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Closed graph and open mapping theorems for normed cones

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Abstract. A quasi-normed cone is a pair (X, p) such that X is a (not necessarily cancellative) cone and q is a quasi-norm on X. The aim of this paper is to prove a closed graph and an open mapping type theorem for quasi-normed cones. This is done with the help of appropriate notions of completeness, continuity and openness that arise in a natural way from the setting of bitopological spaces.

Keywords. Extended quasi-pseudo-metric; quasi-normed cone; continuous mapping; left(right) K-completeness; closed graph; open mapping.

1. Introduction

In the last few years there is a growing interest in the theory of quasi-metric spaces and other related structures such as quasi-normed cones and asymmetric normed linear spaces, because such a theory provides an important tool in the study of several problems in theoretical computer science, approximation theory, applied physics, convex analysis and optimization (see [15], [8], [23], [6], [27], [28], [29], [20], [26], [4], [31] etc.). Many works on general topology and functional analysis have recently been obtained in order to extend the well-known results of the classical theory of normed linear spaces to the framework of asymmetric normed linear spaces and quasi-normed cones. In particular, the dual of an asymmetric normed linear space has been constructed and studied in [10], showing that such a dual space is not a linear space in general but it has a quasi-normed cone structure. Furthermore, in the same reference, it was introduced and characterized the weak topology that is generated by an asymmetric normed linear space and its dual, and, in addition, it was used to prove an extension of the celebrated Alaouglu theorem. Some of the results in [10] have been generalized for quasi-normed cones in [21], [22] and [11], and for topological semicones in [16]. A version of the Heine–Borel theorem for asymmetric linear spaces has been obtained in [7]. Hahn–Banach type theorems for linear functionals on quasi-normed cones (with applications to asymmetric normed linear spaces) have been given in [25] (for related extension theorems see [1] and [30]). The completion of quasi-normed cones has been explored in [18] (compare [9]). The well-known method for generating quotient spaces from normed linear spaces has been generalized for quasi-normed cones in [32].

On the other hand, the notions of preopen set and almost continuous mapping play a very important role in general topology and functional analysis. In particular, they are a key in obtaining open mapping theorems and closed graph theorems on a general context of topological and bitopological spaces (see, for instance, [17], [12], [13] and [3]).

The purpose of this paper is to continue the study of quasi-normed cones and asymmetric normed linear spaces. Thus, in §3 a suitable notion of almost continuity (in the sense of [3]) for mappings between quasi-normed cones is introduced and a closed graph type theorem is obtained. Section 4 is devoted to prove an open mapping type theorem which involves an adapted concept of preopenness from [13]. Finally, extensions of our theorems to the context of quasi-normed monoids are also given.

2. Preliminaries

Throughout this paper the letters $\mathbb{R}$ and $\mathbb{R}^+$ will denote the set of real numbers and the set of nonnegative real numbers, respectively.

Recall that a *monoid* is a semigroup $(X, +)$ with neutral element 0.

According to [6], a *cone* (on $\mathbb{R}^+$) is a triple $(X, +, \cdot)$ such that $(X, +)$ is an Abelian monoid, and $\cdot$ is a function from $\mathbb{R}^+ \times X$ to X such that for all $x, y \in X$ and $r, s \in \mathbb{R}^+$:

(i) $r \cdot (s \cdot x) = (rs) \cdot x$;
(ii) $r \cdot (x + y) = (r \cdot x) + (r \cdot y)$;
(iii) $(r + s) \cdot x = (r \cdot x) + (s \cdot x)$;
(iv) $1 \cdot x = x$;
(v) $0 \cdot x = 0$.

A cone $(X, +, \cdot)$ is called *cancellative* if for all $x, y, z \in X, z+x = z+y$ implies $x = y$.

Obviously, every linear space $(X, +, \cdot)$ can be considered as a cancellative cone when we restrict the operation $\cdot$ to $\mathbb{R}^+ \times X$.

