

## Push-outs of derivations

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**Abstract.** Let  $\mathfrak{A}$  be a Banach algebra and let  $X$  be a Banach  $\mathfrak{A}$ -bimodule. In studying  $\mathcal{H}^1(\mathfrak{A}, X)$  it is often useful to extend a given derivation  $D: \mathfrak{A} \rightarrow X$  to a Banach algebra  $\mathfrak{B}$  containing  $\mathfrak{A}$  as an ideal, thereby exploiting (or establishing) hereditary properties. This is usually done using (bounded/unbounded) approximate identities to obtain the extension as a limit of operators  $b \mapsto D(ba) - b.D(a)$ ,  $a \in \mathfrak{A}$  in an appropriate operator topology, the main point in the proof being to show that the limit map is in fact a derivation. In this paper we make clear which part of this approach is analytic and which algebraic by presenting an algebraic scheme that gives derivations in all situations at the cost of enlarging the module. We use our construction to give improvements and shorter proofs of some results from the literature and to give a necessary and sufficient condition that biprojectivity and biflatness is inherited to ideals.

**Keywords.** Derivation; double centralizer.

### 0. Introduction

For a Banach algebra  $\mathfrak{A}$  and a Banach  $\mathfrak{A}$ -bimodule  $X$  the computation of the bounded Hochschild cohomology group  $\mathcal{H}^1(\mathfrak{A}, X)$  can often be facilitated by extending a given derivation  $D: \mathfrak{A} \rightarrow X$  to some appropriate Banach algebra  $\mathfrak{B}$  containing  $\mathfrak{A}$  as an ideal. Perhaps the first use of this is Proposition 1.11 of [J] where derivations from a Banach algebra with a bounded two-sided approximate identity into a so-called neounital module is extended to the multiplier algebra. In the case of group algebras  $L_1(G)$  this gives the link between dual Banach  $L_1(G)$  modules and  $w^*$ -continuous group actions which is essential in the proof that  $L_1(G)$  has trivial cohomology with coefficients in dual modules if and only if the group  $G$  is amenable (Theorem 2.5 of [J]). In [G1] extension techniques are used to study weak amenability of commutative Banach algebras. In Theorem 7.1 of [GLo] the so-called approximate and essential amenability are established for certain abstract Segal algebras by means of an extension result. In [GLa2] a similar argument is used on symmetric Segal algebras on a SIN group to establish approximate weak amenability and on an amenable group to establish approximate amenability. In the study of weak amenability of Banach algebras  $\mathcal{A}(X)$  of approximable operators on a Banach space  $X$  [B1, G2], it is exploited that derivations can be extended to the Banach algebra  $\mathcal{B}(X)$  of all bounded operators on  $X$ . All these extension techniques are variations over the same extension problem. The data of this problem are a Banach algebra  $\mathfrak{A}$ , a continuous injection of  $\mathfrak{A}$  as an ideal into a larger Banach algebra  $\mathfrak{B}$  and a bounded

derivation  $D: \mathfrak{A} \rightarrow X$  into a Banach  $\mathfrak{A}$ -bimodule. The corresponding diagram problem is

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\iota} & \mathfrak{B} \\ D \downarrow & & \downarrow ? \\ X & \xrightarrow{?} & ? \end{array}, \tag{DP}$$

where we are looking for a Banach  $\mathfrak{B}$ -bimodule  $\tilde{X}$ , a bounded  $\mathfrak{A}$ -bimodule map  $X \xrightarrow{\iota} \tilde{X}$  and a bounded derivation  $\tilde{D}: \mathfrak{B} \rightarrow \tilde{X}$  such that the completed diagram is commutative. Provided we can extend the module action of  $\mathfrak{A}$  on  $X$  to an action of  $\mathfrak{B}$ , the problem (DP) is purely algebraic (the boundedness of  $\tilde{D}$  will be obvious from the construction) and can always be solved. Under various (mainly) topological conditions we shall further investigate exactly how the modules  $X$  and  $\tilde{X}$  are related and how information about  $\tilde{D}$  is transferred to  $D$ .

