Proc. Indian Acad. Sci. (Math. Sci.) Vol. 118, No. 2, May 2008, pp. 207–211.

Some remarks on the local fundamental group scheme

V B MEHTA and S SUBRAMANIAN

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India
E-mail: vikram@math.tifr.res.in; subramnn@math.tifr.res.in

MS received 4 April 2006; revised 3 October 2007

Abstract. We define the local fundamental group scheme and study its properties under base change of the base field.

Keywords. F-trivial vector bundle; local fundamental group scheme.

1. Introduction

The fundamental group-scheme was introduced by Nori in [4] and [5]. In the paper [5], he had made two conjectures. The first conjecture was proved by the present authors in [2]. In an attempt to prove the second conjecture, we had introduced the 'local fundamental group-scheme' which is defined only in characteristic p. In this note, we gather together some necessary and sufficient conditions for it to base-change 'correctly' from one algebraically closed field to another algebraically closed field of characteristic p.

2. The local fundamental group scheme

We define the local fundamental group-scheme, denoted by π^{loc}, as follows:

Let X be a smooth projective variety over an algebraically closed field of characteristic p, and let $F: X \to X$ be the Frobenius map. Let $FT(X)$ denote the category of all vector bundles V on X such that $F^{t*}(V)$ is *trivial* for some integer t. Fix a base point $x \in X$ and let $T: FT(X) \to \mathrm{Vect}(k)$ be given by $V \to V_x$, where V_x denotes the fibre of V at x. One checks easily that $FT(X)$, with the fibre functor T, is a Tannaka category (see [1,5] for Tannaka categories). The corresponding affine group-scheme is defined to be the *local fundamental group-scheme* of X, denoted by $\pi^{\mathrm{loc}}(X)$. Note that $\pi^{\mathrm{loc}}(X)$ is defined only in characteristic p.

Remark 1. We have a canonical surjection $\pi(X) \to \pi^{\mathrm{loc}}(X)$, where $\pi(X)$ is the fundamental group scheme as defined by Nori [5]. Let

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

be an exact sequence of bundles on X, with all three bundles essentially finite, and suppose that V_2 is F-trivial. Pull up V_2 to the local group scheme cover of X where V_2 is trivial. It follows that both V_1 and V_3 are also trivial on that cover, so they are also F-trivial. The surjectivity follows by applying the criterion of Proposition 5, page 121 in [5].

Remark 2. We observe that every vector bundle in $FT(X)$ is of degree zero, and further, every subquotient of degree zero of a F-trivial vector bundle is also in $FT(X)$. This property is crucial in verifying that $FT(X)$ satisfies all the axioms of a Tannaka category.

Lemma 1.1.

(a) *If* $V \in FT(X)$, *then* V *is semistable for any polarization* H *on* X.
(b) *If* V *is stable for* 1 *polarization*, *then* V *is stable with respect to any polarization.*

Proof.

(a) This follows from Remark 1 and Theorem 3.4 of [4], but we provide a sketch proof: if $F^{t*}(V)$ is a trivial vector bundle on X for some t, then it is semistable for any polarization. It follows that V itself is semistable with respect to any polarization.
(b) If $V \in FT(X)$, then there exists

  (1) a principal G-bundle

$$E \to X,$$

  where G is a finite local group-scheme, and E is *reduced* (see Proposition 3, p. 87 and Proposition 3.10, p. 83 of [5])
  (2) and a representation

$$\sigma: G \to GL(n), n = \text{ rank } V,$$

  such that V is associated to the G bundle E by the representation σ. It follows easily that subbundles W of V with $\mu(W) = \mu(V) = 0$ are given by G subspaces W_1 of k^n. Hence if $V \in FT(X)$ and stable with respect to one polarization, then σ is irreducible (we observe that since $\mu(V)$ is zero for $V \in FT(X)$, stability is actually equivalent to the irreducibility of the representation of G defining V) so V is stable with respect to any polarization.

DEFINITION 1.2

Fix positive integers r and t. Let $S(X, r, t)$ denote the *set of isomorphism classes* of stable vector bundles V on X such that rank $V = r$ and $F^{t*}(V)$ is trivial.

We shall show that $S(X, r, t)$ corresponds bijectively to the *closed* points of a scheme $M(X, r, t)$ defined over k.

Theorem. *The following statements are equivalent.*

(1)

$$g: \pi^{\text{loc}}(X_K, x) \to \pi^{\text{loc}}(X, x) \times_k K$$

is an isomorphism, for every extension of algebraically closed fields $K \supset k$.
(2) $S(X; r, t)$ *is a finite set, for all* r *and* t.
(3) *For every extension of algebraically closed fields* $K \supset k$, *every* $V \in S(X_K; r, t)$ *descends to* X, *i.e. is defined over* k.