Let us recall that a *linear mapping* from a cone $(X, +, \cdot)$ to a cone $(Y, +, \cdot)$ is a mapping $f: X \to Y$ such that $f(\alpha \cdot x + \beta \cdot y) = \alpha \cdot f(x) + \beta \cdot f(y)$.

A *quasi-norm* on a cone $(X, +, \cdot)$ is a function $p: X \to \mathbb{R}^+$ such that for all $x, y \in X$ and $r \in \mathbb{R}^+$:

(i) $x = 0$ if and only if there is $-x \in X$ and $p(x) = p(-x) = 0$;
(ii) $p(r \cdot x) = rp(x)$;
(iii) $p(x + y) \leq p(x) + p(y)$.

If the quasi-norm p satisfies: (i′) $p(x) = 0$ if and only if $x = 0$, then p is called a *norm* on the cone $(X, +, \cdot)$.

A quasi-norm defined on a linear space is called an *asymmetric norm* in [8], [9] and [10].

Similarly to [11], *a quasi-normed cone* is a pair (X, p) where X is a cone and p is a quasi-norm on X.

Our main references for quasi-pseudo-metric spaces are [5] and [15].

Let us recall that a *quasi-pseudo-metric* on a set X is a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X$:

(i) $d(x, x) = 0$;
(ii) $d(x, z) \leq d(x, y) + d(y, z)$.

In our context, by a *quasi-metric on X* we mean a quasi-pseudo-metric d on X that satisfies the following condition: $d(x, y) = d(y, x) = 0$ if and only if $x = y$.

We will also consider *extended quasi-(pseudo-)metrics*. They satisfy the above three axioms, except that we allow $d(x, y) = +\infty$ whenever $x \neq y$.

If d is a (n extended) quasi-pseudo-metric on X, then the function d^{-1} defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$ for all $x, y \in X$, is also a (n extended) quasi-pseudo-metric on X, called the conjugate (extended) quasi-pseudo-metric of d.

A (*n extended*) *quasi-pseudo-metric space* is a pair (X, d) such that X is a set and d is a (n extended) quasi-pseudo-metric on X. The (extended) quasi-pseudo-metric space (X, d^{-1}) is called the *conjugate* of (X, d).

If d is a (n extended) quasi(-pseudo)-metric on a set X, then the function d^s defined on $X \times X$ by $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ is a (n extended) (pseudo-)metric on X.

Each extended quasi-pseudo-metric d on a set X induces a topology $\mathcal{T}(d)$ on X which has as a base the family of open d-balls $\{B_d(x, r)\colon x \in X, r > 0\}$, where $B_d(x, r) = \{y \in X\colon d(x, y) < r\}$ for all $x \in X$ and $r > 0$.

According to [2], an extended quasi(-pseudo)-metric d on a set X is said to be *bicomplete* if d^s is a complete extended (pseudo-)metric on X.

Following [19], a sequence in an extended quasi-pseudo-metric space (X, d) is left(resp. right) K-Cauchy if for each $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ (resp. $d(x_m, x_n) < \varepsilon$) for all $m \geq n \geq k$. The extended quasi-pseudo-metric space (X, d) is called left(resp. right) K-sequentially complete if each left(resp. right) K-Cauchy sequence is convergent to some point in (X, d).

Recall that if f is a mapping from the set X to the set Y, the graph of f is the subset $G(f)$ of $X \times Y$ defined by $G(f) = \{(x, f(x))\colon x \in X\}$.

Bitopological spaces appear in a natural way when one considers a (n extended) quasi-pseudo-metric space and its conjugate. As usual, a bitopological space is a triple $(X, \mathcal{T}_1, \mathcal{T}_2)$, where X is a nonempty set and $\mathcal{T}_1, \mathcal{T}_2$ are topologies defined on X (see [14]). A bitopological space is said to be *quasi-pseudo-metrizable* if there exists an extended quasi-pseudo-metric d on X such that $\mathcal{T}(d) = \mathcal{T}_1$ and $\mathcal{T}(d^{-1}) = \mathcal{T}_2$.

