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A geometric approach to the Kronecker problem I: The two row case

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Abstract. Given two irreducible representations μ, ν of the symmetric group S_d, the Kronecker problem is to find an explicit rule, giving the multiplicity of an irreducible representation, λ, of S_d, in the tensor product of μ and ν. We propose a geometric approach to investigate this problem. We demonstrate its effectiveness by obtaining explicit formulas for the tensor product multiplicities, when the irreducible representations are parameterized by partitions with at most two rows.

Keywords. Kronecker product; symmetric group representations; geometric complexity.

1. Introduction

The Kronecker problem for the symmetric group is to determine the multiplicity of an irreducible symmetric group representation, in the tensor product of two such representations.

Let S_d denote the symmetric group on d letters. It is well-known (see [2], [6]) that the complex irreducible representations of S_d are parameterized by integer partitions of d. As we are primarily interested in complex representations, by 'representation' we always mean a complex representation.

Let λ, μ and ν be three partitions of the integer d. Further, let W_λ, W_μ and W_ν be the associated irreducible representations of S_d. Under the natural diagonal action of S_d, $W_\mu \otimes W_\nu$ becomes a representation of S_d. The Kronecker problem, in this context, is to 'compute' the multiplicity with which the representation W_λ occurs inside the representation $W_\mu \otimes W_\nu$. More precisely, let $m_{\lambda\mu\nu}$ denote the multiplicity of W_λ inside $W_\mu \otimes W_\nu$. Then the problem is to determine $m_{\lambda\mu\nu}$.

Our interest in this problem is motivated by the work 'Geometric Complexity Theory' [7,8]. In this work, a strong link is established between the separation of complexity classes in Computer Science and algorithmic problems in Representation Theory. It has been shown there that a good understanding of the 'subgroup restriction problem' (see [8,9]) will be an important step in demonstrating separation of complexity classes, via the proposed approach. The problem of determining tensor product multiplicities is a special, albeit very important case of the 'subgroup restriction problem' [9].

An important case where the tensor product multiplicities is well understood is in general linear group $GL(n, \mathbb{C})$ (or simply $GL(n)$) case. Recall that $GL(n)$ is linearly reductive, and the irreducible polynomial representations of $GL(n)$ are parameterized by partitions with at most n rows [2,15]. Let λ, μ, ν be three partitions with at most n rows and further, let V_λ, V_μ, V_ν be the respective irreducible representations of $GL(n)$. Here, the Kronecker problem is to compute the multiplicity, $c^\lambda_{\mu\nu}$ of V_λ inside $V_\mu \otimes V_\nu$. The celebrated Littlewood–Richardson rule (see [2,13]) gives a complete combinatorial solution to this problem; $c^\lambda_{\mu\nu}$ is interpreted as the number of skew-tableaux with appropriate restrictions. A number of combinatorial interpretations [5,4] are known for the numbers $c^\lambda_{\mu\nu}$. Interestingly, it has also been shown [9] that the problem of checking if $c^\lambda_{\mu\nu}$ is zero can be solved in polynomial time. An analogous result has been conjectured for the multiplicities $m_{\lambda\mu\nu}$. More precisely, it has been conjectured [9] that, given λ, μ and ν, three partitions of the integer d, there is a polynomial time algorithm to check if $m_{\lambda\mu\nu}$ is zero. At present, we do not seem to have a good understanding of the numbers $m_{\lambda\mu\nu}$, except in some special cases. In [12], the authors solve the problem in the case when μ, ν have at most two rows. Explicit formulas for these multiplicities are also known for shapes which are hooks [11]. One of the well-known major open problems [14] is to develop a combinatorial rule, akin to the Littlewood–Richardson rule, for the numbers $m_{\lambda\mu\nu}$.

As the symmetric group representations are self-dual, $m_{\lambda\mu\nu}$ may be, alternately, defined as the multiplicity of the trivial representation inside $W_\lambda \otimes W_\mu \otimes W_\nu$ (thought of as a S_d-representation under the diagonal action). From this interpretation it is clearly evident that the number $m_{\lambda\mu\nu}$ is symmetric in the arguments λ, μ and ν.

In this work we investigate a geometric approach to understanding $m_{\lambda,\mu,\nu}$ based on Proposition 2.3. This proposition is folklore, nevertheless we give the entire proof. We show that the geometric approach easily leads to an explicit formula for $m_{\lambda\mu\nu}$, in the case when all the three partitions λ, μ and ν have at most two rows. The explicit formulas derived in this paper already appear in [12]. However all the previous approaches to explicit formulas are combinatorial, and are based on some very clever manipulations of Schur functions. Here, we make a modest attempt which relies on geometric ideas. We believe that this approach will definitely help in understanding the tensor product multiplicities for many interesting special cases.

2. Geometric approach

As before, we let S_d denote the symmetric group on d letters and $GL(n)$ denote the general linear group of $n \times n$-matrices over complex numbers.

Let λ, μ, ν be three partitions of the integer d. Further, suppose that λ has at most p rows, μ has at most q rows and ν has at most r rows. As mentioned earlier, we have three irreducible S_d-representations W_λ, W_μ and W_ν corresponding to λ, μ and ν. Similarly, we have an irreducible $GL(p)$-representation V_λ, an irreducible $GL(q)$-representation V_μ and an irreducible $GL(r)$-representation V_ν. We use the symbols W and its subscripted versions to denote the symmetric group representations and the symbols V and its subscripted versions to denote the general linear group representations.

DEFINITION 1

We define $m_{\lambda\mu\nu}$ to be the multiplicity of the trivial S_d-representation inside $W_\lambda \otimes W_\mu \otimes W_\nu$.

