2, \mathbb{C})$, described above.

Suppose $f \in C_c^\infty(G)_{m,n}$ and $f_1 = f(X_1; x; X_2)$. Then $f_1 \in C_c^\infty(G)_{r,s}$ for some $r, s \in 2\mathbb{Z}$ and for $\lambda \in i\mathbb{R}$, $\hat{f}_1(\lambda)_{r,s} = \int_G f(X_1; x; X_2)\Phi_\lambda^{r,s}(x^{-1})\mathrm{d}x = \int_G f(x)\Phi_\lambda^{r,s}(X_2; x^{-1}; X_1)\mathrm{d}x = P(\lambda)\hat{f}(\lambda)_{m,n}$ for some polynomial $P(\lambda)$. This and (3.6) imply that for $\nu \in \mathbb{R}$, $\widetilde{\mathcal{A}f_1}(\nu) = P(\nu)\widetilde{\mathcal{A}f}(\nu)$. It is easy to verify that if we replace X_1 and X_2 by D_1 and D_2 where D_1, D_2 are products of finitely many basis elements of $\mathfrak{sl}(2, \mathbb{C})$ and define $\bar{f}(x) = f(D_1; x; D_2)$, then also we have the same conclusion.

Now for the general case $f'(x) = f(g_1; x; g_2)$ can be written as a finite sum $\sum_i f_i(x)$, where each f_i is like $\bar{f}$ above. Hence $\mathcal{A}f' = \sum_i \mathcal{A}f_i$ and $\widetilde{\mathcal{A}f'}(\lambda) = \sum_i \widetilde{\mathcal{A}f_i}(\lambda) = \sum_i P_i(\lambda)\widetilde{\mathcal{A}f}(\lambda)$, by the above argument. Let $\mathcal{P}$ be the polynomial $\sum P_i$ and let $\mathcal{D}$ be the corresponding constant coefficient differential operator on $\mathbb{R}$. Then $\widetilde{\mathcal{A}f'}(\lambda) = \mathcal{P}(\lambda)\widetilde{\mathcal{A}f}(\lambda) = \widetilde{\mathcal{D}\mathcal{A}f}(\lambda)$.

Hence by injectivity of the Euclidean–Fourier transform $\mathcal{A}f' = \mathcal{D}\mathcal{A}f$. This completes the proof. □

Definition of the Abel transform makes sense for spherical L^1-functions of type (m, n), $m, n \in 2\mathbb{Z}$. Indeed we have the following result.

PROPOSITION 3.4

Let f be a function in $L^1(G)_{m,n}$. Then its Abel transform $\mathcal{A}f$ is in $L^1(\mathbb{R}, \mathrm{e}^{|t|})$.

Proof. It is easy to see that $|f(kx)| = |f(x)| = |f(xk)|$.

$$\begin{aligned}\int_{\mathbb{R}} |\mathcal{A}f(t)|\mathrm{e}^t\mathrm{d}t &= \int_{\mathbb{R}} \mathrm{e}^{2t}\left|\int_N f(a_t n)\mathrm{d}n\right|\mathrm{d}t \\ &\le \int_{\mathbb{R}}\int_N |f(a_t n)|\mathrm{d}n\mathrm{e}^{2t}\mathrm{d}t \\ &= \int_K\int_{\mathbb{R}}\int_N |f(ka_t n)|\mathrm{d}n\mathrm{e}^{2t}\mathrm{d}t\mathrm{d}k \\ &= \int_G |f(x)|\mathrm{d}x < \infty\end{aligned}$$

using the Iwasawa decomposition $G = KAN$ which has Jacobian e^{2t}. Again

$$\begin{aligned}\int_{\mathbb{R}} |\mathcal{A}f(t)|\mathrm{e}^{-t}\mathrm{d}t &= \int_{\mathbb{R}}\left|\int_N f(a_t n)\mathrm{d}n\right|\mathrm{d}t \\ &\le \int_{\mathbb{R}}\int_N |f(a_t n)|\mathrm{d}n\mathrm{d}t\end{aligned}$$

$$= \int_K \int_{\mathbb{R}} \int_N |f(a_t nk)| \mathrm{d}n \mathrm{d}t \mathrm{d}k$$

$$= \int_G |f(x)| \mathrm{d}x < \infty$$

using the Iwasawa decomposition $G = ANK$ which has Jacobian 1. Therefore

$$\int_{\mathbb{R}} |\mathcal{A}f(t)| \mathrm{e}^{|t|} \mathrm{d}t = \int_0^{\infty} |\mathcal{A}f(t)| \mathrm{e}^t \mathrm{d}t + \int_{-\infty}^{0} |\mathcal{A}f(t)| \mathrm{e}^{-t} \mathrm{d}t < \infty.$$

That is $\mathcal{A}f(t) \in L^1(\mathbb{R}, \mathrm{e}^{|t|})$. □

In particular if $m = n$ then it follows from the relation (3.6) and (3.7) that $\mathcal{A}f$ is an even function, i.e. $\mathcal{A}$ maps $L^1(G)_{n,n}$ to the even functions in $L^1(\mathbb{R}, \mathrm{e}^{|t|})$.

From Proposition 3.4 above it is clear that for a spherical function $f \in L^1(G)$ of type (m, n), Fourier transform of $\mathcal{A}f$ extends to the strip $|\mathrm{Im}\,\lambda| \le 1$. We also obtain a Riemann–Lebesgue lemma for this weighted L^1-space on $\mathbb{R}$, modifying the classical proof (using simple functions). In view of (3.6) the modified Riemann–Lebesgue lemma for $L^1(\mathbb{R}, \mathrm{e}^{|x|})$ implies the following:

Theorem 3.5 (Riemann–Lebesgue lemma). *Let $f \in L^1(G)$ be a spherical function of type (m, n). Then $\hat{f}(\lambda)_{m,n} \longrightarrow 0$ as $|\lambda| \longrightarrow \infty$ uniformly on $\mathcal{S}_1$.*

Note that the proof of the Riemann–Lebesgue lemma is quite elementary. (The original proof in [7] uses isomorphism of Schwartz spaces.)