### 1. The construction

First we recall standard concepts, notation and elementary results. For further details the reader is referred to [H]. Throughout  $\mathfrak{A}$  is a Banach algebra and  $X$  and  $Y$  are Banach  $\mathfrak{A}$ -bimodules. The  $\mathfrak{A}$ -balanced projective tensor product is denoted by  $-\hat{\otimes}_{\mathfrak{A}}-$ . A Banach  $\mathfrak{A}$  bimodule is called *induced* if exterior multiplication induces a bimodule isomorphism  $\mathfrak{A} \hat{\otimes}_{\mathfrak{A}} X \hat{\otimes}_{\mathfrak{A}} \mathfrak{A} \mapsto X$ . In particular,  $\mathfrak{A}$  is *self-induced* if multiplication induces an isomorphism  $\mathfrak{A} \hat{\otimes}_{\mathfrak{A}} \mathfrak{A} \mapsto \mathfrak{A}$ . We note that, if  $\mathfrak{A}$  has a bounded approximate identity (BAI), then a Banach  $\mathfrak{A}$  module is *neounital* in the sense of [J], if and only if it is induced. The vector space of bounded left(right) Banach module homomorphisms  $\mathfrak{A} \rightarrow X$  is denoted by  ${}_{\mathfrak{A}}\mathbf{h}(\mathfrak{A}, X)$  ( $\mathbf{h}_{\mathfrak{A}}(\mathfrak{A}, X)$ ). These are Banach  $\mathfrak{A}$ -bimodules with the uniform operator norm and module operations given by

$$a.S = a.S(\cdot), \quad S.a = S(a\cdot), \quad a \in \mathfrak{A}, S \in {}_{\mathfrak{A}}\mathbf{h}(\mathfrak{A}, X),$$

$$a.T = T(\cdot)a, \quad T.a = T(\cdot)a, \quad a \in \mathfrak{A}, T \in \mathfrak{A}\mathbf{h}(\mathfrak{A}, X).$$

We note the important instances of hom-tensor duality

$$(\mathfrak{A} \hat{\otimes}_{\mathfrak{A}} X)^* \cong \mathbf{h}_{\mathfrak{A}}(\mathfrak{A}, X^*), \quad (X \hat{\otimes}_{\mathfrak{A}} \mathfrak{A})^* \cong \mathfrak{A}\mathbf{h}(\mathfrak{A}, X^*)$$

with isometric bimodule identifications through

$$\langle a \otimes x, \Phi \rangle = \langle x, S(a) \rangle, \quad \langle x \otimes a, \Psi \rangle = \langle x, T(a) \rangle$$

for  $a \in \mathfrak{A}, x \in X, \Phi \in (\mathfrak{A} \hat{\otimes}_{\mathfrak{A}} X)^*, \Psi \in (X \hat{\otimes}_{\mathfrak{A}} \mathfrak{A})^*, S \in \mathbf{h}_{\mathfrak{A}}(\mathfrak{A}, X^*), T \in \mathfrak{A}\mathbf{h}(\mathfrak{A}, X^*)$ .

We define the left and right annihilators and the annihilator of  $X$  as

$${}_{\mathfrak{A}}\text{ann } X = \{x \in X \mid \mathfrak{A}.x = \{0\}\},$$

$$\text{ann}_{\mathfrak{A}} X = \{x \in X \mid x.\mathfrak{A} = \{0\}\},$$

$$\text{ann } X = {}_{\mathfrak{A}}\text{ann } X \cap \text{ann}_{\mathfrak{A}} X.$$

A *module derivation* is a bounded linear map  $D: \mathfrak{A} \rightarrow X$  satisfying the derivation rule

$$D(ab) = a.D(b) + D(a).b, \quad a, b \in \mathfrak{A}.$$

A derivation is *inner* if it is of the form  $a \mapsto a.x - x.a$  for some  $x \in X$ . The vector space of all derivations  $D: \mathfrak{A} \rightarrow X$  is denoted by  $\mathcal{Z}^1(\mathfrak{A}, X)$  and the vector space of all inner derivations is denoted by  $\mathcal{B}^1(\mathfrak{A}, X)$ . The (first) *Hochschild cohomology group* of  $\mathfrak{A}$  with coefficients in  $X$  is  $\mathcal{H}^1(\mathfrak{A}, X) = \mathcal{Z}^1(\mathfrak{A}, X) / \mathcal{B}^1(\mathfrak{A}, X)$ .

We start by generalizing the notion of double centralizer [J] to module homomorphisms (see also *multipliers* in [S]).

DEFINITION 1.1

A *double centralizer* from  $\mathfrak{A}$  into  $X$  is a pair  $(S, T) \in \mathbf{h}_{\mathfrak{A}}(\mathfrak{A}, X) \times {}_{\mathfrak{A}}\mathbf{h}(\mathfrak{A}, X)$  satisfying

$$a.S(\alpha) = T(a).\alpha, \quad a, \alpha \in \mathfrak{A}.$$

We note that each  $x \in X$  gives rise to a double centralizer  $(L_x, R_x)$ , where  $L_x: a \mapsto x.a$  and  $R_x: a \mapsto a.x$ .