Given a topological space $(X, \mathcal{T})$ and $A \subseteq X$, we denote by $\mathrm{cl}_{\mathcal{T}} A$ (resp. $\mathrm{Int}_{\mathcal{T}}(A)$) the closure (resp. the interior) of A with respect to $\mathcal{T}$.

From now on, given an extended quasi-pseudo-metric space (X, d) and $A \subseteq X$, for the sake of brevity we will denote $\mathrm{cl}_{\mathcal{T}(d)} A$ (resp. $\mathrm{cl}_{\mathcal{T}(d^{-1})} A$) by $\mathrm{cl}_d A$ (resp. $\mathrm{cl}_{d^{-1}} A$) and $\mathrm{Int}_{\mathcal{T}(d)}(A)$ (resp. $\mathrm{Int}_{\mathcal{T}(d^{-1})}(A)$) by $\mathrm{Int}_d(A)$ (resp. $\mathrm{Int}_{d^{-1}}(A)$).

Recently, it was shown that it is possible to generate in a natural way extended quasi-pseudo-metrics from quasi-norms on cones (not necessarily cancellative), extending the well-known result that establishes that each norm on a linear space X induces a metric on X (see [11] and compare [24]). In particular, it was proved that the following result plays a crucial role in our work.

PROPOSITION 1 [24,11]

Let p be a quasi-norm on a cone $(X, +, \cdot)$. Then the function d_p defined on $X \times X$ by

$$d_p(x, y) = \begin{cases} \inf\{p(a)\colon y = x + a\}, & \textit{if } y \in x + X \\ +\infty, & \textit{otherwise} \end{cases}$$

is an extended quasi-pseudo-metric on X. Furthermore, for each $x \in X$, $r \in \mathbb{R}^+ \backslash \{0\}$ and each $\varepsilon > 0$, $rB_{d_p}(x, \varepsilon) = rx + \{y \in X\colon p(y) < r\varepsilon\}$, and the translations with respect to $+$ and $\cdot$ are $\mathcal{T}(d_p)$-open.

The Sorgenfrey topology and the Alexandroff topology are obtained as particular cases of the above construction (see [11] and [24]).

It is obvious, from Proposition 1, that every quasi-normed cone (X, p) can be considered as a quasi-pseudo-metrizable bitopological space endowed with the topologies $\mathcal{T}(d_p)$ and $\mathcal{T}(d_p^{-1})$.

A quasi-normed cone is *bicomplete* provided that the induced extended quasi-pseudo-metric is bicomplete.

We end the section recalling that for subsets A and B of a quasi-normed cone (X, p) it is clear that $\mathrm{cl}_{d_p} A + \mathrm{cl}_{d_p} B \subset \mathrm{cl}_{d_p}(A + B)$ (resp. $\mathrm{cl}_{d_p^{-1}} A + \mathrm{cl}_{d_p^{-1}} B \subset \mathrm{cl}_{d_p^{-1}}(A + B)$.

3. Closed graph theorem for normed cones

The result for normed linear spaces usually called the closed graph theorem is formulated as follows.

Theorem 2. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two Banach spaces. Then every closed graph linear mapping from X into Y is continuous.*

A natural way of extending the preceding result to the context of normed cones consists of replacing Banach spaces by bicomplete quasi-normed cones. Thus one may conjecture that the next result would be desirable.

"*Let (X, p) and (Y, q) be two bicomplete quasi-normed cones. Then every linear mapping $f\colon X \to Y$ with closed graph in $(X \times Y, d_p^s \times d_q^s)$ is continuous from $(X, \mathcal{T}(d_p))$ to $(Y, \mathcal{T}(d_q))$.*"

However the following example shows that such a result does not hold in our context.