Abel transform of Schwartz class functions. Apart from integrable functions, the Abel transform also makes sense for any function in the Harish-Chandra L^p-Schwartz spaces $\mathcal{C}^p(G)$, for $0 \le p \le 2$ (see p. 13 of [3] for definition of $\mathcal{C}^p(G)$). As $\mathcal{C}^p(G) \subset \mathcal{C}^2(G)$, it is enough to consider only $\mathcal{C}^2(G)$. Now existence of the Abel transform for $f \in \mathcal{C}^2(G)$ follows from the following result: for any $q > 1$, the integral $\int_N \Xi(a_t n)(1 + \sigma(a_t n))^{-q} \mathrm{d}n$ converges for any real t (see Theorems 6.2.3 and 6.2.4 of [10]). It is not difficult to see that the relation (3.6) between the Fourier transform is also valid for any function $f \in \mathcal{C}^2(G)$. In fact the image of $\mathcal{C}^p(G)_{m,n}$ under the Abel transform is a weighted Schwartz space on $\mathbb{R}$ (see [1] for the case $m = n = 0$, which can be modified for a general (m, n)).

Kernel and inverse of Abel transform. It is clear from the Paley–Wiener theorem (p. 47 of [3]) that for a K-biinvariant function $f \in C_c^{\infty}(G)$, and for every $m \in 2\mathbb{Z}$, there is a function $g \in C_c^{\infty}(G)_{m,m}$ such that $\mathcal{A}f = \mathcal{A}g$. Proposition 3.1 also reveals that unless we restrict the Abel transform on function spaces of fixed left and right K-types, it has a huge kernel. We will see now that even when we consider only functions of fixed left and right types, the Abel transform is not injective.

Consider the space $\mathcal{C}^2(G)$. A function $f \in \mathcal{C}^2(G)$ is called a cusp form if Fourier transform of f with respect to a unitary representation π of G is zero whenever π is induced from a representation which is trivial on the subgroup N. Recall that the unitary principal series representations are examples of such π's. It follows that the space of cusp forms are generated by the matrix coefficients of the discrete series representations $\Psi_k^{m,n}$, in particular the matrix coefficients $\Psi_k^{m,n}$ are cusp forms. Suppose that f is a spherical cusp form of type (m, n). Then by definition $\hat{f}_{m,n}(\lambda) = 0$ for all $\lambda \in i\mathbb{R}$. Now (3.6) shows that the Euclidean–Fourier transform of $\mathcal{A}f$ is zero on $\mathbb{R}$. By injectivity of the Fourier transform on $\mathbb{R}$, this implies that $\mathcal{A}f \equiv 0$, i.e. the cusp forms are in the kernel of the Abel transform. Therefore, one cannot retrieve a spherical function of type (m, n) from its Abel transform, whenever there is discrete series relevant to m, n. If $m \cdot n \leq 0$ then there is no discrete series for a (m, n)-type function. In these cases it is possible to find an inversion formula of the Abel transform, which will be an analogue of Theorem 7.1 in [15]. In general however using the dual of Abel transform we get back only the *principal part* f_{P} of a function f from its Abel transform Af. The principal part f_{P} is the continuous portion of the inversion formula, which is also known as wave-packet (see p. 50 of [3]). We shall illustrate this below.

Let L be the pseudo differential operator on $\mathbb{R}$ with amplitude $\mu(i\nu)$, where $\mu(i\nu)$ is the Plancherel density (see (2.5)). Precisely for $g \in C_c^\infty(\mathbb{R})$, $\widetilde{Lg}(\nu) = \mu(i\nu)\tilde{g}(\nu)$, $\nu \in \mathbb{R}$. Then for $f \in C_c^\infty(G)_{m,n}$,

$$L\mathcal{A}f(t) = \frac{1}{2\pi}\int_{i\mathbb{R}} \mu(i\nu)\hat{f}_{m,n}(i\nu)\mathrm{e}^{i\nu t}\mathrm{d}\nu, \text{ as } \widetilde{\mathcal{A}f}(\nu) = \hat{f}_{m,n}(i\nu) \text{ (see (3.6))}.$$

Hence

$$\begin{aligned}
&\mathcal{A}^*_{-m,-n}L\mathcal{A}f(x)\\
&= \frac{1}{2\pi}\int_K\int_{i\mathbb{R}} \mu(\lambda)\hat{f}_{m,n}(\lambda)\mathrm{e}^{\lambda H(xk^{-1})}\mathrm{e}^{-H(xk^{-1})}\,\mathrm{d}\lambda\, e_{-m}(K(xk^{-1}))e_n(k^{-1})\mathrm{d}k\\
&= \frac{1}{2\pi}\int_{i\mathbb{R}} \hat{f}_{m,n}(\lambda)\Phi_{-\lambda}^{-m,-n}(x^{-1})\mu(\lambda)\mathrm{d}\lambda\\
&= \frac{1}{2\pi}\int_{i\mathbb{R}} \hat{f}_{m,n}(\lambda)\Phi_{\lambda}^{n,m}(x)\mu(\lambda)\mathrm{d}\lambda \text{ (since } \Phi_{-\lambda}^{-m,-n}(x^{-1}) = \Phi_{\lambda}^{n,m}(x))\\
&= 2\pi f_{\mathrm{P}}(x) \text{ (see Theorem 10.2 of [3])}.
\end{aligned}$$

As mentioned earlier, when $m \cdot n \leq 0$, then there is no discrete series representations relevant for (m, n)-type functions and hence $f_{\mathrm{P}} = f$; i.e. in that case we get back the whole function.