Proof. Before we prove that the statements are equivalent, we discuss in brief some generalities on Tannaka categories. Let $\mathcal{C}'$ be the Tannaka category of F-trivial vector bundles on X. Then $\mathcal{C}'$ is isomorphic to the category of rational, finite dimensional representations of $\pi^{\text{loc}}(X, x)$. We consider the category $\mathcal{C}'_1$ defined as follows:

A vector bundle V on X_K is an object of $\mathcal{C}'_1$ if there is an F-trivial vector bundle V_1 on X such that V is a degree zero subquotient of $V_1 \otimes_k K$ on X_K. The morphisms of $\mathcal{C}'_1$ are vector bundle homomorphisms and it is easily seen that with the tensor product of vector bundles as the tensor operation in the category and the fibre functor which associates to a bundle V its fibre over the base point x of X_K, $\mathcal{C}'_1$ is a Tannaka category.

Let R' denote the Tannaka category of rational, finite dimensional representations of $\pi^{\text{loc}}(X, x) \times_k K$ over K. We can define a functor from R' to $\mathcal{C}'_1$ as follows:

If V is an object of R' namely a representation of $\pi^{\text{loc}}(X, x) \times_k K$, then there is a finite, local group scheme G over k and a surjection $\pi^{\text{loc}}(X, x) \to G$ such that V is actually a representation of $G_K = G \times_k \operatorname{spec} K$. The homomorphism $\pi^{\text{loc}}\ (X, x) \to G$ defines a principal G-bundle $\pi \colon E \to X$ on X. Let $\pi \colon E_K \to X_K$ be the base change of this bundle to K so that $E_K \to X_K$ is a principal G_K bundle. The representation V of G_K associates to this principal bundle a vector bundle (which is F-trivial) denoted by $\tilde{V}$. The G_K module V is a subquotient of $K[G]^{\oplus n}$, where $K[G]$ denotes the coordinate ring of G_K. As $K[G] = k[G] \otimes_k K$ where $k[G]$ denotes the coordinate ring of G over k, if we let V_1 be the vector bundle on X associated to the G-bundle $\pi \colon E \to X$ by the representation of G on $k[G]^{\oplus n}$, we obtain that $\tilde{V}$ is a degree zero subquotient of $V_1 \otimes_k K$. Hence $\tilde{V}$ is an object of $\mathcal{C}'_1$. It is easily seen that the functor from R' to $\mathcal{C}'_1$ defined as above is an equivalence of Tannaka categories, so the affine group scheme associated to $\mathcal{C}'_1$ is $\pi^{\text{loc}}\ (X, x) \times_k K$.

Let $\mathcal{D}'$ denote the Tannaka category of all F-trivial vector bundles on X_K. Since every object of $\mathcal{C}'_1$ is F-trivial $\mathcal{C}'_1$ is a Tannaka subcategory of $\mathcal{D}'$. The affine group scheme associated to $\mathcal{D}'$ is $\pi^{\text{loc}}\ (X_K, x)$ and the functor $\mathcal{C}' \subset \mathcal{D}'$ induces the canonical homomorphism

$$g \colon \pi^{\text{loc}}(X_K, x) \to \pi^{\text{loc}}(X, x) \times_k K.$$

An application of Proposition 5, p. 121 of [5] shows that g is surjective. Now we prove that (1), (2) and (3) are equivalent.

1 ⇒ 3. If g is an isomorphism, then it is, in particular, a closed immersion, and arguing as in Proposition 3.1 of [2], we see that any stable F-trivial bundle over X_K descends to X.

3 ⇒ 1. We now assume that every stable F-trivial bundle on X_K is defined over X. We have to show that g is a closed immersion. We use Proposition 2.21(b) in [1] for this purpose. We have to show that any object of $\mathcal{D}'$, namely an F-trivial bundle on X_K, is a subquotient of an object of $\mathcal{C}'_1$. Since any F-trivial vector bundle on X_K is semistable, it has a filtration by stable F-trivial bundles on X_K. By hypothesis, the stable F-trivial bundles descend to X, and we are left with an extension of stable bundles. For simplicity, let V be an object of $\mathcal{D}'$, which is an extension.

$$0 \to V_1 \to V \to V_2 \to 0,$$

where V_1 and V_2 are stable F-trivial bundles on X_K. The extension bundle V defines an element of $\operatorname{Ext}^1_{\mathcal{O}_{X_K}}(V_2, V_1)$. However, we have

$$\operatorname{Ext}^1_{\mathcal{O}_{X_K}}(V_2, V_1) = \operatorname{Ext}^1_{\mathcal{O}_X}(V_2, V_1) \otimes_k K.$$