Example 3. On $\mathbb{R}$ define the function u by $u(x) = x \vee 0$. Then u is clearly a quasi-norm on $\mathbb{R}$ whose induced (extended) quasi-metric is given by $d_u(x, y) = (y - x) \vee 0$. The topology $\mathcal{T}(d_u)$ is the so-called upper topology on $\mathbb{R}$. Moreover $(\mathbb{R}, u)$ is a bicomplete quasi-normed cone because d_u^s is exactly the Euclidean metric on $\mathbb{R}$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = -x$. It is clear that f is linear with closed-graph in $(\mathbb{R} \times \mathbb{R}, d_u^s \times d_u^s)$ but it is not continuous at 0.

In the light of the preceding example and taking into account that every quasi-normed cone can be considered as a quasi-pseudo-metrizable bitopological space, we recall the necessary notions from bitopological spaces with the aim of proving a closed graph type theorem in our context.

According to [12] a mapping $f\colon (X, \mathcal{T}) \to (Y, \mathcal{S})$ is said to be *almost continuous* at $x \in X$ if and only if for each open V containing $f(x)$, $\mathrm{cl}_{\mathcal{T}}(f^{-1}(V))$ is a neighborhood of x. Moreover, if f is almost continuous at each point of X, then f is called *almost continuous*. We observe that a continuous mapping f is always almost continuous.

Following [3] we say that a mapping f from the bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ to the bitopological space $(Y, \mathcal{S}_1, \mathcal{S}_2)$ is (i, j)-*lower almost continuous* at $x \in X$ if for each subset $V \in \mathcal{S}_i$, with $f(x) \in V$, $\mathrm{cl}_{\mathcal{T}_j}(f^{-1}(V))$ is a neighborhood of x with respect to $\mathcal{T}_i$, where $i \neq j$ and $i, j \in \{1, 2\}$. Thus, f is called (i, j)-*lower almost continuous* if f is (i, j)-lower almost continuous at each point of X.

Next we adapt the former notion of almost continuity to our asymmetric framework.

DEFINITION 4

Let (X, p) and (Y, q) be two quasi-normed cones. A mapping $f: X \to Y$ is (d_p^{-1}, d_q)-*almost continuous* at $x \in X$ if for each open subset $V \in \mathcal{T}(d_q)$ such that $f(x) \in V$, $\mathrm{cl}_{d_p^{-1}}(f^{-1}(V))$ is a neighborhood of x with respect to $\mathcal{T}(d_p)$. We will say that f is (d_p^{-1}, d_q)-*almost continuous* if f is (d_p^{-1}, d_q)-almost continuous at each point of X.

The next lemma, which will be useful later on, characterizes continuity of linear mappings defined between quasi-normed cones. We omit its proof because it is straightforward from its counterpart for Banach spaces.

Lemma 5. *Let* (X, p) *and* (Y, q) *be two quasi-normed cones and let* $f: X \to Y$ *be a linear mapping. Then the following statements are equivalent*:

(1) f *is continuous.*
(2) f *is continuous at* 0.

The following technical result is due to Cao and Reilly [3].

Lemma 6. *Let* (X, d_1) *and* (Y, d_2) *be two (extended) quasi-pseudo-metric spaces and let* $f: X \to Y$ *be a mapping with closed graph in* $(X \times Y, d_1^{-1} \times d_2^{-1})$. *If* (Y, d_2^{-1}) *is right* K*-sequentially complete and for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $B_{d_1}(x, \delta) \subseteq \mathrm{cl}_{d_1^{-1}}(f^{-1}(B_{d_2}(y, \varepsilon)))$ *for all pairs* $(x, y) \in G(f)$, *then* $\mathrm{cl}_{d_1^{-1}}(f^{-1}(B_{d_2}(y, \varepsilon))) \subseteq f^{-1}(B_{d_2}(y, \eta))$ *for each point* $y \in Y$ *and for all* $\eta > \varepsilon > 0$.