We also notice that $\mathcal{A}^*$ is not inverse of $\mathcal{A}$, indeed $\mathcal{A}^*\mathcal{A}$ is a pseudo differential operator on G, given by

$$\mathcal{A}^*_{-m,-n}\mathcal{A}f(x) = \int_{i\mathbb{R}} \hat{f}_{m,n}(\lambda)\frac{1}{\mu(\lambda)}\Phi_{\lambda}^{n,m}(x)\mu(\lambda)\mathrm{d}\lambda \text{ for } f \in C_c^\infty(G)_{m,n}.$$

Remark 3.6. Though Abel transform can be defined for arbitrary functions, the results in this section involve only spherical functions of arbitrary but fixed (left and right) types. Suppose we take a function in $C_c^\infty(G)$ with no restriction on K-types. As mentioned earlier we can decompose f as $f = \sum_{m,n} f_{m,n}$ where each $f_{m,n}$ is of type (m, n). Consequently $\mathcal{A}f = \sum_{m,n}\mathcal{A}f_{m,n}$ and hence for $\nu \in \mathbb{R}$, $\tilde{f}(\nu) = \sum_{m,n}\hat{f}(i\nu)_{m,n}$. This clearly shows that we can not reconstruct the operator Fourier transform from the Abel transform as the information regarding the individual matrix coefficients of the Fourier transform is lost. Thus the Abel transform is useful when the functions are of *pure type*. However in §5 we will show that we can still use the Abel transform to deal with a problem which involves non K-finite functions.

We conclude this section with a discussion on definition of Abel transform and its dual on distributions.

Abel transform and its dual of distributions. A compactly supported distribution of type (m, n) on G is by definition an element in $C^\infty(G)'_{m,n}$, the dual space of $C^\infty(G)_{m,n}$. A distribution of type (m, n) on G is an element in $C_c^\infty(G)'_{m,n}$. Recall that for $f \in C^\infty(\mathbb{R})$, $\mathcal{A}^*_{-m,-n} f \in C^\infty(G)_{m,n}$. Motivated by Proposition 3.2 we can define the Abel transform of $w \in C^\infty(G)'_{m,n}$ by

$$\langle \mathcal{A}w, f\rangle_{\mathbb{R}} = \langle w, \mathcal{A}^*_{-m,-n} f\rangle_G, \quad \text{for all } f \in C^\infty(\mathbb{R}).$$

Here $\langle \cdot, \cdot\rangle_{\mathbb{R}}$ and $\langle \cdot, \cdot\rangle_G$ are L^2-inner products on $\mathbb{R}$ and G respectively. Clearly, $\mathcal{A}w$ is a compactly supported distribution on $\mathbb{R}$. This is implicit in [2] where Bagchi and Sitaram defined Abel transform for compactly supported K-biinvariant distributions. Notice that the *slice* (see (3.6) and (3.7)) is in-built in this definition. Precisely for w as above and $\nu \in \mathbb{C}$, the Euclidean–Fourier transform of $\mathcal{A}w$ at ν is the (m, n)-th spherical Fourier transform of w at $i\nu$.

On the other hand, let w_1 be a distribution on $\mathbb{R}$. Let us recall that $f \in C_c^\infty(G)_{m,n}$ implies $\mathcal{A}f \in C_c^\infty(\mathbb{R})$. Using this $\mathcal{A}^*_{m,n} w_1$ can be defined as

$$\langle \mathcal{A}^*_{m,n} w_1, f\rangle = \langle w_1, \mathcal{A}f\rangle, \quad \text{for all } f \in C^\infty(G)_{m,n}.$$

Clearly $\mathcal{A}^*_{m,n} w_1$ is a distribution of type (m, n) on G.

4. Abel transform of complex measures

We recall that the complex measures are bounded. It is possible to define Abel transform of any bounded measure, though it is useful only when the measure is of some *pure type*. (See Remark 3.6).

A complex measure μ on G is of type (m, n) if for every Borel set E and $k_\theta, k_\phi \in K$, $\mu(k_\theta E k_\phi) = \mathrm{e}^{im\theta}\mathrm{e}^{in\phi}\mu(E)$, equivalently if $\mathrm{d}\mu(k_\theta x k_\phi) = \mathrm{e}^{-im\theta}\mathrm{e}^{-in\phi}\mathrm{d}\mu(x)$. From this definition it follows that a measure μ is of type (m, n), precisely if $\mu(f) = 0$ for any $f \in C_c(G)$ such that $f_{m,n} \equiv 0$. In particular, a measure μ is of type $(0, 0)$ if it is K-biinvariant, i.e. for E as above and for $k_1, k_2 \in K$, $\mu(k_1 E k_2) = \mu(E)$.

For a complex measure μ on G, its (m, n)-th component is given by $\mathrm{d}\mu_{m,n}(x) = \int_{K\times K} \mathrm{d}\mu(k_\theta x k_\phi)$.

We will restrict our attention here only to the K-biinvariant complex measures for brevity. However, results in this section are extendable for complex measures of type (m, n), as we have seen in the case of functions.

Let μ be a K-biinvariant complex measure. We define the Abel transform of μ through its volume element as

$$\mathrm{d}\mathcal{A}\mu(a_t) = \mathrm{e}^{-t}\int_N \mathrm{d}\mu(a_t n).$$

Notice that if $\mathrm{d}\mu(x) = f(x)\mathrm{d}x$ where $f \in L^1(G//K)$ and $\mathrm{d}x$ is the Haar measure of G, then from the definition above it follows that $\mathrm{d}\mathcal{A}\mu(a) = \mathcal{A}f(a)\mathrm{d}a$.

We have the following analogue of (3.6):

$$(\mathcal{A}\mu)\tilde{}(i\lambda) = \hat{\mu}(\lambda) = \mu(\phi_\lambda) = \int_G \phi_\lambda(x^{-1})\mathrm{d}\mu(x). \tag{4.8}$$

As μ is a bounded measure and ϕ_λ is bounded by 1 on $\mathcal{S}_1$, Fourier transform of μ extends to the (closed) strip $\mathcal{S}_1$ and in particular Fourier transform exists at ± 1. We start with the spherical Fourier transform of μ:

$$\hat{\mu}(\lambda) = \int_G \phi_\lambda(x^{-1})\mathrm{d}\mu(x) = \int_G \int_K \mathrm{e}^{-(\lambda+1)H(xk)}\mathrm{d}k\mathrm{d}\mu(x).$$

Using Fubini's theorem and the K-biinvariance of μ we have

$$\hat{\mu}(\lambda) = \int_G \mathrm{e}^{-(\lambda+1)H(y)}\mathrm{d}\mu(y).$$

We use the Iwasawa decomposition $G = KAN$ and write $y = ka_t n$ to obtain

$$\hat{\mu}(\lambda) = \int_{A\times N} \mathrm{e}^{-(\lambda+1)t}\mathrm{d}\mu(a_t n).$$

