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On the finiteness properties of Matlis duals of local cohomology modules

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Abstract. Let R be a complete semi-local ring with respect to the topology defined by its Jacobson radical, $\mathfrak{a}$ an ideal of R, and M a finitely generated R-module. Let $D_R(-) := \mathrm{Hom}_R(-, E)$, where E is the injective hull of the direct sum of all simple R-modules. If n is a positive integer such that $\mathrm{Ext}^j_R(R/\mathfrak{a}, D_R(H^t_{\mathfrak{a}}(M)))$ is finitely generated for all $t > n$ and all $j \geqslant 0$, then we show that $\mathrm{Hom}_R(R/\mathfrak{a}, D_R(H^n_{\mathfrak{a}}(M)))$ is also finitely generated. Specially, the set of prime ideals in $\mathrm{Coass}_R(H^n_{\mathfrak{a}}(M))$ which contains $\mathfrak{a}$ is finite.

Next, assume that $(R, \mathfrak{m})$ is a complete local ring. We study the finiteness properties of $D_R(H^r_{\mathfrak{a}}(R))$ where r is the least integer i such that $H^i_{\mathfrak{a}}(R)$ is not Artinian.

Keywords. Local cohomology modules; cofinite modules; associated primes; coassociated primes; filter regular sequences; Matlis duality functor.

1. Introduction

Throughout this paper, R denotes a commutative Noetherian ring with non-zero identity and M is a finitely generated R-module. Also, we use $\mathbb{N}_0$ (respectively $\mathbb{N}$) to denote the set of non-negative (respectively positive) integers. Our terminology follows the textbook [3] on local cohomology.

We begin by recalling the Grothendieck's conjecture for finiteness of local cohomology module $H^i_{\mathfrak{a}}(M)$ of M with respect to $\mathfrak{a}$ (see Exposé XIII, 1.1 of [8]).

Conjecture. If $\mathfrak{a}$ is an ideal of R and M is a finitely generated R-module, then $\mathrm{Hom}_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ is finitely generated, for all $i \in \mathbb{N}_0$.

This conjecture was shown to be false by Hartshorne [9]. He showed that $H^2_{\mathfrak{a}}(R)$ has an infinite dimensional Socle and, in particular, $\mathrm{Hom}_R(R/\mathfrak{a}, H^2_{\mathfrak{a}}(R))$ must be non-finitely generated, where $R = K[x, y, u, v]/(xy - uv)$ and $\mathfrak{a} = (x, y)$. He also defined an R-module M to be $\mathfrak{a}$-cofinite if $\mathrm{Supp}_R(M) \subseteq V(\mathfrak{a})$ and $\mathrm{Ext}^j_R(R/\mathfrak{a}, M)$ is finitely generated for all $j \in \mathbb{N}_0$. Therefore, he asked when the local cohomology modules of a finitely generated module are $\mathfrak{a}$-cofinite. In this regard, the best known result is that, for a finitely generated R-module M and an ideal $\mathfrak{a}$ of R, if either $\mathfrak{a}$ is principal or R is local with $\dim(R/\mathfrak{a}) = 1$, then $H^i_{\mathfrak{a}}(M)$'s are $\mathfrak{a}$-cofinite. These results are proved in Theorem 1 of [14] and Theorem 1 of [6], respectively.

It is easy to see that a $\mathfrak{a}$-cofinite module has only finitely many associated primes. Huneke [11] asked whether the number of associated prime ideals of $H^i_{\mathfrak{a}}(M)$ is always finite. If R

is regular local containing a field, then $H^i_{\mathfrak{a}}(R)$ has only finitely many associated primes for all $i \in \mathbb{N}_0$ (cf. [12] (in the positive characteristic) and [20] (in the characteristic zero)).

On the other hand, Brodmann and Lashgari [2] and Khashyarmanesh and Salarian [18] have shown that the first non-finitely generated local cohomology module has only finitely many associated primes. For some other work on this question, we refer the reader to [22], [23], [28], [7], [10], [15] and [16].

However, examples given by Singh [27] (in the non-local case) and Katzman [13] (in the local case) show that there exist local cohomology modules of Noetherian rings with infinitely many associated primes.

There have been four attempts to dualize the theory of associated prime ideals by Macdonald [21], Chambless [4], Zöschinger [31] and Yassemi [30]. In [30], it is shown that, in case the ring R is Noetherian local, these definitions are equivalent.

Now, let E be the injective hull of the direct sum of all simple R-modules and $D_R(-)$ be the functor $\mathrm{Hom}_R(-, E)$, which is a natural generalization of Matlis duality functor to non-local rings (see [26]). Following [30], for a local ring R, we define a prime ideal $\mathfrak{p}$ to be a coassociated prime of M if $\mathfrak{p}$ is an associated prime of $D_R(M)$. Concerning this, we are led to the following questions which are duals of the above-mentioned questions 'in some sense'.

- If R is local, is the number of coassociated primes of $H^i_{\mathfrak{a}}(M)$ always finite?
- When is the R-module $\mathrm{Ext}^j_R(R/\mathfrak{a}, D_R(H^i_{\mathfrak{a}}(M)))$ finitely generated?

For a complete Noetherian local ring $(R, \mathfrak{m})$, Delfino and Marley [6], investigated the set of coassociated prime ideals of $H^d_{\mathfrak{a}}(M)$, where M is a finitely generated R-module of dimension d.

In this paper, after some backgrounds in §2, we provide partial answers to the above questions. A new point of view is the use of the concept of filter regular sequences which is a natural generalization of the concept of regular sequences. Such sequences determine a complex which involves the Matlis dual of local cohomology modules (see (3.1)). We exploit our complex to study the above questions that yields the results which we mention some in details here.