Thus, the element representing V in $\mathrm{Ext}^1_{\mathcal{O}_{X_K}}(V_2, V_1)$ can be written as $\sum_i \alpha_i e_i$, where $\alpha_i \in K$ and $e_i \in \mathrm{Ext}^1_{\mathcal{O}_X}(V_2, V_1)$. By the construction of Baer sum in Ext^1 (see Definition 3.4.4, p. 78 of [6]), we see that $\sum \alpha_i e_i$ represents a subquotient of extensions of V_2 by V_1 on X_k. Hence we obtain that V is a subquotient of a F-trivial bundle defined over X. The case where there are more than two stable factors in the stable filtration of V is similar.

2 ⇔ 3. For proving this equivalence, we recall some basic facts, details of which may be found in Chapter 5 of [3]. We do not follow all the notations in [3], in particular, we use p for the characteristic of the base field. Let $Q(E/P)$ denote the scheme of quotient sheaves of a fixed vector bundle E on X, such that the Hilbert polynomial of the quotient sheaves is the fixed polynomial P. Then $Q(E/P)$ is a projective algebraic scheme. There exists a coherent sheaf U over $Q(E/P) \times X$ and a surjective homomorphism of sheaves $p_X^* E \to U$ such that U is flat over $Q(E/P)$ and has the universal property for flat quotients of E with Hilbert polynomial P. When X is a smooth projective curve, we consider $E = \mathcal{O}_X^{\mathcal{N}}$ and the polynomial $P(m) = r(mh - g + 1)$, where h is a positive integer denoting the degree of an ample line bundle on X. We let $Q = Q(E/P)$ and let

$$R = \{q \in Q \text{ such that } U_q \text{ is locally free and } H^0(\mathcal{O}_X^N) \to H^0(U_q) \text{ is an isomorphism}\}.$$

The group $GL(N)$ acts on Q and on U. Let

$$R^s = \{q \in R \text{ such that } U_q \text{ is stable}\}.$$

Let O_t be the open subset of R^s (for the definition of R^s, see §4, p. 141 of [3]) defined by

$$O_t = \{x \in R^s | F^{t*}(U_x) \text{ is semistable on } X \text{ where } F \text{ is the Frobenius of } X\}.$$

Let Y_t be the closed subset of O_t defined by

$$\{y \in O_t | F^{t*}U_y \otimes \mathcal{O}_X(-mhp^t) \text{ is trivial}\}$$

and we give it the reduced subscheme structure. Note that Y_t is closed in O_t, and not necessarily closed in R^s. As O_t is open and $GL(N)$ invariant in R^s, the quotient $O_t/GL(N)$ exists as a scheme. Since Y_t is closed and $GL(N)$ invariant in O_t, the quotient $Y_t/GL(N)$ exists as a scheme, and let this scheme be denoted by $M(X, r, t)$. The k-rational points of $M(X, r, t)$ are in bijection with $S(X, r, t)$. It is further clear that for an extension of algebraically closed fields $K \supset k$ the schemes O_t, Y_t and $M(X, r, t)$ base change to the corresponding schemes over K. In particular,

$$M(X_K, r, t) \cong M(X, r, t) \times_k K.$$

In general, one always has

$$M(X_K, r, t)(k) \subset M(X_K, r, t)(K).$$

If strict equality holds, then as $M(X, r, t)$ is quasi-compact, $M(X, r, t)$ is the spectrum of an Artin ring on k. In that case, every element of $S(X_K, r, t)$ is defined over k. Further, we see that if every element of $S(X_K, r, t)$ is defined over k, then $S(X, r, t)$ is finite. □

Acknowledgement

We thank the referee for comments on an earlier version of this paper which helped in clarifying some points and in improving the exposition.

References

[1] Deligne P and Milne J, Tannakian categories in Hodge cycles, motives and Shimura varieties, Lecture Notes in Mathematics 900 (Springer) (1982)

[2] Mehta V B and Subramanian S, On the fundamental group scheme, *Invent. Math.* **148** (2002) 143–150

[3] Newstead P E, Introduction to moduli problems and orbit spaces, TIFR Lecture Notes (1978)

[4] Nori M V, Representations of the fundamental group, *Compositio Math.* **33** (1976) 29–41

[5] Nori M V, The fundamental group scheme, *Proc. Indian Acad. Sci.* (*Math. Sci.*) **91** (1982) 73–122

[6] Weibel Charles A, An introduction to homological algebra (Cambridge University Press) (1994)