At $\lambda = \pm 1$, we have

$$\hat{\mu}(1) = \int_{A\times N} \mathrm{e}^{-2t}\mathrm{d}\mu(a_t n) = \int_A \mathrm{e}^{-t}\mathrm{e}^{-t} \int_N \mathrm{d}\mu(a_t n) = \int_A \mathrm{e}^{-t}\mathrm{d}\mathcal{A}\mu(t) < \infty$$

and

$$\hat{\mu}(-1) = \int_{A\times N} \mathrm{d}\mu(a_t n) = \int_A \mathrm{e}^{t}\mathrm{e}^{-t} \int_N \mathrm{d}\mu(a_t n) = \int_A \mathrm{e}^{t}\mathrm{d}\mathcal{A}\mu(t) < \infty.$$

Hence,

$$\int_{\mathbb{R}} \mathrm{e}^{|t|}\mathrm{d}\mathcal{A}\mu(t) = \int_0^\infty \mathrm{e}^{t}\mathrm{d}\mathcal{A}\mu(t) + \int_{-\infty}^0 \mathrm{e}^{-t}\mathrm{d}\mathcal{A}\mu(t) < \infty$$

generalizes Proposition 3.4. This shows that the Euclidean–Fourier transform of $\mathcal{A}\mu$ extends to $\mathcal{S}_1$. Note also that as μ is K-biinvariant, $\mathrm{d}\mu(x) = \mathrm{d}\mu(x^{-1})$, $\hat{\mu}(\lambda) = \hat{\mu}(-\lambda)$ and hence $\mathcal{A}\mu$ is symmetric about the origin.

We define convolution of two K-biinvariant complex measures μ_1 and μ_2 as

$$\mu_1 * \mu_2(f) = \int_{G\times G} f(xy)\mathrm{d}\mu_2(y)\mathrm{d}\mu_1(x), \text{ for } f \in C_c(G//K).$$

Using (2.3) it can be verified that for $\lambda \in i\mathbb{R}$, $(\mu_1 * \mu_2)\hat{}(\lambda) = \widehat{\mu_1}(\lambda)\widehat{\mu_2}(\lambda)$. Now using (4.8), the right-hand side of this equality is $\widetilde{\mathcal{A}\mu_1}(i\lambda)\widetilde{\mathcal{A}\mu_2}(i\lambda)$ which is the Euclidean–Fourier transform of $\mathcal{A}\mu_1 * \mathcal{A}\mu_2$ at $i\lambda$. Using (4.8) again for the left-hand side we see that it is $(\mathcal{A}(\mu_1 * \mu_2))\tilde{}(i\lambda)$. Thus we have $(\mathcal{A}(\mu_1 * \mu_2))\tilde{}(i\lambda) = (\mathcal{A}\mu_1 * \mathcal{A}\mu_2)\tilde{}(i\lambda)$ for $\lambda \in i\mathbb{R}$. By the injectivity of Fourier transform we conclude that $\mathcal{A}\mu_1 * \mathcal{A}\mu_2 = \mathcal{A}\mu_1 * \mathcal{A}\mu_2$.

5. Applications

In a recent article [15], Helgason has listed some nice applications of the Abel transform on various problems which involve K-biinvariant functions and distributions. First we shall add to his list another application of the Abel transform which will deal with (n, n)-type measures.

Let $p_t = \frac{1}{2\sqrt{\pi t}} e^{-x^2/t}$, $t > 0$ be the Gauss kernel on $\mathbb{R}$. Then $\tilde{p}_t(\nu) = e^{-\frac{t}{4}\nu^2}$ for all $\nu \in \mathbb{C}$. A theorem of Cramér (1936) states that if for two probability measures μ_1 and μ_2 on $\mathbb{R}$, which are symmetric about the origin, $\mu_1 * \mu_2 = p_t$, then either one of μ_1 and μ_2 is δ_0 or $\mu_1 = p_{s_1}$ and $\mu_2 = p_{s_2}$ where $s_1 + s_2 = t$.

In the symmetric space G/K, the object corresponding to the Gauss kernel is the heat kernel $\{h_t\}_{t>0}$ which constitutes the fundamental solution of the heat equation $\Delta u = \frac{\partial}{\partial t} u$, of the Laplace–Beltrami operator Δ of G/K. It is well-known that h_t is a K-biinvariant function, defined through its spherical Fourier transform $\hat{h}_t(\lambda) = e^{\frac{t}{4}(\lambda^2-1)}$. Therefore by (4.8) $\widetilde{\mathcal{A}h_t}(i\lambda) = e^{\frac{t}{4}(\lambda^2-1)}$. We state the following analogue of Cramér's theorem due to Graczyk [12] for the symmetric space G/K.

Theorem 5.1 (Graczyk). *Let μ_1, μ_2 be two K-biinvariant probability measures on G/K, neither of which is a Dirac measure at eK. If $\mu_1 * \mu_2 = h_t$ for some $t > 0$, then $\mu_1 = h_{t_1}$ and $\mu_2 = h_{t_2}$, for $t_1, t_2 > 0$ such that $t_1 + t_2 = t$.*

Let us define h_t^n, a spherical function of type (n, n) on G by prescribing its Fourier transform $\widehat{h_t^n}(\lambda)_{n,n} = \int_G h_t^n(x)\Phi_\lambda^{n,n}(x)dx = e^{\frac{t}{4}(\lambda^2-1)}$ for all $\lambda \in \mathbb{C}$. Using isomorphism of Schwartz spaces (see [3]) we see that h_t^n is a function of type (n, n) in $\mathcal{C}^p(G)$ for any $p \in [0, 2]$. Indeed h_t^n is the unique (n, n)-type solution of the (pseudo) heat equation $\left(\Omega - \frac{\partial}{\partial t}\right) f = 0$ such that $\hat{f}(-1) = 1$ (which is a generalization of the condition $\int_G f(x)dx = 1$ in the K-biinvariant case). Here Ω is the Casimir operator of G. If we consider the right n-type functions on G, then h_t^n takes up the role of h_t (see [19] and [20] for details). We call h_t^n, the (n, n)-th heat kernel of G. Now suppose μ_1, μ_2 are two (n, n)-type complex measures such that $\hat{\mu}_j(-1) = 1$, $j = 1, 2$ and $\mu_1 * \mu_2 = h_t^n$. Motivated by Cramér's theorem, one can ask: What condition on μ_j will ensure that both of them are (n, n)-th heat kernel at t_1 and t_2 with $t_1 + t_2 = t$? Note that (n, n)-type functions and measures are necessarily complex valued. Also for a (n, n) type function f (or measure μ), $\int_G f(x)dx = 0 (\int_G d\mu(x) = 0)$ unless $n = 0$. Therefore they cannot be probabilities. It is interesting to note that the Abel transform for such a function can still be a positive function (or positive Radon measure). The (n, n)-th heat kernel h_t^n is one such example.