For a double centralizer  $(S, T)$  and  $a \in \mathfrak{A}$  we have  $a.S = L_{Ta}$ ,  $a.T = R_{Ta}$ ,  $S.a = L_{Sa}$ ,  $T.a = R_{Sa}$ .

Thus, norming  $\mathbf{h}_{\mathfrak{A}}(\mathfrak{A}, X) \times {}_{\mathfrak{A}}\mathbf{h}(\mathfrak{A}, X)$  by  $\|(U, V)\| = \max\{\|U\|, \|V\|\}$  the set of double centralizers is a Banach sub-bimodule of  $\mathbf{h}_{\mathfrak{A}}(\mathfrak{A}, X) \times {}_{\mathfrak{A}}\mathbf{h}(\mathfrak{A}, X)$ , which we denote by  $\Delta(X)$ . The map  $\iota_X: x \mapsto (L_x, R_x)$ ,  $x \in X$  is a bounded bimodule homomorphism of  $X$  into  $\Delta(X)$  with the property that  $\mathfrak{A}.\Delta(X) + \Delta(X).\mathfrak{A} \subseteq \iota_X(X)$ .

With the definition to follow we have stipulated a setting that covers most applications.

DEFINITION 1.2

Let  $(\mathfrak{A}, |\cdot|)$  and  $(\mathfrak{B}, \|\cdot\|)$  be Banach algebras. We say that  $\mathfrak{B}$  is an *envelope* of  $\mathfrak{A}$  if

- (1)  $\mathfrak{A}$  is an ideal in  $\mathfrak{B}$  with bounded inclusion  $\iota: \mathfrak{A} \rightarrow \mathfrak{B}$ ;
- (2) there is  $C > 0$  so that  $\max\{|ba|, |ab|\} \leq C|a|\|b\|$ ,  $a \in \mathfrak{A}$ ,  $b \in \mathfrak{B}$ .

We can now solve the diagram problem (DP). In cases of interest the  $\mathfrak{A}$ -module action on  $X$  can be extended to an action of  $\mathfrak{B}$ , so we shall assume from the outset that  $X$  is a  $\mathfrak{B}$ -bimodule.

**Theorem 1.3.** *Let  $\mathfrak{B}$  be an envelope of  $\mathfrak{A}$  and let  $X$  be a Banach  $\mathfrak{B}$ -bimodule. Then  $X$  is naturally a Banach  $\mathfrak{A}$ -bimodule and  $\Delta(X)$  is naturally a Banach  $\mathfrak{B}$ -bimodule. To each bounded derivation  $D: \mathfrak{A} \rightarrow X$  there is a bounded derivation  $\tilde{D}: \mathfrak{B} \rightarrow \Delta(X)$ , such that the diagram*

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\iota} & \mathfrak{B} \\ D \downarrow & & \downarrow \tilde{D} \\ X & \xrightarrow{\iota_X} & \Delta(X) \end{array}$$

is commutative. Furthermore, each  $(S, T) \in \Delta(X)$  determines a derivation  $a \mapsto S(a) - T(a)$ ,  $a \in \mathfrak{A}$  of  $\mathfrak{A}$  into  $X$ .

Suppose that  $\overline{\mathfrak{A}^2} = \mathfrak{A}$ . Then  $\tilde{D}$  is uniquely determined by  $D$  and in case  $\tilde{D}$  is inner, there is  $(S, T) \in \Delta(X)$  such that  $D(a) = S(a) - T(a)$ ,  $a \in \mathfrak{A}$ .

*Proof.* We make  $\Delta(X)$  a Banach  $\mathfrak{B}$ -bimodule by defining

$$b.(L, R) = ({}^bL, {}^bR) \quad \text{and} \quad (L, R).b = (L^b, R^b), \quad b \in \mathfrak{B}, (L, R) \in \Delta(X),$$

with

$${}^bL(a) = b.L(a), {}^bR(a) = R(ab), L^b(a) = L(ba), R^b(a) = R(a).b, \quad a \in \mathfrak{A}.$$

We define  $\tilde{D}$  by

$$\tilde{D}(b) = (\mathcal{L}(b), \mathcal{R}(b)), \quad b \in \mathfrak{B},$$

where

$$\mathcal{L}(b)(a) = D(ba) - b.D(a), \mathcal{R}(b)(a) = D(ab) - D(a).b, \quad a \in \mathfrak{A}, b \in \mathfrak{B}.$$

We check that we can complete the diagram as required. Let  $a, c \in \mathfrak{A}$  and  $b, b_1, b_2 \in \mathfrak{B}$ .