Theorem 7 (Closed graph theorem). *Let* (X, p) *and* (Y, q) *be two quasi-normed cones such that the extended quasi-metric space* (Y, d_q^{-1}) *is right* K*-sequentially complete. If* $f: X \to Y$ *is a linear mapping with closed graph in* $(X \times Y, d_p^{-1} \times d_q^{-1})$ *which is* (d_p^{-1}, d_q)-*almost continuous at* 0, *then* $f: X \to Y$ *is continuous.*

Proof. Since f is (d_p^{-1}, d_q)-almost continuous at 0, for each $\varepsilon > 0$,

$$\mathrm{cl}_{d_p^{-1}}(f^{-1}(B_{d_q}(0, \varepsilon))$$

is a neighborhood of 0 with respect to $\mathcal{T}(d_p)$. Hence, there exists a $\delta > 0$ such that

$$B_{d_p}(0, \delta) \subseteq \mathrm{cl}_{d_p^{-1}}(f^{-1}(B_{d_q}(0, \varepsilon)).$$

Moreover for each pair $(x, f(x)) \in G(f)$ it is easy to check that

$$x + f^{-1}(B_{d_q}(0, \varepsilon)) \subseteq f^{-1}(B_{d_q}(f(x), \varepsilon)). \tag{1}$$

By (1) we deduce immediately that

$$\mathrm{cl}_{d_p^{-1}}(x) + \mathrm{cl}_{d_p^{-1}}(f^{-1}(B_{d_q}(0, \varepsilon)) \subseteq \mathrm{cl}_{d_p^{-1}}(f^{-1}(B_{d_q}(f(x), \varepsilon)).$$

It follows from $B_{d_p}(x,\delta) = x + B_{d_p}(0,\delta)$ that

$$B_{d_p}(x,\delta) \subset \mathrm{cl}_{d_p^{-1}}(x) + \mathrm{cl}_{d_p^{-1}}(f^{-1}(B_{d_q}(0,\varepsilon)) \subseteq \mathrm{cl}_{d_p^{-1}}(f^{-1}(B_{d_q}(f(x),\varepsilon)).$$

Now we show that $f\colon X \to Y$ is continuous at 0. Indeed, by Lemma 6 we have

$$\mathrm{cl}_{d_p^{-1}}\left(f^{-1}\left(B_{d_q}\left(0,\frac{\varepsilon}{2}\right)\right)\right) \subseteq f^{-1}(B_{d_q}(0,\varepsilon)),$$

for each $\varepsilon > 0$. Since f is (d_p^{-1}, d_q)-almost continuous at 0, $\mathrm{cl}_{d_p^{-1}}\big(f^{-1}\big(B_{d_q}\big(0,\frac{\varepsilon}{2}\big)\big)\big)$ is a neighborhood of 0 with respect to $\mathcal{T}(d_p)$. Whence we obtain that $f^{-1}(B_{d_q}(0,\varepsilon))$ is a neighborhood of 0 with respect to $\mathcal{T}(d_p)$. Thus,

$$f(f^{-1}(B_{d_q}(0,\varepsilon))) \subseteq B_{d_q}(0,\varepsilon)$$

so that $f\colon X \to Y$ is continuous at 0. Finally, by Lemma 5, we conclude that f is continuous on X. The proof is complete. ■

4. Open mapping theorem for normed cones

Let us recall the notion of an open mapping. A mapping $f\colon (X,\mathcal{T}_1) \to (Y,\mathcal{T}_2)$ acting between topological spaces is said to be open if $f(U) \in \mathcal{T}_2$ for all $U \in \mathcal{T}_1$.

The well-known open mapping theorem for normed linear spaces is given in the following way.