We define the measure δ_n on G, which is supported on K by $\delta_n(k_\theta) = \frac{1}{2\pi} e^{-in\theta} d\theta$. We will call δ_n as the Dirac measure of type n. If $n = 0$ then $\delta_n = \delta_{eK}$, the Dirac measure of the symmetric space G/K. The (n, n)-th Fourier transform of δ_n is 1 and is given by

$$\hat{\delta}_n(\lambda)_{n,n} = \delta_n(\Phi_\lambda^{n,n}) = \int_G \delta_n(x)\Phi_\lambda^{n,n}(x^{-1})dx = \frac{1}{2\pi}\int_K \Phi_\lambda^{n,n}(k_\theta)e^{in\theta}d\theta$$

$$= \frac{1}{2\pi}\int_K \Phi_\lambda^{n,n}(e)d\theta = 1.$$

It is clear that $\hat{\delta}_n(\lambda)_{i,j} = 0$ whenever either $i \neq n$ or $j \neq n$. It is also easy to verify that $f * \delta_n = f_{\cdot,n}$ and $\delta_n * f = f_{n,\cdot}$, where $f_{\cdot,n}$ and $f_{n,\cdot}$ respectively are the right and left n-th component of f. With this preparation we offer the following generalization of Cramér's theorem. For $n = 0$, this is equivalent to Theorem 5.1.

Theorem 5.2. *Let μ_1 and μ_2 be two complex measures of right type n such that $\widehat{\mu_j}_{n,n}$ are continuous positive definite functions with $\widehat{\mu_j}(-1) = 1$. If $\mu_1 * \mu_2 = h_t^n$ then there are only these three possibilities*:

(i) $\mu_1 = \delta_n$ *and* $(\mu_2)_{n,\cdot} = h_t^n$,
(ii) $\mu_2 = \delta_n$ *and* $\mu_1 = h_t^n$,
(iii) $\mu_1 = h_{t_1}^n$ *and* $(\mu_2)_{n,\cdot} = h_{t_2}^n$ *for some positive* t_1, t_2 *with* $t_1 + t_2 = t$.

Proof. Possibilities (i) and (ii) are obvious. So we are left with the case $\mu_1 \neq \delta_n$ and $(\mu_2)_{n,\cdot} \neq \delta_n$.

It follows from the equality $\mu_1 * \mu_2 = h_t$ that μ_1 is of type (n, n). Let us first assume that μ_2 is also of type (n, n). To make the notation simpler we will write $\widehat{\mu_j}$ for $\hat{\mu}_{n,n}$. We have from the hypothesis $\widehat{\mu_1}(\lambda) \cdot \widehat{\mu_2}(\lambda) = \mathrm{e}^{(t/4)(\lambda^2-1)}$ for $\lambda \in i\mathbb{R}$. In particular, $\widehat{\mu_1}(0)\widehat{\mu_2}(0) = \mathrm{e}^{-(t/4)}$. Therefore

$$(\widehat{\mu_1}(0))^{-1}\widehat{\mu_1}(\lambda) \cdot (\widehat{\mu_2}(0))^{-1}\widehat{\mu_2}(\lambda) = \mathrm{e}^{(t/4)\lambda^2} \text{ for } \lambda \in i\mathbb{R}. \tag{5.1}$$

As $\mu_1 * \mu_2 = h_t^n$ and both μ_1 and μ_2 are of type (n, n), by Proposition 3.1 we have $\mathcal{A}\mu_1 * \mathcal{A}\mu_2 = \mathcal{A}h_t$. Let us denote $\mathrm{e}^{t/4}\mathcal{A}h_t$ by h. Then $\tilde{h}(-i\lambda) = \mathrm{e}^{(t/4)\lambda^2}$ for $\lambda \in i\mathbb{R}$, i.e. h is a Gaussian. Let $\nu_j(x) = (\widehat{\mu_j}(0))^{-1}\mathcal{A}\mu_j$ for $j = 1, 2$. Then

(1) $\nu_1 * \nu_2 = h$ by (5.1).
(2) $\int_{\mathbb{R}} \mathrm{d}\nu_j = 1$ as $\int_{\mathbb{R}} \mathrm{d}\mathcal{A}\mu_j = \widetilde{\mathcal{A}\mu_j}(0) = \widehat{\mu_j}(0)$.
(3) The measures ν_j are symmetric about the origin as $\tilde{\nu}_j = (\widehat{\mu_j}(0))^{-1}\widehat{\mu_j}$ is an even function.
(4) Since, $\widehat{\mu_j}$ are continuous positive definite functions, by Euclidean Bochner's theorem ν_j are positive Radon measures and hence are probabilities by (2).

We apply Euclidean Cramér's theorem (see p. 525 of [8]) to get $\nu_1 = p_{t_1}$ and $\nu_2 = p_{t_2}$ where $t_1 + t_2 = t$ and $\widetilde{p_{t_j}}(-i\lambda) = \mathrm{e}^{(t_j/4)\lambda^2}$, for $j = 1, 2$.

Now since $\widehat{\mu_j}(1) = 1 = \widetilde{\mathcal{A}\mu_j}(-i)$ and $\widetilde{\mathcal{A}\mu_j}(-i) = \widehat{\mu_j}(0)\widetilde{p_{t_j}}(-i) = \widehat{\mu_j}(0)\mathrm{e}^{t_j/4}$, we have $\widehat{\mu_j}(0) = \mathrm{e}^{-t_j/4}$. Therefore $\widehat{\mu_j}(\lambda) = \widetilde{\mathcal{A}\mu_j}(-i\lambda) = \widehat{\mu_j}(0)\widetilde{p_{t_j}}(-i\lambda) = \mathrm{e}^{((\lambda^2-1)/4)t_j}$. That is $\mu_j = h_{t_j}^n, = 1, 2$.