The double centralizer property:

$$\begin{aligned} \mathcal{L}(b)(ac) &= D(bac) - b.D(ac) \\ &= ba.D(c) + D(ba).c - b.(a.D(c) + D(a).c) \\ &= (D(ba) - b.D(a)).c = \mathcal{L}(b)(a).c, \end{aligned}$$

$$\begin{aligned} \mathcal{R}(b)(ac) &= D(acb) - D(ac).b \\ &= a.D(cb) + D(a).cb - (D(a).c + a.D(c)).b \\ &= a.(D(cb) - D(c).b) = a.\mathcal{R}(b)(c) \end{aligned}$$

and

$$\begin{aligned} a.\mathcal{L}(b)(c) &= a.(D(bc) - b.D(c)) = D(abc) - D(a).bc - ab.D(c) \\ &= D(ab).c - D(a).bc = (D(ab) - D(a).b).c = \mathcal{R}(b)(a).c. \end{aligned}$$

The derivation property:

$$\begin{aligned} ({}^{b_1}\mathcal{L}(b_2) + \mathcal{L}(b_1)^{b_2})(a) &= b_1.\mathcal{L}(b_2)(a) + \mathcal{L}(b_1)(b_2a) \\ &= b_1.(D(b_2a) - b_2.D(a)) \\ &\quad + D(b_1b_2a) - b_1.D(b_2a) \\ &= D(b_1b_2a) - b_1b_2.D(a) \\ &= \mathcal{L}(b_1b_2)(a) \end{aligned}$$

and

$$\begin{aligned}
 ({}^{b_1}\mathcal{R}(b_2) + \mathcal{R}(b_1)^{b_2})(a) &= \mathcal{R}(b_2)(ab_1) + \mathcal{R}(b_1)(a).b_2 \\
 &= D(ab_1b_2) - D(ab_1).b_2 \\
 &\quad + (D(ab_1) - D(a).b_1).b_2 \\
 &= D(ab_1b_2) - D(a).b_1b_2 \\
 &= \mathcal{R}(b_1b_2)(a).
 \end{aligned}$$

Since clearly  $(L_{D(a)}, R_{D(a)}) = (\mathcal{L}(a), \mathcal{R}(a))$  the diagram is commutative.

It is immediate to verify that each  $(S, T) \in \mathbf{\Delta}(X)$  defines a derivation by  $a \mapsto S(a) - T(a)$  (see also [GLa2]).

Now suppose that  $\overline{(\mathfrak{A})^2} = \mathfrak{A}$ , and let  $\tilde{D}: b \mapsto (\mathcal{L}(b), \mathcal{R}(b))$  be any derivation that makes the diagram commutative. From the derivation identities

$$L_{D(ab)} = {}^a\mathcal{L}(b) + L_{D(a)}, \quad R_{D(bc)} = {}^bR_{D(c)} + \mathcal{R}(b)^c \quad a, c \in \mathfrak{A}, b \in \mathfrak{B}$$

we get  $\mathcal{L}(b)(ac) = D(bac) - b.D(ac)$  and  $\mathcal{R}(b)(ac) = D(acb) - D(ac).b$  so  $\tilde{D}$  is uniquely determined by  $D$ .

If  $\tilde{D}(a) = (S, T).a - a.(S, T)$ ,  $a \in \mathfrak{A}$ , then

$$(L_{D(a)}, R_{D(a)}) = (L_{S(a)-T(a)}, R_{S(a)-T(a)}), \quad a \in \mathfrak{A},$$

so for all  $a_1, a_2 \in \mathfrak{A}$  we have

$$\begin{aligned}
 D(a_1).a_2 &= (S(a_1) - T(a_1)).a_2 = S(a_1a_2) - T(a_1).a_2; \\
 a_1.D(a_2) &= a_1.(S(a_2) - T(a_2)) = T(a_1).a_2 - T(a_1a_2)
 \end{aligned}$$

so that  $D(a_1a_2) = S(a_1a_2) - T(a_1a_2)$ . Since  $\overline{\mathfrak{A}^2} = \mathfrak{A}$ , we get  $D(a) = S(a) - T(a)$ ,  $a \in \mathfrak{A}$ .

Very often one is interested in dual modules. The property of being a dual module is preserved by the double centralizer construction. (This also follows easily from the description as multipliers in [S].)