Theorem 8. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two Banach spaces. Then every closed graph linear mapping from X onto Y is open.*

It seems natural that one may ask if the preceding result is still true when bicomplete quasi-normed cones substitute Banach spaces. Hence, as in case of the closed graph type theorem, one may conjecture that the desired result for normed cones can be postulated in the next terms.

"*Let (X,p) and (Y,q) be two bicomplete quasi-normed cones. Then every linear mapping f from X onto Y with closed graph in $(X \times Y, d_p^s \times d_q^s)$ is open.*"

Unfortunately such a result does not hold as it is shown in the next example.

Example 9. Consider the quasi-normed cone $(\mathbb{R}, u)$ and the linear function $f\colon \mathbb{R} \to \mathbb{R}$ as given in Example 3. It is clear that f has a closed graph in $(\mathbb{R}\times\mathbb{R}, d_u^s \times d_u^s)$ and that f is onto. However it is not hard to see that f is not open.

In order to get an analogue of the open mapping theorem in our context we introduce the bitopological concepts which will be the key in the remainder of this section.

A subset A of a topological set $(X,\mathcal{T})$ is called preopen if $A \subseteq \mathrm{Int}_{\mathcal{T}}(\mathrm{cl}_{\mathcal{T}} A)$ (see [17]). In [13] the preopeness has been generalized to the context of bitopological spaces. A subset A of a bitopological space $(X,\mathcal{T}_1,\mathcal{T}_2)$ is said to be (i,j)-*preopen* if $A \subseteq \mathrm{Int}_{\mathcal{T}_i}(\mathrm{cl}_{\mathcal{T}_j} A)$.

A mapping $f\colon (X,\mathcal{T}_1,\mathcal{T}_2) \to (Y,\mathcal{S}_1,\mathcal{S}_2)$ will be say (i,j)-*almost open* if $f(U)$ is (i,j)-preopen in $(Y,\mathcal{S}_1,\mathcal{S}_2)$ for each $U \in \mathcal{T}_i$, where $i \neq j$ and $i,j \in \{1,2\}$. For a deeper discussion of (i,j)-almost open and for a fuller treatment of the general case of (i,j)-almost open multifunctions acting between bitopological spaces we refer the reader to [3].

Now we adapt the notion of (i,j)-preopenness to our context.

DEFINITION 10

Let (X, p) and (Y, q) be two quasi-normed cones.

(1) A subset A of X is said to be (d_p, d_p^{-1})-preopen if $A \subseteq \mathrm{Int}_{d_p}(\mathrm{cl}_{d_p^{-1}} A)$.

(2) A mapping $f: (X, p) \to (Y, q)$ is called (d_p, d_p^{-1})-almost open if $f(U)$ is (d_q, d_q^{-1})-preopen in $(Y, \mathcal{T}(d_q), \mathcal{T}(d_q^{-1}))$ for each $U \in \mathcal{T}(d_p)$.

The proof of the next useful lemma was given by Cao and Reilly in [3].

Lemma 11. *Let (X, d_1) and (Y, d_2) be two (extended) quasi-pseudo-metric spaces and let $f: X \to Y$ be a mapping with closed graph in $(X \times Y, d_1^{-1} \times d_2^{-1})$. If (X, d_1^{-1}) is right K-sequentially complete and for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $B_{d_2}(y, \delta) \subseteq \mathrm{cl}_{d_2^{-1}}(f(B_{d_1}(x, \varepsilon)))$ for all pairs $(x, y) \in G(f)$, then $\mathrm{cl}_{d_2^{-1}}(f(B_{d_1}(x, \varepsilon))) \subseteq f(B_{d_1}(x, \eta))$ for each point $x \in X$ and for all $\eta > \varepsilon > 0$.*

Theorem 12 (Open mapping theorem). *Let (X, p) and (Y, q) be two quasi-normed cones such that the extended quasi-metric space (X, d_p^{-1}) is right K-sequentially complete. If $f: X \to Y$ is a (d_p, d_p^{-1})-almost open linear mapping with closed graph in $(X \times Y, d_p^{-1} \times d_q^{-1})$, then f is open.*