2.1 *Schur–Weyl duality*

Below, we recall the well-known Schur–Weyl duality [15,3] which is a key to the geometric approach that we propose here. Consider the vector space $\mathbb{C}^n$ under the natural action of

$GL(n)$. This action naturally induces an action of $GL(n)$ on the space $X = (\mathbb{C}^n)^{\otimes d}$. More precisely, let $g \in GL(n)$ and $v = v_1 \otimes v_2 \otimes \cdots \otimes v_d \in X$. Then

$$g \cdot v = g \cdot v_1 \otimes g \cdot v_2 \otimes \cdots \otimes g \cdot v_d.$$

This action is extended linearly to all of X. Under this action X becomes a left $GL(n)$-module. We also have the following, natural, S_d-action on X. Let $\sigma \in S_d$. Then

$$\sigma \cdot v = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(d)}.$$

This action, with its linear extension to X, equips X with a left S_d-module structure. It is easily seen that the two actions (of $GL(n)$ and S_d) on X commute with each other, and X can be thought of as a left $GL(n) \times S_d$-module. Now, the Schur–Weyl duality asserts the following multiplicity-free decomposition of X into irreducible $GL(n) \times S_d$-representations:

$$X = (\mathbb{C}^n)^{\otimes d} = \bigoplus_{\chi} V_\chi \otimes W_\chi.$$

In the above expression, χ runs over all partitions of integer d with at most n rows.

Recall that [2] when $\chi = (d)$, i.e. χ is the partition with only one part d, $V_\chi = \mathrm{Sym}^d(\mathbb{C}^n)$ as $GL(n)$-modules, and W_χ is the trivial S_d-module. From this, the following well-known result follows.

$$X^{S_d} = \mathrm{Sym}^d(\mathbb{C}^n) \text{ as } GL(n)\text{-modules.}$$

2.2 *Our approach*

Let $n = pqr$ and U denote the vector space $\mathbb{C}^p \otimes \mathbb{C}^q \otimes \mathbb{C}^r$, identified with $\mathbb{C}^n$. In the above notation we have

$$X = (\mathbb{C}^n)^{\otimes d} = U^{\otimes d} = (\mathbb{C}^p \otimes \mathbb{C}^q \otimes \mathbb{C}^r)^{\otimes d}.$$

As above, X is equipped with a left $GL(n) \times S_d$-module structure, and $X^{S_d} = \mathrm{Sym}^d(\mathbb{C}^n)$ as $GL(n)$-modules.

Now we give another description of the space X^{S_d} of S_d-invariant vectors in X. Towards this, we have a natural action of $G = GL(p) \times GL(q) \times GL(r)$ on U induced by the fundamental actions of $GL(p)$, $GL(q)$, $GL(r)$ on $\mathbb{C}^p$, $\mathbb{C}^q$, $\mathbb{C}^r$ respectively. This action gives rise to a natural action of G on X. We also have a natural action of $S_d^{\times 3} = S_d \times S_d \times S_d$ on X; the three permutations acting independently on p-vectors, q-vectors and r-vectors respectively.

More precisely, let $v = (v_1^p \otimes v_1^q \otimes v_1^r) \otimes (v_2^p \otimes v_2^q \otimes v_2^r) \otimes \cdots \otimes (v_d^p \otimes v_d^q \otimes v_d^r) \in X$, $g = (g_1, g_2, g_3) \in G$ and $\sigma = (\sigma_p, \sigma_q, \sigma_r) \in S_d^{\times 3}$. Then

$$\begin{aligned} g \cdot v &= (g_1 \cdot v_1^p \otimes g_2 \cdot v_1^q \otimes g_3 \cdot v_1^r) \otimes (g_1 \cdot v_2^p \otimes g_2 \cdot v_2^q \otimes g_3 \cdot v_2^r) \\ &\quad \otimes \cdots \otimes (g_1 \cdot v_d^p \otimes g_2 \cdot v_d^q \otimes g_3 \cdot v_d^r), \\ \sigma \cdot v &= (v^p_{\sigma_p(1)} \otimes v^q_{\sigma_q(1)} \otimes v^r_{\sigma_r(1)}) \otimes (v^p_{\sigma_p(2)} \otimes v^q_{\sigma_q(2)} \otimes v^r_{\sigma_r(2)}) \\ &\quad \otimes \cdots \otimes (v^p_{\sigma_p(d)} \otimes v^q_{\sigma_q(d)} \otimes v^r_{\sigma_r(d)}). \end{aligned}$$

These actions are extended linearly to all of X and they clearly commute with each other. An important observation is that the S_d-action on X defined earlier coincides with the induced action of S_d, thought of as a diagonal subgroup of $S_d^{\times 3}$, induced from the $S_d^{\times 3}$-action defined above.

As X may also be thought of as $(\mathbb{C}^p)^{\otimes d} \otimes (\mathbb{C}^q)^{\otimes d} \otimes (\mathbb{C}^r)^{\otimes d}$, by repeated application of Schur–Weyl duality, we get

$$X = \left(\bigoplus_{\lambda} V_\lambda \otimes W_\lambda\right) \otimes \left(\bigoplus_{\mu} V_\mu \otimes W_\mu\right) \otimes \left(\bigoplus_{\nu} V_\nu \otimes W_\nu\right)$$

$$= \bigoplus_{\lambda,\mu,\nu} (V_\lambda \otimes V_\mu \otimes V_\nu) \otimes (W_\lambda \otimes W_\mu \otimes W_\nu).$$

The above decomposition is a multiplicity-free decomposition of X as a $G \times S_d^{\times 3}$-module. Now, we compute the space of S_d-invariant vectors in X, by restriction to the diagonal subgroup, S_d, of $S_d^{\times 3}$. Recall that, by Definition 1, $(W_\lambda \otimes W_\mu \otimes W_\nu)^{S_d} = \mathbb{C}^{m_{\lambda\mu\nu}}$. Therefore

$$X^{S_d} = \bigoplus_{\lambda,\mu,\nu} m_{\lambda\mu\nu}(V_\lambda \otimes V_\mu \otimes V_\nu) \text{ as } G\text{-modules.}$$

Thus, keeping in mind that $X^{S_d} = \mathrm{Sym}^d(\mathbb{C}^n)$, we have, as G-modules,

$$\mathrm{Sym}^d(\mathbb{C}^p \otimes \mathbb{C}^q \otimes \mathbb{C}^r) = \bigoplus_{\lambda,\mu,\nu} m_{\lambda\mu\nu}(V_\lambda \otimes V_\mu \otimes V_\nu).$$

We record this in the lemma below.