**PROPOSITION 1.4**

*Let  $X^*$  be the dual of a Banach  $\mathfrak{A}$ -bimodule  $X$ . Then there is an isometric bimodule isomorphism*

$$\mathbf{\Delta}(X^*) \cong ((\mathfrak{A} \hat{\otimes}_{\mathfrak{A}} X \oplus X \hat{\otimes}_{\mathfrak{A}} \mathfrak{A})/N)^*,$$

where  $\oplus$  denotes  $\ell_1$ -direct sum and  $N = \text{clspan}\{(a \otimes_{\mathfrak{A}} x.\alpha, -a.x \otimes_{\mathfrak{A}} \alpha) \mid a, \alpha \in \mathfrak{A}, x \in X\}$ .

If  $q: \mathfrak{A} \hat{\otimes}_{\mathfrak{A}} X \oplus X \hat{\otimes}_{\mathfrak{A}} \mathfrak{A} \rightarrow (\mathfrak{A} \hat{\otimes}_{\mathfrak{A}} X \oplus X \hat{\otimes}_{\mathfrak{A}} \mathfrak{A})/N$  is the quotient map and  $\mu: (\mathfrak{A} \hat{\otimes}_{\mathfrak{A}} X \oplus X \hat{\otimes}_{\mathfrak{A}} \mathfrak{A}) \rightarrow X$  is the canonical map induced by  $(a \otimes_{\mathfrak{A}} x, x' \otimes_{\mathfrak{A}} a') \mapsto a.x + x'.a'$ , then  $q^* \circ \iota_{X^*} = \mu^*$ . In particular,  $\iota_{X^*}$  is bounded below, if and only if  $\mu$  is surjective.

*Proof.* We note that  $N \subseteq \ker \mu$ . The statement is then a consequence of the natural isometric isomorphisms  $\mathbf{h}_{\mathfrak{A}}(\mathfrak{A}, X^*) \cong (\mathfrak{A} \hat{\otimes}_{\mathfrak{A}} X)^*$  and  ${}_{\mathfrak{A}}\mathbf{h}(\mathfrak{A}, X^*) \cong (X \hat{\otimes}_{\mathfrak{A}} \mathfrak{A})^*$  and the definition of double centralizers.

In the important case of induced modules the description is particularly nice.

**COROLLARY 1.5**

*Suppose that  $X$  is an induced  $\mathfrak{A}$ -module. Then  $\iota_{X^*}: X^* \rightarrow \mathbf{\Delta}(X^*)$  is an isomorphism.*

*Proof.* In view of the proposition above we must prove  $\ker \mu \subseteq N$ . Since  $X$  is induced a generic element in  $\ker \mu$  has the form  $k = (\sum_n a_n \otimes x_n \cdot c_n, \sum_n a'_n \cdot x'_n \otimes c'_n)$  with  $\sum_n a_n \otimes x_n \otimes c_n = -\sum_n a'_n \otimes x'_n \otimes c'_n$  in  $\mathfrak{A} \hat{\otimes}_{\mathfrak{A}} X \hat{\otimes}_{\mathfrak{A}} \mathfrak{A}$ . It follows that  $\sum_n a'_n \cdot x'_n \otimes c'_n = -\sum_n a_n \cdot x_n \otimes c_n$  so that  $k = \sum_n (a_n \otimes x_n \cdot c_n, -a_n \cdot x_n \otimes c_n)$ , which is clearly in  $N$ .

In the following we investigate further properties of the map  $\iota_X$ . Clearly  $\iota_X$  is injective, if and only if  $\text{ann } X = \{0\}$ . It turns out that in the case of injectivity the pair  $(\iota_X, \mathbf{\Delta}(X))$  is a universal element (in a certain category, only to be implicitly specified) with the consequence that our solution to the diagram problem is actually a push-out. Henceforth we shall only work with annihilator-free modules. To which extent this is a limitation is illustrated by the following remark.

*Remark 1.6.* For any left Banach  $\mathfrak{A}$  module  $X$  there is a smallest closed submodule  $\mathcal{N}$  such that  ${}_{\mathfrak{A}}\text{ann}(X/\mathcal{N}) = \{0\}$ . To see this, let  $\mathcal{M}$  be the set of closed submodules  $N \subseteq X$  such that