When μ_2 is not (n, n)-type then $\mu_1 * \mu_2 = \mu_1 * (\mu_2)_{n,\cdot}$ and we can proceed as above with μ_1 and $(\mu_2)_{n,\cdot}$ instead of μ_1 and μ_2. □

It is also possible to formulate a Cramér's theorem for complex measures on G of pure types. If a measure μ is of type (m, n) say, then by $\hat{\mu}$ we mean the (m, n)-th Fourier coefficient of μ.

PROPOSITION 5.3

Let μ_1 *and* μ_2 *be two complex measures of pure* (*left and right*) *types on G such that* $\widehat{\mu_j}$ *is continuous positive definite functions with* $\widehat{\mu_j}(-1) = 1$, $j = 1, 2$. *If* $\mu_1 * \mu_2 = h_t^n$ *then either one of* μ_1 *and* μ_2 *is* δ_n *or* $\mu_1 = h_{t_1}^n$ *and* $\mu_2 = h_{t_2}^n$ *for some positive* t_1, t_2 *with* $t_1 + t_2 = t$.

Proof. It is clear that the left type of μ_1 and right type of μ_2 are n. Also, right type of μ_1 is equal to the left type of μ_2, since $\mu_1 * \mu_2 \not\equiv 0$. We suppose that μ_1, μ_2 are of type (n, r) and (r, n) respectively for some $r \in 2\mathbb{Z}$. We assume that $r \neq n$.

Case 1. $n = 0$. This is not possible since if $r \neq 0$ then $\Phi_{-1}^{0,r} \equiv 0$ (see Proposition 7.1 of [3]) and consequently $\widehat{\mu_1}(-1) = 0$.

Case 2. $n \neq 0$. As $\mu_1 * \mu_2 = h_t^n$, $\widehat{\mu_1}(l)\widehat{\mu_2}(l) \neq 0$ for all discrete series representations π_l containing the vector e_n. If $n > r$ then $\widehat{\mu_1}(l) = 0$ for l in $r < l < n, l \in 2\mathbb{Z}+1$ as the discrete series representation π_l for such l does not contain e_r. If $n < r$ we can change the role of n and r in the argument above and conclude that $\widehat{\mu_2}(l) = 0$ for l in $n < l < r, l \in 2\mathbb{Z}+1$ (see p. 16 of [3]).

Thus the only possibility is $r = n$. Rest of the argument is same as that in the theorem above. □

Our next application of the Abel transform is on a problem which deals with L^2 functions on the group without any K-finiteness assumption.

The following theorem is an example of an *uncertainty principle* (see [13], [16] and [9] for a comprehensive survey on uncertainty principle). See also [25, 6, 19, 24, 26] and the references therein.

Theorem 5.4 (Gelfand–Shilov). *Let $f \in L^2(\mathbb{R})$. Suppose f satisfies*

$$(1) \qquad \int_{\mathbb{R}} |f(x)| e^{\frac{(\alpha|x|)^p}{p}} dx < \infty,$$

$$(2) \qquad \int_{\mathbb{R}} |\tilde{f}(y)| e^{\frac{(\beta|y|)^q}{q}} dy < \infty, \tag{5.2}$$

where $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $q \leq 2$ and $\alpha\beta > (\sin\frac{\pi}{2}(q-1))^{1/q}$. Then $f = 0$ almost everywhere.

For $p = q = 2$ this is a special case of the Cowling–Price theorem [5]. For $p \neq 2$, it can be extracted from [18, 11]. We shall prove its analogue for G. Our method will be reducing the problem to the Euclidean case using the Abel transform and applying Theorem 5.4.

Theorem 5.5. *Let $f \in L^2(G)$. Suppose f satisfies*

$$(1) \qquad \int_G |f(x)| e^{\frac{(\alpha\sigma(x))^p}{p}} \Xi(x) dx < \infty,$$

$$(2) \qquad \int_{i\mathbb{R}} \|\hat{f}(\sigma, \lambda)\|_2 e^{\frac{(\beta|\lambda|)^q}{q}} d\mu(\sigma, \lambda) < \infty, \tag{5.3}$$

where $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha\beta > \left(\sin\frac{\pi}{2}(\gamma-1)\right)^{1/\gamma}$ where $\gamma = \min\{p, q\}$. Then $f = 0$ almost everywhere. Here $\|\cdot\|_2$ is the Hilbert–Schmidt norm.

Proof. We will only deal with the case $p \neq 2$. The case $p = 2$ follows from the Beurling's theorem [21].

From condition (1) it follows that f is also an integrable function. We have $f = \sum_{m,n\in\mathbb{Z}} f_{m,n}$ in the sense of distributions on G. Note that for each $m, n \in 2\mathbb{Z}$, $f_{m,n}$ is in $L^1(G) \cap L^2(G)$, since $f \in L^1(G) \cap L^2(G)$. Let us take two arbitrary $m, n \in 2\mathbb{Z}$. As $|f_{m,n}(x)| \leq \int_{K\times K} |f(k_1 x k_2)| dk_1 dk_2$, $\sigma(x)$ and $\Xi(x)$ are K-biinvariant functions and the Haar measure dx is invariant under the transformation $x \mapsto k_1 x k_2$. We can substitute f by $f_{m,n}$ in (1) of (5.3). Also in (2) of (5.3) we can substitute $\|\hat{f}(\lambda)\|_2$ by $|\hat{f}_{m,n}(\lambda)|$ as $\|\hat{f}(\lambda)\|_2^2 = \sum_{m,n} |\hat{f}_{m,n}(\lambda)|^2$. Thus we get

$$\text{(1)} \qquad \int_G |f_{m,n}(x)| \mathrm{e}^{\frac{(\alpha\sigma(x))^p}{p}} \Xi(x) \mathrm{d}x < \infty,$$

$$\text{(2)} \qquad \int_{i\mathbb{R}} |\hat{f}(\lambda)_{m,n}| \mathrm{e}^{\frac{(\beta|\lambda|)^q}{q}} \mathrm{d}\mu(\lambda) < \infty. \tag{5.4}$$