Lemma 2. *With the notation as above, $m_{\lambda\mu\nu}$ is the multiplicity of the G-module $V_\lambda \otimes V_\mu \otimes V_\nu$ inside the G-module* $\mathrm{Sym}^d(\mathbb{C}^p \otimes \mathbb{C}^q \otimes \mathbb{C}^r)$.

The G-action on U naturally gives rise to a G-action on the space $\mathbb{C}[U]$ of polynomial functions on U: for $f \in \mathbb{C}[U]$ and $g \in G$, $g \cdot f \in \mathbb{C}[U]$ is such that, $(g \cdot f)(u) = f(g^{-1} \cdot u)$ for $u \in U$. Note that the space $\mathbb{C}[U]_d$ of homogeneous polynomial functions of degree d is G-stable. Moreover, $\mathbb{C}[U]_d = \mathrm{Sym}^d(U^*)$ as G-modules. Since $\mathrm{Sym}^d(U^*) = \mathrm{Sym}^d(U)^*$ as G-modules (this follows, for instance, from the easily checked fact that, as G-modules, $(U^*)^{\otimes d} = (U^{\otimes d})^*$) and G is linearly reductive, we also have the following decomposition:

$$\mathbb{C}[U]_d = \bigoplus_{\lambda,\mu,\nu} m_{\lambda\mu\nu}(V_\lambda^* \otimes V_\mu^* \otimes V_\nu^*) \text{ as } G\text{-modules.}$$

The following proposition summarizes what we have seen so far.

PROPOSITION 3

Let $G = GL(p) \times GL(q) \times GL(r)$, and d be a positive integer. Consider the natural action of G on $U = \mathbb{C}^p \otimes \mathbb{C}^q \otimes \mathbb{C}^r$ and the induced action on the space of homogeneous functions of degree d, $\mathbb{C}[U]_d$. Further, let λ, μ, ν be three partitions of d such that λ, μ, ν have at most p, q and r rows respectively. Then,

$$m_{\lambda\mu\nu} = \text{the multiplicity of the irreducible } G\text{-module } V_\lambda^* \otimes V_\mu^* \otimes V_\nu^* \text{ in } \mathbb{C}[U]_d.$$

Our geometric approach to understanding $m_{\lambda\mu\nu}$ is based on the previous proposition. The proposition interprets $m_{\lambda\mu\nu}$ as the multiplicity of an appropriate irreducible G-module inside the space of functions on U. This strongly suggests that, understanding the geometry of G-action on U should reveal interesting information about $m_{\lambda\mu\nu}$. Observe that U is the space of tri-vectors and we have a very natural action of G on tri-vectors. Thus, what we are looking for is the geometry of G-action on tri-vectors. Naturally, this will involve a good understanding of the ring of invariants, orbits of the tri-vectors and their orbit-closures and the related algebraic geometry. This appears to be a difficult problem in general. However, we expect the problem to be tractable at least for small/special values of the various parameters involved. We strongly believe that the approach would lead to a better understanding of $m_{\lambda\mu\nu}$ for small values of p, q and r and also, for special families of shapes λ, μ and ν such as rectangular shapes.

The rest of this paper is devoted to the geometric analysis of the case when $p = q = r = 2$. The main purpose of this exercise is to demonstrate the fruitfulness of the geometric approach. In this case, we have a complete understanding of the orbits, their orbit closures and the ring of invariants. This allows us to derive an explicit formula for $m_{\lambda\mu\nu}$ when all λ, μ and ν have at most two rows (note that, there is no restriction on d of which λ, μ and ν are partitions of). The analysis already points out certain subtleties and throws some interesting questions. We believe that, based on a similar analysis, albeit, more involved, one should be able to analyze all the cases with $p, q \leq 3$ and $r = 2$.

3. $GL(2) \times GL(2) \times GL(2)$-action on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

In this section, we analyze the action of $G = GL(2) \times GL(2) \times GL(2)$ on $U = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. As a first step, we classify the orbits for this action.

3.1 *Orbits and orbit-closures*

We identify U with the space $M(2) \oplus M(2)$, of pairs (A, B), of 2×2-matrices. Next, we look at the following three actions of $GL(2)$ on this space.

Fix $(A, B) \in M(2) \oplus M(2)$. The first (left) action is given as follows: for $g_1 \in GL(2)$, $g_1 \cdot (A, B) = (g_1 A, g_1 B)$. The second (left) action is given as follows: for $g_2 \in GL(2)$, $g_2 \cdot (A, B) = (Ag_2^t, Bg_2^t)$ (here, g_2^t denotes the transpose of the matrix g_2). In order to describe the third (left) action, we let

$$g_3 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then $g_3 \cdot (A, B) = (a_{11}A + a_{12}B, a_{21}A + a_{22}B)$. These three actions are mutually commuting and hence, we get an action of G on $M(2) \oplus M(2)$. It can be easily verified that this action is 'compatible' with the standard action of G on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. We find it convenient to work with this action in order to determine the orbits geometrically.