Proof. Since f is (d_p, d_p^{-1})-almost open, $f(B_{d_p}(0, \varepsilon))$ is (d_q, d_q^{-1})-preopen in $(Y, \mathcal{T}(d_q), \mathcal{T}(d_q^{-1}))$ for each $\varepsilon > 0$. Thus

$$f(B_{d_p}(0, \varepsilon)) \subseteq \mathrm{Int}_{d_q}(\mathrm{cl}_{d_q^{-1}}(f(B_{d_p}(0, \varepsilon)))).$$

Hence, $\mathrm{cl}_{d_q^{-1}}(f(B_{d_p}(0, \varepsilon)))$ is a neighborhood of 0 with respect to $\mathcal{T}(d_q)$. So there exists $\delta > 0$ such that

$$B_{d_q}(0, \delta) \subseteq \mathrm{cl}_{d_q^{-1}}(f(B_{d_p}(0, \varepsilon))).$$

On the other hand, it is easy to see that for each pair $(x, f(x)) \in G(f)$ we have

$$f(x) + f(B_{d_p}(0, \varepsilon)) \subseteq f(B_{d_p}(x, \varepsilon)).$$

Consequently we obtain

$$\mathrm{cl}_{d_q^{-1}}(f(x)) + \mathrm{cl}_{d_q^{-1}}(f(B_{d_p}(0, \varepsilon))) \subseteq \mathrm{cl}_{d_q^{-1}}(f(B_{d_p}(x, \varepsilon))).$$

Whence

$$B_{d_q}(f(x), \delta) \subseteq \mathrm{cl}_{d_q^{-1}}(f(x)) + \mathrm{cl}_{d_q^{-1}}(f(B_{d_p}(0, \varepsilon))) \subseteq \mathrm{cl}_{d_q^{-1}}(f(B_{d_p}(x, \varepsilon))),$$

because of $B_{d_q}(f(x), \delta) = f(x) + B_{d_q}(0, \delta)$. It follows that $\mathrm{cl}_{d_q^{-1}}(f(B_{d_p}(x, \varepsilon)))$ is a neighborhood of $f(x)$ with respect to $\mathcal{T}(d_q)$.

Next we show that $f: X \to Y$ is an open mapping. Indeed, let $U \in \mathcal{T}(d_p)$ such that $x \in U$. Then there exists $\delta > 0$ with $B_{d_p}(x, \delta) \subseteq U$, so that $f(B_{d_p}(x, \delta)) \subseteq f(U)$. We now apply Lemma 11 to obtain

$$\mathrm{cl}_{d_p^{-1}}\left(f\left(B_{d_p}\left(x, \frac{\delta}{2}\right)\right)\right) \subseteq f(B_{d_p}(x, \delta)) \subseteq f(U).$$

Immediately we deduce that $f(U)$ is a neighborhood of $f(x)$ with respect to $\mathcal{T}(d_q)$. Therefore we have showed that f is open. This completes the proof. ■

We conclude the paper by noting that although the above theorems, Theorem 7 and Theorem 12, have been obtained for quasi-normed cones, in the proof of such theorems the product by scalars does not play any role. Consequently the assumption that (X, p) is a quasi-normed cone can be relaxed, Theorem 7 and Theorem 12 being true if we exchange 'quasi-normed' cone from the hypothesis by 'quasi-normed monoid'. We will now show how to dispense with the unnecessarily restrictive assumption.

Similarly to the cone case, the pair (X, p) will be called a *quasi-normed monoid* if X is a monoid (not necessarily cancellative) and p is a quasi-norm on X, i.e. $p\colon X \to \mathbb{R}^+$ is a function such that for all $x, y \in X$:

(i) $x = 0$ if and only if there is $-x \in X$ and $q(x) = q(-x) = 0$;
(ii) $q(x + y) \leq q(x) + q(y)$.