We will now show that condition (1) in (5.4) implies

$$I = \int_G |f_{m,n}(x)| \mathrm{e}^{\frac{(\alpha\sigma(x))^p}{p}} \mathrm{e}^{-H(x)} \mathrm{d}x < \infty. \tag{5.5}$$

For $k \in K$, we substitute x by xk^{-1} in I, use the right K-invariance of $|f_{mn}(x)|$ and $\sigma(x)$ and the right invariance of the Haar measure to obtain

$$I = I(k) = \int_G |f_{m,n}(x)| \mathrm{e}^{\frac{(\alpha\sigma(x))^p}{p}} \mathrm{e}^{-H(xk^{-1})} \mathrm{d}x.$$

Therefore,

$$I = \int_K I(k) \mathrm{d}k = \int_K \int_G |f_{m,n}(x)| \mathrm{e}^{\frac{(\alpha\sigma(x))^p}{p}} \mathrm{e}^{-H(xk^{-1})} \mathrm{d}x.$$

Since $\int_K \mathrm{e}^{-H(xk^{-1})} \mathrm{d}k = \Xi(x^{-1}) = \Xi(x)$, we conclude from (1) of (5.4) that $I < \infty$.

Now we use the Iwasawa decomposition $G = KAN$ which has Jacobian e^{2t} and write $x = ka_t n$ in (5.5). Using the left K invariance of $|f_{m,n}(x)|$ and $\sigma(x)$ and as $H(ka_t n) = t$ we obtain

$$\int_{\mathbb{R}\times N} |f_{m,n}(a_t n)| \mathrm{e}^{\frac{(\alpha\sigma(a_t n))^p}{p}} \mathrm{e}^{-t} \mathrm{e}^{2t} \mathrm{d}t \mathrm{d}n < \infty.$$

Since $\sigma(a_t n) \geq \sigma(a_t) = |t|$ we have

$$\int_{\mathbb{R}\times N} |f_{m,n}(a_t n)| \mathrm{e}^{\frac{(\alpha|t|)^p}{p}} \mathrm{e}^{t} \mathrm{d}t \mathrm{d}n = \int_{\mathbb{R}} \mathcal{A}(|f_{m,n}|(t) \mathrm{e}^{\frac{(\alpha|t|)^p}{p}} \mathrm{d}t < \infty.$$

Notice that $\mathcal{A}|f_{m,n}|(t) \geq |\mathcal{A} f_{m,n}(t)|$. Thus we have

$$\int_{\mathbb{R}} |\mathcal{A}(f_{m,n}(t)| \mathrm{e}^{\frac{(\alpha|t|)^p}{p}} \mathrm{d}t < \infty. \tag{5.6}$$

Now we take up condition (2) of (5.4). We observe that by (2.5) there exists $R > 0$ such that $\mu(i\lambda) \geq \frac{1}{2}$ say for all $\lambda \in \mathbb{R}$ with $|\lambda| \geq R$, because $\mu(i\lambda) \longrightarrow \infty$ as $|\lambda| \longrightarrow \infty$. Therefore, $\int_{\{\lambda\in i\mathbb{R}\colon |\lambda|>\mathbb{R}\}} |\hat{f}(\lambda)_{m,n}| \mathrm{e}^{\frac{(\beta|\lambda|)^q}{q}} \mathrm{d}\lambda < \infty$. Also, as f is integrable and hence by Riemann–Lebesgue lemma $\hat{f}_{m,n}(\lambda)$ is a continuous function of λ, $\int_{-iR}^{iR} |\hat{f}(\lambda)_{m,n}| \mathrm{e}^{\frac{(\beta|\lambda|)^q}{q}} \mathrm{d}\lambda < \infty$. That is

$$\int_{i\mathbb{R}} |\hat{f}(\lambda)_{m,n}| \mathrm{e}^{\frac{(\beta|\lambda|)^q}{q}} \mathrm{d}\lambda < \infty. \tag{5.7}$$

In view of (3.6), the above inequality can be rewritten as

$$\int_{\mathbb{R}} |\mathcal{F}\mathcal{A} f_{m,n}(\lambda)| \mathrm{e}^{\frac{(\beta|\lambda|)^q}{q}} \mathrm{d}\lambda < \infty. \tag{5.8}$$

As $f_{m,n} \in L^1(G)$, $\mathcal{A}f_{m,n} \in L^1(\mathbb{R})$ (see Proposition 3.4). Again as $f_{m,n} \in L^2(G)$, by Plancherel theorem $\hat{f}_{m,n} \in L^2(i\mathbb{R}, \mu(\lambda)\mathrm{d}\lambda)$. As mentioned above $\hat{f}_{m,n}$ is continuous and $\mu(\lambda)$ is an increasing function. Therefore $\hat{f}_{m,n} \in L^2(i\mathbb{R}, \mathrm{d}\lambda)$. From this, (3.6) and the Euclidean Plancherel theorem we conclude that $\mathcal{A}f_{m,n} \in L^2(\mathbb{R})$. Thus $\mathcal{A}f_{m,n} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

The inequalities (5.6) and (5.8) are the two conditions of Theorem 5.4 for the pair $(\mathcal{A}f_{m,n}, \hat{f}_{m,n})$. Note that if necessary, interchanging the role of the function with its Euclidean–Fourier transform in Theorem 5.4, we can assume without loss of generality that $q < p$ or equivalently $q < 2$.

Thus by Theorem 5.4 we conclude that $\hat{f}(\lambda)_{m,n} = 0$ for $\lambda \in i\mathbb{R}$ or equivalently $\mathcal{A}f_{m,n} \equiv 0$, i.e. $f_{m,n}$ is a cusp form (see kernel of Abel transform in §3). That is $f_{m,n}$ can have nonzero Fourier transform only at the discrete representations.

We state a result extracted from [6]. Suppose f is of sufficiently rapid decay such that $\hat{f}_{m,n}$ extends to an entire function on $\mathbb{C}$. Now if $\hat{f}_{m,n} \equiv 0$ on the unitary principle series, then $\hat{f}_{m,n}(k) = 0$ for all discrete series.