Observe that for the action of $(g_1, g_2) \in GL(2) \times GL(2)$ on $C \in M(2)$, given by $(g_1, g_2) \cdot C = g_1 C g_2^t$, the rank of matrix C completely characterizes its orbit. Thus, there are precisely three orbits, corresponding to rank 0, rank 1 and rank 2 matrices.

Now we begin to classify orbits for the G-action on $U = M(2) \oplus M(2)$. Define

$$Z = \{(A, B) | A \text{ and } B \text{ are linearly dependent as elements of } M(2) = \mathbb{C}^4\}.$$

Clearly, both Z and its complement in U are G-stable. So, to classify orbits in U, it suffices to classify the orbits in Z and the orbits in $U\backslash Z$ separately.

We first classify orbits in Z. Note that, any $(A, B) \in Z$ can be brought to $(C, 0)$ by using the third action of $GL(2)$. Now, any $(C, 0) \in Z$ can be brought, depending on the rank of C, to one of the following three pairs, by the first and second action: $(0, 0)$, $(I, 0)$ or $(J, 0)$ where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, in Z, we have three orbits. Moreover, rank 0 matrices are in closure of rank 1 matrices, which, in turn, are in the closure of rank 2 matrices.

Now we turn our attention to $U\backslash Z$, the complement of Z. Let $(A, B) \in U\backslash Z$. Consider the 2-dimensional plane spanned by A and B in $M(2) = \mathbb{C}^4$. Given any (C, D), which is another (ordered) basis of this plane, we can go from (A, B) to (C, D) using the third action. Therefore, the G-orbits in $U\backslash Z$ are in one-to-one correspondence with the orbits of 2-dimensional planes in $M(2) = \mathbb{C}^4$ (under the induced first and second actions on 2-dimensional planes).

At this point, we find it convenient to work with (projective) lines in $\mathbb{P}^3$ (projectivization of $M(2)$). Note that the first and second action of $GL(2)$ on $M(2)$ translate to the following linear action of $GL(2) \times GL(2)$ on $M(2)$: for $(g_1, g_2) \in GL(2) \times GL(2)$ and $A \in M(2)$, $(g_1, g_2) \cdot A = g_1 A g_2^t$. This action, being linear, descends to the corresponding projectivization $\mathbb{P}^3$ and also, to lines in $\mathbb{P}^3$.

For the sake of clarity, we write down this action below. Let $H = GL(2) \times GL(2)$. Then, in homogeneous co-ordinates, we have: for $(g_1, g_2) \in H$ and $[A] \in \mathbb{P}^3$, $(g_1, g_2) \cdot [A] = [g_1 A g_2^t]$. Moreover, via (g_1, g_2), the line passing through points $[A]$ and $[B]$ of $\mathbb{P}^3$ is sent to the line passing through $[g_1 A g_2^t]$ and $[g_1 B g_2^t]$. Recall that, we want to classify the lines in $\mathbb{P}^3$ for this H-action. Towards this, consider the H-stable hypersurface DET in $\mathbb{P}^3$ given by the vanishing of the determinant 'function' (recall that $\mathbb{P}^3$ is the projectivization of the space of 2×2-matrices).

As DET is H-stable, we have the following three H-stable subsets of lines in $\mathbb{P}^3$: the ones entirely contained in DET, the ones tangent to the DET and proper chords to the DET. Note that any line in $\mathbb{P}^3$ intersects DET (DET being a projective subvariety).

Observe that H acts transitively on DET and also on its complement in $\mathbb{P}^3$. Let ℓ be a line in $\mathbb{P}^3$ not entirely contained in DET. We may assume, without loss of generality, that ℓ passes through $[I]$. Let $[A]$ be a point in which ℓ intersects DET. Clearly, A (or any other representative for $[A]$) has rank 1. Note that, the subgroup $K = \{(g, (g^{-1})^t) | g \in GL(2)\}$ of H fixes the point $[I]$. Now, we use K to move $[A]$. It is not difficult to see that $[A]$ can be brought to one of the following two points of $\mathbb{P}^3$:

$$J_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad J_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Roughly speaking, we just move A to its Jordan canonical form via K.

Thus, we have shown that any line ℓ not entirely contained in DET, has one of the following lines in its orbit

$$\ell_1 = \text{the line joining } [I] \text{ and } [J_1], \quad \text{or} \quad \ell_2 = \text{the line joining } [I] \text{ and } [J_2].$$

An easy calculation reveals that ℓ_1 is a proper chord to the DET while ℓ_2 is a tangent to the DET hypersurface. Moreover, the orbit of ℓ_1 has the orbit of ℓ_2 in its closure.

Now we focus on the lines which are entirely contained in DET. We may assume, without loss of generality, that ℓ passes through $[J]$ where

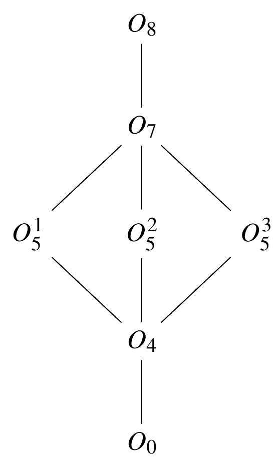
$$J = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then it is easy to see that any line completely contained in DET, and passing through $[J]$, can be brought to one of the following lines:

$$\ell_1 = \text{the line joining } [J] \text{ and } \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right], \quad \text{or} \quad \ell_2 = \text{the line joining } [J] \text{ and } \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right].$$

So there are four H-orbits of lines in $\mathbb{P}^3$, which give rise to four G-orbits in $U \setminus Z$. Therefore, in total, we have seven G-orbits in $U = M(2) \oplus M(2)$.