The quasi-norm q is called a norm if it satisfies in addition:

(i′) $q(x) = 0$ if and only if $x = 0$.

Several examples of quasi-normed monoids which arise in a natural way in some branches of theoretical computer science, such as algorithmic complexity theory (dual complexity space) and denotational semantics (domain of words) has been studied in [24].

A slight modification of the construction given in Proposition 1 allows us to obtain a general method to generate extended quasi-pseudo-metrics from quasi-norms defined on monoids (for a deeper discussion, see [24]).

PROPOSITION 13 (Proposition 4 in [24])

Let p be a quasi-norm on a monoid $(X, +)$. Then the function d_p defined on $X \times X$ by

$$d_p(x, y) = \begin{cases} \inf\{p(a)\colon y = x + a\}, & \text{if } y \in x + X \\ +\infty, & \text{otherwise} \end{cases}$$

is an extended quasi-pseudo-metric on X. Furthermore for each $x \in X$ and each $\varepsilon > 0$, $B_{d_p}(x, \varepsilon) = x + \{y \in X\colon p(y) < \varepsilon\}$, and the translations with respect to $+$ are $\mathcal{T}(d_p)$-open.

Let us recall that a *homomorphism* from a monoid $(X, +)$ to a monoid $(Y, +)$ is a mapping $f\colon X \to Y$ such that $f(x + y) = f(x) + f(y)$ for all $x, y \in X$. Replacing linear mappings between quasi-normed cones by homomorphisms between quasi-normed monoids in Lemma 5 we can characterize the continuity in the following easy way.

Lemma 14. *Let (X, p) and (Y, q) be two quasi-normed monoids and let $f\colon X \to Y$ be a homomorphism such that $f(0) = 0$. Then f is continuous on X if and only if f is continuous at* 0.

Proof. Suppose f is continuous at $0 \in X$. Now we prove that f is continuous on X. Let $y \in X$. Thus for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(B_{d_p}(0, \delta)) \subset B_{d_q}(0, \varepsilon).$$

Moreover

$$f(B_{d_p}(y, \delta)) = f(y) + f(B_{d_p}(0, \delta)) \subset f(y) + B_{d_q}(0, \varepsilon)$$

because f is a homomorphism and

$$B_{d_p}(y, \delta) = y + B_{d_p}(0, \delta).$$

Consequently

$$f(B_{d_p}(y, \delta)) \subset B_{d_q}(f(y), \varepsilon).$$

Therefore f is continuous at $y \in X$ and, thus, f is continuous on X. ■

Note that we obtain the equivalence (1), (2) in Lemma 5 as an immediate consequence of the preceding one.

Finally, if we interchange quasi-normed cone by quasi-normed monoid in Definitions 4 and 10 the same proofs in Theorems 7 and 12 remain valid for homeomorphisms preserving the neutral element, obtaining in this way related results to those for quasi-topological groups given in [3].

Theorem 15. *Let (X, p) and (Y, q) be two quasi-normed monoids such that the extended quasi-pseudo-metric space (Y, d_q^{-1}) is right K-sequentially complete. If $f\colon X \to Y$ is a homomorphism with closed graph in $(X \times Y, d_p^{-1} \times d_q^{-1})$ which is (d_p^{-1}, d_q)-almost continuous at* 0 *and such that $f(0) = 0$, then $f\colon X \to Y$ is continuous.*

Theorem 16. *Let (X, p) and (Y, q) be two quasi-normed monoids such that the extended quasi-pseudo-metric space (X, d_p^{-1}) is right K-sequentially complete. If $f\colon X \to Y$ is a (d_p, d_p^{-1})-almost open homomorphism with closed graph in $(X \times Y, d_p^{-1} \times d_q^{-1})$ such that $f(0) = 0$, then f is open.*

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