Therefore $f_{m,n}$ cannot satisfy condition (1) of (5.4) unless it is zero. In view of the decomposition $f = \sum_{m,n \in 2\mathbb{Z}} f_{m,n}$ in the sense of distribution, we conclude that $f = 0$ almost everywhere. □

Remark 5.6.

1. If we restrict our attention to functions whose right type is fixed, then we can start from a weaker assumption. We will illustrate this below.
   Suppose $f \in L^2(G)$ is a function of right type n for some $n \in 2\mathbb{Z}$ which satisfies

$$\int_G |f(x)| \mathrm{e}^{\frac{\alpha|H(x)|^p}{p}} \mathrm{e}^{-H(x)} \mathrm{d}x < \infty \tag{5.9}$$

   and the condition (2) of Theorem 5.5. Then we can show that $f = 0$ almost everywhere. Here is a sketch of the proof.

   Note that for any $m \in 2\mathbb{Z}$ the left m-th component of f also satisfies (5.9) as H is left K-invariant. Thus without loss of generality we can assume that f is an (m, n)-type function. Therefore $|f|$ is a K-biinvariant function. It is clear that $f \in L^1(G)$. We want to show that $\int_{\mathbb{R}} \mathcal{A}|f|(t)\mathrm{e}^{\frac{(\alpha|t|)^p}{p}} \mathrm{d}t < \infty$. But using the dual of Abel transform this is equivalent to $\int_G |f(x)| \mathcal{A}^*(\mathrm{e}^{\frac{(\alpha|\cdot|)^p}{p}})(x)\mathrm{d}x < \infty$. Using the definition of $\mathcal{A}^*$ and the right K-invariance of f, we see that this is precisely condition (5.9). This completes the proof.

   We recall that for this class of functions condition (1) in Theorem 5.5 is equivalent to (5.4). Now since $|H(x)| \leq \sigma(x)$ for all $x \in G$, it is clear that (5.9) is weaker than condition (1) in Theorem 5.5.
2. There is some misunderstanding regarding the implication of Beurling's theorem. As noted in [4], even on $\mathbb{R}^n$, Beurling's theorem does not imply the Gelfand–Shilov theorem. However the following weak version of it does follow from Beurling's theorem: A function $f \in L^2(G)$ which satisfies conditions (1) and (2) of Theorem 5.5 is 0 almost everywhere if $\alpha \cdot \beta > 1$ (see [21] for a proof of Beurling's theorem on $SL(2, \mathbb{R})$).
3. Some other well-known uncertainty theorems like Hardy, Morgan, Cowling–Price etc. follow from this Gelfand–Shilov theorem. The mutual dependencies of these

uncertainty theorems can be schematically displayed as follows:

$$\begin{array}{ccc} \text{Gelfand–Shilov} & \Rightarrow & \text{Cowling–Price} \\ \Downarrow & & \Downarrow \\ \text{Morgan's} & \Rightarrow & \text{Hardy's} \end{array}.$$

Hardy's, Morgan's and Cowling–Price theorems were proved independently on certain semisimple Lie groups and for their symmetric spaces in recent years by many authors (see [9, 6, 26] and the references therein for statements and proofs).

6. Concluding remarks

So far we have considered the Abel transform for functions (distributions, measures) of arbitrary but pure spherical type. As a next step, one will be interested to generalize the results for functions having a fixed right type, but which perhaps have more than one isotypic component on the left, and in particular the functions which may not be K-finite on left.

Recall that $C_c^\infty(G/K)_n$ is the subspace of right n-type functions in $C_c^\infty(G)$ for some $n \in 2\mathbb{Z}$. We have pointed out (see Remark 3.6) that though it is possible to define Abel transform of such functions, it may not be useful. In particular, Abel transform is not injective for such class of functions. As a remedy, we will consider *Radon transform* for such functions. This Radon transform is an analogue of the Radon transform on $\mathbb{R}^n$.

We define Radon transform for $f \in C_c^\infty(G/K)_n$ by

$$\mathcal{R}f(k,t) = \mathrm{e}^t \int_N f(ka_t n)\mathrm{d}n.$$

For f as above, we define its Helgason–Fourier transform by (see [22])

$$\tilde{f}(\lambda,k,n) = \int_G f(x)\mathrm{e}^{(\lambda-1)H(x^{-1}k)} e_n(K(x^{-1}k))\mathrm{d}x, \ \text{ for } \lambda \in \mathfrak{a}^*, k \in K \text{ and } n \in 2\mathbb{Z}. \tag{6.1}$$

For a spherical function f of type (m,n), its Radon transform is related to its Abel transform by $\langle \mathcal{R}f(\cdot,t), e_m\rangle_{L^2(K)} = \mathcal{A}f(t)$ and its Helgason–Fourier transform is related to the matrix coefficients of its group Fourier transform by $\langle \tilde{f}(\lambda,k,n), e_m\rangle_{L^2(K)} = \hat{f}_{m,n}(\lambda)$.

The following analogue of (3.7) is true for $f \in C_c^\infty(G/K)_n$.

$$\tilde{f}(\lambda,k,n) = \int_{\mathbb{R}} \mathcal{R}f(k,t)\mathrm{e}^{-\lambda t} = \mathcal{F}\mathcal{R}f(k,\cdot)(-i\lambda) \text{ for } \lambda \in i\mathbb{R}. \tag{6.2}$$

That is, the Helgason–Fourier transform of a function in $C_c^\infty(G/K)_n$ factors as Euclidean–Fourier transform of its Radon transform.

The definition of Radon transform makes sense for function in $L^1(G/K)_n$. Indeed if $f \in L^1(G/K)_n$ then $\mathcal{R}f(k,\cdot) \in L^1(\mathbb{R}, \mathrm{e}^{|t|})$ for almost every $k \in K$. Relation (6.2) is valid for any $f \in L^1(G/K)_n$ for almost every $k \in K$ and for all $\lambda \in \mathcal{S}_1$. A Riemann–Lebesgue lemma can be proved as a consequence. In a forthcoming paper we shall take up various mapping properties of the Abel and Radon transform for functions of type (m,n) and right type n respectively.

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