Now we switch back to U defined to be $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The following figure depicts all the orbits in U along with their closures. The orbits are ordered under the relation $\leq$ defined as follows: $O \leq O'$ if $O \subseteq \overline{O'}$. The left part of the figure depicts the Hasse diagram for this partial order. On the right part of the figure, we have also written candidate points in the corresponding orbits on the left. Here we have used the symbols 0 and 1 as a basis of $\mathbb{C}^2$ and omitted the tensor symbols. Thus, for example, the orbit O_5^2 contains the point $1 \otimes 0 \otimes 0 + 0 \otimes 0 \otimes 1 \in U$. The expressions for candidate points are arrived at from the analysis done above. The symbols used to denote the orbits also encode the dimension of the orbit in the subscript. Thus, there is one orbit of dimensions 8, 7, 4 and 0 each, and three orbits of dimension 5. These dimensions may be obtained by computing the dimensions of the stabilizers of the candidate points given on the right. We omit the details.



O_8	$000 + 111$
O_7	$x = 100 + 010 + 001$
O_5^1, O_5^2, O_5^3	$010 + 001$, $100 + 001$, $100 + 010$
O_4	000
O_0	$*$

For later use, we let $x = 100+010+001$ denote a candidate point in the orbit $Gx = O_7$. Further, we set $X = \overline{O_7} \subset U$. The next subsection is devoted to a detailed geometric study of X.

3.2 *Geometry of X*

It turns out that X is a hypersurface in U, given by the vanishing of a homogeneous semi-invariant function f, of degree 4. In order to show this, we work with the earlier identification of U with $M(2) \oplus M(2)$.

Recall that the orbit O_7 corresponds to lines in $\mathbb{P}^3$ which are tangent to the DET hypersurface. Using this, one can easily write down the equation of f as follows. Let $(A, B) \in M(2) \oplus M(2)$ and consider the quadratic equation in variable t,

$$\det(A + tB) = 0.$$

The coefficients of this equation are quadratic in the entries of A and B. If $(A, B) \in O_7$, then the 'line' $A+tB$ is tangent to the DET hypersurface and therefore, the discriminant of this equation vanishes. The discriminant of the above equation is a degree four homogeneous function on U which we denote by f. It is easily shown that f is an irreducible semi-invariant and it generates the ideal of X. In fact,

$$(g_1, g_2, g_3) \cdot f = (\det(g_1)\det(g_2)\det(g_3))^{-2} f.$$

It can also be easily shown that f generates the ring of invariants for the semi-simple part $\hat{G} = SL(2) \times SL(2) \times SL(2)$ of G.

$$\mathbb{C}[U]^{\hat{G}} = \mathbb{C}[f].$$

Recall that, from the analysis of orbits, we have $X = O_7 \cup O_5^1 \cup O_5^2 \cup O_5^3 \cup O_4 \cup O_0$. Clearly O_7 is open in X and it has co-dimension 2 in X as there is no six dimensional orbit. Since X is an irreducible hypersurface, this implies that X is a normal variety and, so $\mathbb{C}[X] = \mathbb{C}[O_7]$.

Let H be the stabilizer of x. Then $O_7 = G/H$. Consider the map ϕ: $G \to O_7 \hookrightarrow X$ given by $\phi(g) = g \cdot x$. This map induces a G-map ϕ: $G/H \to X$. At the co-ordinate ring level, ϕ^*: $\mathbb{C}[X] \to \mathbb{C}[O_7] \to \mathbb{C}[G/H]$ is a G-isomorphism (thanks to the normality of X, the natural restriction map $\mathbb{C}[X] \to \mathbb{C}[O_7]$ is a G-isomorphism).

3.3 *G-module structure of* $\mathbb{C}[X]$

Since $\mathbb{C}[X] = \mathbb{C}[G/H]$ as G-modules, we can obtain the G-module structure of $\mathbb{C}[X]$ from that of $\mathbb{C}[G/H]$. Towards this, we recall the fundamental Peter–Weyl theorem [3]. Consider the left action of $G \times G$ on G given as: for $(g_1, g_2) \in G \times G$, $g \in G$, $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$. Further, consider the induced action of $G \times G$ on $\mathbb{C}[G]$. Then, the Peter–Weyl theorem (see [3]) says that, as $G \times G$-modules

$$\mathbb{C}[G] = \oplus_\chi K_\chi \otimes K_\chi^*.$$

It is well-known [3] that $\mathbb{C}[G/H] = \mathbb{C}[G]^H$ where the H-action on G is via restriction of the $G \times G$-action to the subgroup $\{e\} \times H$. Therefore, as a G-module (the left copy $G \times \{e\} \subset G \times G$),

$$\mathbb{C}[G/H] = \mathbb{C}[G]^H = \oplus_\chi K_\chi \otimes (K_\chi^*)^H.$$

As $\mathbb{C}[X] = \mathbb{C}[G/H]$, the multiplicity of G-module K_χ inside $\mathbb{C}[X]$ is the dimension of $(K_\chi^*)^H$.