$$\forall x \in X \quad \mathfrak{A}.x \subseteq N \implies x \in N.$$

Put  $\mathcal{N} = \bigcap_{N \in \mathcal{M}} N$ . Then one checks that  $\mathcal{N}$  has the desired property. The module  $\mathcal{N}$  may be constructed by a transfinite procedure as follows. For any left submodule  $N \subseteq X$  we set  $N: \mathfrak{A} = \{x \in X \mid \mathfrak{A}.x \subseteq N\}$ . Let  $N_1 = \{0\}: \mathfrak{A}$ . For a successor ordinal  $\beta = \alpha + 1$  we set  $N_\beta = N_\alpha: \mathfrak{A}$  and for a limit ordinal  $\gamma$  we set  $N_\gamma = \overline{\bigcup_{\alpha < \gamma} N_\alpha}$ . Then  $\mathcal{N} = \bigcup_{\text{card}(\alpha) \leq \text{card}(X)} N_\alpha$ . Note that by a cardinality argument any given sequence in  $\mathcal{N}$  is contained in some  $N_\gamma$ , so  $\mathcal{N}$  is automatically closed. Replacing  $\mathfrak{A}$  by its opposite algebra or by its enveloping algebra we get the same statements for right modules and bimodules.

These elementary observations are part of the following description of the double centralizer module as a universal object.

**PROPOSITION 1.7**

*Let  $X$  be a Banach  $\mathfrak{A}$ -bimodule with  $\text{ann } X = \{0\}$ . The map  $\iota_X: x \mapsto (L_x, R_x)$  is a bounded  $\mathfrak{A}$ -bimodule monomorphism from  $X$  into  $\mathbf{\Delta}(X)$ . The pair  $(\iota_X, \mathbf{\Delta}(X))$  is a universal object in the sense: If  $\tilde{X}$  is a Banach  $\mathfrak{A}$ -bimodule and  $j: X \rightarrow \tilde{X}$  is a bounded  $\mathfrak{A}$ -bimodule monomorphism with*

$$\mathfrak{A}.\tilde{X} + \tilde{X}.\mathfrak{A} \subseteq j(X), \tag{*}$$

*then there is a unique bounded  $\mathfrak{A}$ -bimodule homomorphism  $\tilde{j}: \tilde{X} \rightarrow \mathbf{\Delta}(X)$  such that  $\iota_X = \tilde{j} \circ j$ .*

*Proof.* Since  $\text{ann } X = \{0\}$  the assignment  $x \mapsto (L_x, R_x)$  defines a monomorphism which, by the paragraphs after Definition 1.1, satisfies (\*). Let  $j: X \rightarrow \tilde{X}$  be a monomorphism satisfying (\*). By this property  $L_{\tilde{x}}(\mathfrak{A}) + R_{\tilde{x}}(\mathfrak{A}) \subseteq j(X)$  for any  $\tilde{x} \in \tilde{X}$ , so we may define  $S = j^{-1} \circ L_{\tilde{x}}$  and  $T = j^{-1} \circ R_{\tilde{x}}$ . By the closed graph theorem  $S$  and  $T$  are bounded linear maps, and since  $j$  is a bimodule homomorphism, the pair  $(S, T) \in \Delta(X)$ . Define  $\tilde{j}(\tilde{x}) = (S, T)$ . It is clear that  $\iota_X = \tilde{j} \circ j$ , and yet an application of the closed graph theorem tells that  $\tilde{j}$  is bounded. It remains to establish uniqueness. Hence suppose that  $\tilde{j}_1 \circ j = \tilde{j}_2 \circ j = \iota_X$ . Let  $\tilde{x} \in \tilde{X}$  and set  $(S_i, T_i) = \tilde{j}_i(\tilde{x})$ ,  $i = 1, 2$ . For any  $a \in \mathfrak{A}$  we have  $(L_{T_i a}, R_{T_i a}) = a \cdot \tilde{j}_i(\tilde{x}) = \tilde{j}_i(a \cdot \tilde{x}) = \iota_X(j^{-1}(a \cdot \tilde{x}))$ ,  $i = 1, 2$ . Thus  $\iota_X(T_1 a) = \iota_X(T_2 a)$ , and, since  $\iota_X$  is a monomorphism,  $T_1 = T_2$ . By considering  $\tilde{j}_i(\tilde{x}) \cdot a$  we get  $S_1 = S_2$  as well.

## 2. Examples

The examples to follow serve to illustrate a unified approach to various results in the literature about inherited cohomology.

### 2.1 Amenability

Extension techniques were first used in [J], where derivations were extended from Banach algebras with bounded approximate identities to their double centralizer algebras. Recall that a Banach algebra is *amenable* if  $\mathcal{H}^1(\mathfrak{A}, X^*) = \{0\}$  for all Banach  $\mathfrak{A}$ -bimodules  $X$ .