Recall that the homogeneous ideal $I(X) = f\mathbb{C}[U]$ where f is the degree four homogeneous semi-invariant. Therefore, we have the following exact sequence of G-modules and G-maps

$$0 \to f\mathbb{C}[U]_{d-4} \to \mathbb{C}[U]_d \to \mathbb{C}[X]_d \to 0.$$

Further, recall that $m_{\lambda\mu\nu}$ = multiplicity of $V_\lambda^* \otimes V_\mu^* \otimes V_\nu^*$ inside $\mathbb{C}[U]_d$. We now define $n_{\lambda\mu\nu}$ = multiplicity of $V_\lambda^* \otimes V_\mu^* \otimes V_\nu^*$ inside $\mathbb{C}[X]_d$. Note that, as λ, μ, ν are partitions of d, by degree constraints, $n_{\lambda\mu\nu}$ is also the multiplicity of $V_\lambda^* \otimes V_\mu^* \otimes V_\nu^*$ inside all of $\mathbb{C}[X]$. Therefore,

$$n_{\lambda\mu\nu} = \text{ dimension of } (V_\lambda \otimes V_\mu \otimes V_\nu)^H.$$

An easy calculation shows that, H consists of matrices of the form

$$\left(\begin{pmatrix} x & a \\ 0 & \frac{1}{yz} \end{pmatrix}, \begin{pmatrix} y & b \\ 0 & \frac{1}{xz} \end{pmatrix}, \begin{pmatrix} z & c \\ 0 & \frac{1}{xy} \end{pmatrix}\right), \qquad \text{where} \quad a + b + c = 0.$$

It turns out that there is at most 1 H-invariant (up to scalars) in $V_\lambda \otimes V_\mu \otimes V_\nu$. The following proposition characterizes exact conditions on λ, μ and ν under which the dimension of the space of H-invariants is one.

PROPOSITION 4

Let $\lambda = (d-a, a)$, $\mu = (d-b, b)$, $\nu = (d-c, c)$. *Without loss of generality, we assume* $a \leq b \leq c$. *If* $a+b-c \geq 0$ *and* $(a+b+c)-d \leq 0$, *then* $n_{\lambda\mu\nu} = 1$. *Otherwise,* $n_{\lambda\mu\nu} = 0$.

We postpone the proof of this proposition to a later subsection. Here, we use the proposition to obtain the G-module structure of $\mathbb{C}[U]$.

3.4 *G-module structure of* $\mathbb{C}[U]$

Recall the exact G-sequence

$$0 \to f\mathbb{C}[U]_{d-4} \to \mathbb{C}[U]_d \to \mathbb{C}[X]_d \to 0.$$

With $\delta = (2, 2)$, V_δ is the one-dimensional $GL(2)$-representation where $g \in GL(2)$ acts via the scalar $\det(g)^2$. As

$$(g_1, g_2, g_3) \cdot f = (\det(g_1)\det(g_2)\det(g_3))^{-2} f,$$

$f \cdot V_{\lambda'}^* \otimes V_{\mu'}^* \otimes V_{\nu'}^* = V_{\lambda'+\delta}^* \otimes V_{\mu'+\delta}^* \otimes V_{\nu'+\delta}^*$ as a G-module.

Set $\partial\lambda = \lambda - \delta$ (remove a 2×2 square from the diagram of λ) whenever possible. Then, $f \cdot V_{\partial\lambda}^* \otimes V_{\partial\mu}^* \otimes V_{\partial\nu}^* = V_\lambda^* \otimes V_\mu^* \otimes V_\nu^*$. Thus, under the multiplication by f, the G-modules in $\mathbb{C}[U]_{d-4}$ of type $V_{\partial\lambda}^* \otimes V_{\partial\mu}^* \otimes V_{\partial\nu}^*$ get translated to the G-modules of type $V_\lambda^* \otimes V_\mu^* \otimes V_\nu^*$. This immediately yields the following formula:

$$m_{\lambda\mu\nu} = n_{\lambda\mu\nu} + m_{\partial\lambda\partial\mu\partial\nu}.$$

By repeated application of the above formula, we get

$$m_{\lambda\mu\nu} = n_{\lambda\mu\nu} + n_{\partial\lambda\partial\mu\partial\nu} + n_{\partial^2\lambda\partial^2\mu\partial^2\nu} + \cdots .$$

Let $\lambda = (d-a, a)$, $\mu = (d-b, b)$, $\nu = (d-c, c)$ with $a \leq b \leq c$. Then, as the partition $\partial^j\lambda = (d-a-2j, a-2j)$ and so on, by Proposition 4 applied to $\partial^j\lambda$, $\partial^j\mu$, $\partial^j\nu$, we see that $n_{\partial^j\lambda\partial^j\mu\partial^j\nu}$ is 1 if $a+b-c \geq 2j$ and $(a+b+c)-d \leq 2j$. Otherwise $n_{\partial^j\lambda\partial^j\mu\partial^j\nu}$ is 0. Thus, $n_{\partial^j\lambda\partial^j\mu\partial^j\nu}$ is 1 if j lies in the interval $[\lceil(a+b+c-d)/2\rceil, \lfloor(a+b-c)/2\rfloor]$; else it is 0. This coupled with the above derived formula for $m_{\lambda\mu\nu}$ immediately proves our main theorem, which appeared first in [12].

Theorem 5. *Let* $\lambda = (d-a, a)$, $\mu = (d-b, b)$, $\nu = (d-c, c)$. *Without loss of generality, we assume* $a \le b \le c$. *Then*

$$m_{\lambda\mu\nu} = \textit{ size of } [\lceil (a+b+c-d)/2 \rceil, \lfloor (a+b-c)/2 \rfloor].$$

The next subsection is devoted to the proof of Proposition 4.

3.5 *The proof of Proposition* 4

Fix $\lambda = (d-a, a)$, $\mu = (d-b, b)$, $\nu = (d-c, c)$ with $a \le b \le c$. We are interested in computing H-invariants in the G-module $V_\lambda \otimes V_\mu \otimes V_\nu$. Let V abbreviate $V_\lambda \otimes V_\mu \otimes V_\nu$. Recall that H consists of matrices of the form

$$\left(\begin{pmatrix} x & a \\ 0 & \frac{1}{yz} \end{pmatrix}, \begin{pmatrix} y & b \\ 0 & \frac{1}{xz} \end{pmatrix}, \begin{pmatrix} z & c \\ 0 & \frac{1}{xy} \end{pmatrix} \right), \quad \text{where} \quad a+b+c=0.$$

Let $\mathfrak{g}$, $\mathfrak{h}$ be the Lie algebras for the groups G and H respectively. Then, the G-module V gives rise to a $\mathfrak{g}$-module V. We first compute $\mathfrak{h}$-invariants for $\mathfrak{g}$-module V. Note that, $\mathfrak{h}$-invariants are vectors annihilated by $\mathfrak{h}$.

As $\mathfrak{g} = \mathfrak{gl}_2 \oplus \mathfrak{gl}_2 \oplus \mathfrak{gl}_2$, we have the standard elements $e, f \in \mathfrak{gl}_2$ whose actions on V_λ, V_μ and V_ν are easily described. Recall that

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The main idea here is to find a nice basis for $\mathfrak{h}$ in terms of e's and f's so that $\mathfrak{h}$-action for the basis elements can be explicitly described. Further, we write down linear conditions for $\mathfrak{h}$-invariance for the basis and analyze the solutions to these linear conditions.

From the form of elements of H, it is easily checked that the nilpotent part of $\mathfrak{h}$ admits the following basis:

$$e_1 = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) = (e, -e, 0),$$

$$e_2 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \right) = (0, e, -e).$$

We set $a' = d - 2a$, $b' = d - 2b$, $c' = d - 2c$. Then, V_λ has a basis $[0], [1], \ldots [a']$. Similarly, with an abuse of notation, V_μ has a basis $[0], \ldots, [b']$ and V_ν has a basis $[0], \ldots, [c']$. Further, we have $e \cdot [0] = 0$ and $e \cdot [i+1] = [i]$. Thus, for the tensor basis, $\{[i] \otimes [j] \otimes [k] \mid 0 \le i \le a', 0 \le j \le b', 0 \le k \le c'\}$, the actions of e_1 and e_2 are given as follows:

$$e_1 \cdot ([i] \otimes [j] \otimes [k]) = [i-1] \otimes [j] \otimes [k] - [i] \otimes [j-1] \otimes [k],$$

$$e_2 \cdot ([i] \otimes [j] \otimes [k]) = [i] \otimes [j-1] \otimes [k] - [i] \otimes [j] \otimes [k-1].$$

In order to analyze the action of toral part of $\mathfrak{h}$, one could also work at the group level. Let

$$t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}.$$

Then, for $[i] \in V_\lambda$, $t \cdot [i] = (t_1t_2)^a t_1^{a'-i} t_2^i [i]$. Similarly, for $[j] \in V_\mu$, $t \cdot [j] = (t_1t_2)^b t_1^{b'-j} t_2^j [j]$, and, for $[k] \in V_\nu$, $t \cdot [i] = (t_1t_2)^c t_1^{c'-k} t_2^k [k]$.

Note that, the following matrices form a maximal torus, say T, of H:

$$\left(\begin{pmatrix} x & 0 \\ 0 & \frac{1}{yz} \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & \frac{1}{xz} \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & \frac{1}{xy} \end{pmatrix} \right).$$

The action of an element t in the above form on $[i] \otimes [j] \otimes [k]$ simplifies to

$$t \cdot ([i] \otimes [j] \otimes [k]) = (xyz)^{d-(a+b+c)-(i+j+k)}([i] \otimes [j] \otimes [k]).$$

Thus, for $[i] \otimes [j] \otimes [k]$ to be invariant under T, it must necessarily be the case that $d-(a+b+c) = i+j+k$. Note that, as $[i]\otimes[j]\otimes[k]$ is a simultaneous eigenbasis of T, any T-invariant vector is a linear combination of $\{[i]\otimes[j]\otimes[k] \mid i+j+k = d-(a+b+c)\}$.

Note that, we have $0 \le i + j + k \le a' + b' + c' = 3d - (2a + 2b + 2c)$. Further, given an integer l such that $0 \le l \le 3d - (2a + 2b + 2c)$, there exists $[i]$, $[j]$, $[k]$ such that $i + j + k = l$. Therefore, a T-invariant vector exists iff $0 \le d - (a + b + c) \le 3d - (2a + 2b + 2c)$. The condition $d - (a + b + c) \le 3d - (2a + 2b + 2c)$ can be rewritten as $a + b + c \le 2d$. Observe that as λ, μ, ν are partitions, $2a \le d$, $2b \le d$ and $2c \le d$. Using these facts, it follows that, we always have $a + b + c \le 2d$. Therefore, the condition $d - (a + b + c) \le 3d - (2a + 2b + 2c)$ is redundant. Therefore, we conclude that a T-invariant vector exists iff $0 \le d - (a + b + c)$, equivalently, $(a + b + c) - d \le 0$ (this is one of the conditions stated in Proposition 4). Further, in this case, any T-invariant vector is a linear combination of $\{[i] \otimes [j] \otimes [k] \mid i + j + k = d - (a + b + c)\}$.

For further discussion, we assume that $(a + b + c) - d \le 0$. Recall that, as $a \le b \le c$, $c' \le b' \le a'$. Now, we turn our attention to those T-invariant vectors which are killed by the nilpotent part of $\mathfrak{h}$, that is, by both e_1 and e_2.

For l such that $0 \le l \le 3d - (2a + 2b + 2c)$, let V_l denote the subspace of V spanned by $\{[i] \otimes [j] \otimes [k] \mid i + j + k = l\}$. With $l_0 = d - (a + b + c)$, V_{l_0} is the space of all T-invariant vectors. Observe that e_1 and e_2 both map V_{l_0} to V_{l_0-1}.