**Theorem 2.1 (Proposition 5.1 of [J]).** *Let  $\mathfrak{B}$  be amenable and let  $\mathfrak{A} \subseteq \mathfrak{B}$  be a closed two-sided ideal. Then  $\mathfrak{A}$  is amenable, if (and only if)  $\mathfrak{A}$  has a BAI.*

*Proof.* The only if part is a standard fact about amenable Banach algebras. As observed in [J], to prove amenability of  $\mathfrak{A}$ , we only have to look at derivations into modules  $X^*$  where  $X$  is a neounital  $\mathfrak{A}$ -module. The module action on such a module is naturally extended to an action of  $\mathfrak{B}$  for instance by noting that  $X \cong \mathfrak{A} \hat{\otimes}_{\mathfrak{A}} X \hat{\otimes}_{\mathfrak{A}} \mathfrak{A}$ . By Corollary 1.5 we have  $X^* = \Delta(X^*)$ , so the conclusion follows immediately by amenability of  $\mathfrak{B}$ .

### 2.2 Weak amenability

A Banach algebra is *weakly amenable* if  $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*) = \{0\}$ . For commutative Banach algebras this is equivalent to  $\mathcal{H}^1(\mathfrak{A}, X) = \{0\}$  for all symmetric modules  $X$ . In this situation the hereditary properties of weak amenability are simple.

**Theorem 2.2 (Corollary 1.3 of [G]).** *Let  $\mathfrak{A}$  be a commutative Banach algebra and let  $\mathfrak{B}$  be a commutative weakly amenable envelope. Then  $\mathfrak{A}$  is weakly amenable, if (and only if)  $\mathfrak{A}^2 = \mathfrak{A}$ .*

*Proof.* Let  $D: \mathfrak{A} \rightarrow \mathfrak{A}^*$  be a derivation. Since  $\mathfrak{B}$  is commutative  $\Delta(\mathfrak{A})$  is symmetric, so the push-out derivation  $\tilde{D}$  is the zero map by weak amenability of  $\mathfrak{B}$ . Since  $\overline{\mathfrak{A}^2} = \mathfrak{A}$ , the map  $\iota_{\mathfrak{A}^*}$  is a monomorphism, whence  $D = 0$ .

As above the ‘only if’ part is a standard fact about weakly amenable Banach algebras.

### 2.3 Approximately inner derivations

The notion of approximately inner derivations was introduced in [GLo]. A derivation  $D: \mathfrak{A} \rightarrow X$  is *approximately inner* if  $D$  belongs to the strong operator closure of  $\mathcal{B}^1(\mathfrak{A}, X)$ , i.e. if there is a net  $x_\gamma \in X$  such that  $D(a) = \lim_\gamma a \cdot x_\gamma - x_\gamma \cdot a$ ,  $a \in \mathfrak{A}$ . The Banach algebra  $\mathfrak{A}$  is termed *approximately amenable*, if for all Banach  $\mathfrak{A}$ -bimodules  $X$  all derivations  $D: \mathfrak{A} \rightarrow X^*$  are approximately inner, and *approximately weakly amenable* if all derivations  $D: \mathfrak{A} \rightarrow \mathfrak{A}^*$  are approximately inner. We have the generalization of Corollary 2.3 of [GLo].

**Theorem 2.3.** *Let  $\mathfrak{A}$  have an approximately amenable envelope  $\mathfrak{B}$ . If  $X$  is  $\mathfrak{A}$ -induced, then every derivation  $D: \mathfrak{A} \rightarrow X^*$  is approximately inner. In particular, if  $\mathfrak{A}$  has a BAI, then  $\mathfrak{A}$  is approximately amenable.*

*Proof.* The first part is an immediate consequence of Corollary 1.5. If  $\mathfrak{A}$  has a BAI, we only have to consider derivations into duals of neounital modules (Proposition 2.5 of [GLo]). But neounital modules over Banach algebras with a BAI are exactly the induced modules.

### 2.4 Abstract Segal algebras

Let  $\mathfrak{S} \subseteq \mathfrak{A}$  be an abstract symmetric Segal algebra of a Banach algebra  $\mathfrak{A}$ , i.e.  $\mathfrak{A}$  is an envelope of  $\mathfrak{S}$  and  $\mathfrak{S}$  is dense in  $\mathfrak{A}$  (see [Bu]). From Theorem 2.2 it follows that, if  $\mathfrak{A}$  is commutative and weakly amenable, then  $\mathfrak{S}$  is weakly amenable when  $\overline{\mathfrak{S}^2} = \mathfrak{S}$ , thus strengthening Theorem 3.3 of [GLa1].