3.5.1 *Two-dimensional calculation.* The analysis of vectors killed by e_1 and e_2 is mainly based on the following interesting 'two-dimensional' calculation. Towards this, let W_1 and W_2 be spaces of dimensions $p+1$ and $q+1$ respectively with $p \le q$. By abuse of notation, we let $\{[0], \ldots, [p]\}$ and $\{[0], \ldots [q]\}$ be bases of W_1 and W_2 respectively. Thus, $W = W_1 \otimes W_2$ has a natural basis $\{[i] \otimes [j] \mid 0 \le i \le p, 0 \le j \le q\}$.

For l such that $0 \le l \le p + q + 2$, let W_l denote the subspace of W spanned by $\{[i] \otimes [j] \mid i + j = l\}$. Clearly, W_l is of dimension $l+1$. Now, we define the nilpotent operator e_W on W by the following formula:

$$e_W([i] \otimes [j]) = [i-1] \otimes [j] - [i] \otimes [j-1].$$

It is clear that e_W maps W_l to W_{l-1}.

It can be shown, by elementary linear algebra, that for l such that

$q+1 < l \le p + q + 2$, $\quad e_W$ restricted to W_l is injective;

$p+1 < l \le q+1$, $\quad e_W$ is an isomorphism between W_l and W_{l-1};

$0 \le l \le p+1$, $\quad e_W$ restricted to W_l is surjective.

Moreover, in the last case, that is, when $0 \leq l \leq p+1$, all the vectors in W_l killed by e_W are scalar multiples of

$$\sum_{i+j=l} [i] \otimes [j].$$

Now we revisit the operators e_1 and e_2 on V. Note that, V is the 'three-dimensional' analogue of W above. Further, both e_1 and e_2 operate only along two 'directions'. Thus, it is possible to apply the two-dimensional analysis above to both of them.

Recall that V has the natural basis $\{[i] \otimes [j] \otimes [k] \mid 0 \leq i \leq a', 0 \leq j \leq b', 0 \leq k \leq c'\}$ with $c' \leq b' \leq a'$. Based on the two-dimensional analysis, one can show that, there exists a non-zero T-invariant vector in V_{l_0} which is killed by both e_1 and e_2 iff $l_0 \leq c'$. Further, in this case, all the vectors in V_{l_0} which are killed by both e_1 and e_2 are scalar multiples of

$$\sum_{i+j+k=l_0} [i] \otimes [j] \otimes [k].$$

The condition $l_0 \leq c'$ translates to $d - (a + b + c) \leq d - 2c$ which is equivalent to $a + b - c \geq 0$ (this is the 'other' condition stated in Proposition 4).

Therefore, the space of $\mathfrak{h}$-invariants in V is one-dimensional if $a + b - c \geq 0$ and $(a + b + c) - d \leq 0$. Otherwise there is no non-zero $\mathfrak{h}$-invariant in V. It can also be argued that $\mathfrak{h}$-invariant vectors in V indeed give rise to H-invariant vectors in V. This completes the proof of Proposition 4.

4. Discussion

In this paper, we have proposed a geometric approach to the problem of tensor product multiplicities for the symmetric group, and shown its effectiveness by obtaining explicit formulas for $m_{\lambda\mu\nu}$ in the case of representations parametrized by two row shapes (λ, μ and ν have at most two rows). We believe that similar ideas would be useful to obtain the multiplicities $m_{\lambda\mu\nu}$ when λ, μ have at most three rows and ν has at most two rows. For this problem one needs to analyze the natural $GL(3) \times GL(3) \times GL(2)$-action on $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2$. A geometric analysis, similar to the one we have done in this paper, allows one to classify orbits and orbit closures in this case. One can also arrive at a good understanding of the ring of invariants. The main issue here is that the algebraic geometry of these orbit closures is more complex, and needs to be done more carefully. For example, normality of certain closures (which plays an important role here) is no longer true in this case.

A classification of orbits for the natural action of $GL(3) \times GL(3) \times GL(3)$ on $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ has also been carried out in [10]. The motivation in [10] is however very different, and it is not clear how to use the results from [10] to address our problem.

There are some more cases where the geometric approach seems to give a better understanding of $m_{\lambda\mu\nu}$. In these cases, typically, some of the shapes λ, μ or ν are restricted to rectangles. Again the geometry involved is rich, and has connections to some classical results in invariant theory, such as the Artin–Procesi theorem. This is carried out in [1].

For $m_{\lambda\mu\nu}$ with general λ, μ and ν, one needs a better understanding of $GL(p) \times GL(q) \times GL(r)$ action on $\mathbb{C}^p \otimes \mathbb{C}^q \otimes \mathbb{C}^r$. It looks promising to study the related $GL(p) \times GL(q)$

action on the Grassmanian $\mathrm{Gr}(r, pq)$. This may turn out to be tractable at least for small values of r.

We conclude this section with another related problem. Consider the natural algebraic group morphism from $GL(p) \times GL(q)$ to $GL(pq)$ induced by the natural action of $GL(p) \times GL(q)$ on $\mathbb{C}^p \otimes \mathbb{C}^q$ identified with $\mathbb{C}^{pq}$. In terms of matrices, $(A, B) \in GL(p) \times GL(q)$ maps to $A \otimes B \in GL(pq)$ where $\otimes$ denotes the Kronecker tensor product of matrices. Via this morphism, a $GL(pq)$-module becomes a $GL(p) \times GL(q)$-module. Now, given an irreducible $GL(pq)$-module V_λ, the problem is to decompose V_λ as a $GL(p) \times GL(q)$-module. Let $d^\lambda_{\mu\nu}$ denote the multiplicity of the irreducible $GL(p) \times GL(q)$-module $V_\mu \otimes V_\nu$ in V_λ. It is known that $d^\lambda_{\mu\nu} = m_{\lambda\mu\nu}$. Therefore, the formulas derived here also give us an explicit decomposition of an irreducible $GL(4)$-module, given by a partition with at most two rows, into irreducible $GL(2) \times GL(2)$-modules.

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