In [GLa2] derivations from concrete symmetric Segal algebras are investigated. A concrete symmetric Segal algebra has an approximate identity (AI) which is a BAI for  $L^1(G)$ . The results Theorem 3.1 and Remark 3.4 of this reference follow from Theorem 2.4.

**Theorem 2.4 [GLa2].** *Suppose that  $\mathfrak{A}$  is amenable and  $X$  is a Banach  $\mathfrak{A}$ -bimodule. Then every derivation  $D: \mathfrak{S} \rightarrow X^*$  has the form  $D(a) = S(a) - T(a)$ ,  $a \in \mathfrak{S}$ . If  $\mathfrak{S}$  has an AI which is a BAI for  $\mathfrak{A}$ , then every derivation  $D: \mathfrak{S} \rightarrow X$  is approximately inner and in particular  $\mathfrak{S}$  is approximately weakly amenable.*

*Proof.* Since  $\overline{\mathfrak{S}^2} = \mathfrak{S}$  the first part follows from Theorem 1.3 and Proposition 1.4. For the proof being approximable inner, we may as in [GLa2] reduce the task to show that, if  $X$  is neounital over  $\mathfrak{A}$ , then every derivation  $D: \mathfrak{S} \rightarrow X^*$  is a  $w^*$  limit of inner derivations. This reduction is obtained by an appeal to the reduction to neounital modules in [J] combined with a standard application of Goldstine's theorem and Hahn–Banach's theorem. Hence let  $(e_\gamma)_\Gamma$  be an AI for  $\mathfrak{S}$ . Since  $\mathfrak{A}$  is amenable, there is  $(S, T) \in \Delta(X^*)$  so that  $\tilde{D}(a) = a \cdot (S, T) - (S, T) \cdot a$ ,  $a \in \mathfrak{S}$ . Since  $X$  is neounital, we get  $D(a) = w^* - \lim_\gamma a \cdot T(e_\gamma) - T(e_\gamma) \cdot a$ ,  $a \in \mathfrak{S}$ .

### 2.5 Biprojective and biflat Banach algebras

Biprojective and biflat Banach algebras can be described in terms of derivations, as demonstrated in [S]. Biprojectivity and biflatness are nicely inherited by ideals.

**Theorem 2.5.** *Let  $\mathfrak{A}$  be a closed ideal of a biprojective (biflat) Banach algebra  $\mathfrak{B}$ . Then  $\mathfrak{A}$  is biprojective (biflat) if and only if  $\mathfrak{A}$  is self-induced.*



*Proof.* The property of being self-induced is well-known to hold for all biflat, and hence all biprojective Banach algebras. By Theorem 5.9 of [S]  $\mathfrak{A}$  is biprojective if and only if  $\mathcal{H}^1(\mathfrak{A}, \Delta(X)) = \{0\}$  for all Banach  $\mathfrak{A}$  bimodules  $X$ . In fact, to establish biprojectivity it suffices to check for  $X$  being the kernel of the multiplication  $\mathfrak{A} \hat{\otimes} \mathfrak{A} \rightarrow \mathfrak{A}$ . Clearly this  $X$  is a Banach  $\mathfrak{B}$ -bimodule, so we may apply Theorem 1.3. By Lemma 5.2(iv) of [S] the map  $\iota_{\Delta(X)}: \Delta(X) \rightarrow \Delta(\Delta(X))$  is an isomorphism, since  $\mathfrak{A}$  is self-induced. Using that  $\mathfrak{B}$  is biprojective it is possible to prove directly that  $\mathcal{H}^1(\mathfrak{B}, \Delta(X)) = \{0\}$  and therefore that  $\mathcal{H}^1(\mathfrak{A}, \Delta(X)) = \{0\}$ . However, we prefer to use Theorem 1.3 once more, now applied to the identity map  $\mathfrak{B} \rightarrow \mathfrak{B}$  and  $\Delta(X)$  viewed as a  $\mathfrak{B}$  module. Invoking the universal property of the push-out we conclude that every derivation  $D: \mathfrak{A} \rightarrow \Delta(X)$  factors as  $D = j \circ \tilde{D} \circ \iota$  where  $\tilde{D}: \mathfrak{B} \rightarrow \Delta(\Delta(X))_{\mathfrak{B}}$  and  $\iota: \mathfrak{A} \rightarrow \mathfrak{B}$  is the inclusion. The subscript indicates that double centralizers are with respect to  $\mathfrak{B}$ . Since  $\mathfrak{B}$  is biprojective,  $\tilde{D}$  is inner, thus giving that  $D$  is inner.

By applying the above proof to the dual module  $X^*$  we get the statement about biflatness.

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