In §3, we show that whenever R is a complete semi-local ring with respect to the topology defined by its Jacobson radical and n is a positive integer such that $\mathrm{Ext}^j_R(R/\mathfrak{a}, D_R(H^t_{\mathfrak{a}}(M)))$ is finitely generated for all $t > n$ and all $j \in \mathbb{N}_0$, then $\mathrm{Hom}_R(R/\mathfrak{a}, D_R(H^n_{\mathfrak{a}}(M)))$ is also finitely generated. This implies that, over a complete local ring R, for a positive integer n, the set $\mathrm{Coass} H^n_{\mathfrak{a}}(M) \cap V(\mathfrak{a})$ is finite if $H^t_{\mathfrak{a}}(M)$ is Artinian for all $t > n$.

The least non-Artinian local cohomology module was investigated by Melkersson [24] and Lü and Tang [19]. They introduced the concept of the filter-depth of M with respect to $\mathfrak{a}$ as follows:

$$f\text{-depth}(\mathfrak{a}, M) := \min\{\mathrm{depth}_{\mathfrak{a}R_{\mathfrak{p}}} M_{\mathfrak{p}} \colon \mathfrak{p} \in \mathrm{Supp}_R(M/\mathfrak{a}M)\backslash\{\mathfrak{m}\}\}.$$

In the final section, among the other things, we study the finiteness properties of $\mathrm{Ext}^j_R(R/\mathfrak{a}, D_R(H^n_{\mathfrak{a}}(R)))$ where $n = f\text{-depth}(\mathfrak{a}, R)$.

2. Background

First of all, we mention a generalization of the concept of regular sequences which is needed in this paper. We say that a sequence $x_1, \ldots, x_n$ of elements of R is an $\mathfrak{a}$-filter regular sequence on M, if $x_1, \dots, x_n \in \mathfrak{a}$ and

$$\operatorname{Supp}_R\left(\frac{(x_1, \dots, x_{i-1})M :_M x_i}{(x_1, \dots, x_{i-1})M}\right) \subseteq V(\mathfrak{a})$$

for all $i = 1, \dots, n$, where $V(\mathfrak{a})$ denotes the set of prime ideals of R containing $\mathfrak{a}$. The concept of an $\mathfrak{a}$-filter regular sequence on M is a generalization of the filter regular sequence which has been studied in [5], [29] and has led to some interesting results. Note that both concepts coincide if $\mathfrak{a}$ is the maximal ideal in the local ring. Also note that $x_1, \dots, x_n$ is a weak M-sequence if and only if it is an R-filter regular sequence on M. It is easy to see that the analogue of Appendix 2(ii) of [29] holds true whenever R is Noetherian, M is finitely generated and $\mathfrak{m}$ is replaced by $\mathfrak{a}$; so that, if $x_1, \dots, x_n$ is an $\mathfrak{a}$-filter regular sequence on M, then there is an element $x_{n+1} \in \mathfrak{a}$ such that $x_1, \dots, x_n, x_{n+1}$ is an $\mathfrak{a}$-filter regular sequence on M. Thus, for a positive integer n, there exists an $\mathfrak{a}$-filter regular sequence on M of length n.

PROPOSITION 2.1 (see Proposition 1.2 of [17] and Proposition 2.3 of [1])

Let $x_1, \dots, x_n$ $(n \geqslant 1)$ be an $\mathfrak{a}$-filter regular sequence on M. Then there are the following isomorphisms

$$H^i_{\mathfrak{a}}(M) \cong \begin{cases} H^i_{(x_1,\dots,x_n)}(M), & \text{for } 0 \le i < n, \\ H^{i-n}_{\mathfrak{a}}(H^n_{(x_1,\dots,x_n)}(M)), & \text{for } n \le i. \end{cases}$$

Now, we give a brief review of Matlis duality theorem. To do this, we recall the definition of Matlis duality functor.

DEFINITION 2.2 (see [25] and [26])

For a commutative ring R, let Σ_R be the direct sum

$$\oplus_{\mathfrak{m}\in \operatorname{MaxSpec}(R)} R/\mathfrak{m}$$

of all simple R-modules, E_R be the injective hull of Σ_R, and $D_R(-)$ be the functor $\operatorname{Hom}_R(-, E_R)$.

Note that $D_R(-)$ is a natural generalization of Matlis duality functor to non-local rings.

PROPOSITION 2.3 (see [25])

Let R be a semi-local Noetherian ring and M be an Artinian R-module. Then $D_R(M)$ is finitely generated over the completion of R.

We end this section by the following remarks, which are needed in the sequel.

Remarks 2.4 (see Exercise 1.1.2 and Remark 2.2.17 of [3])

(i) Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R. Then, for every R-module N, we have the following equality

$$H^0_{\mathfrak{a}}(H^0_{\mathfrak{b}}(N)) = H^0_{\mathfrak{a}+\mathfrak{b}}(N).$$

(ii) Let x be an arbitrary element of R. Then, for every R-module N, there exists an exact sequence

$$0 \longrightarrow H^0_{(x)}(N) \longrightarrow N \longrightarrow N_x \longrightarrow H^1_{(x)}(N) \longrightarrow 0,$$

where N_x denotes the module of fractions of N with respect to the multiplicatively closed subset $\{x^i | i \in \mathbb{N}_0\}$.

3. Finiteness properties of associated primes of Matlis dual of local cohomology modules

There are basic exact sequences which we will use often in this paper. We separate these exact sequences in the following:

Remark 3.1. For a non-negative integer n, assume that $x_1, \dots, x_{n+1}$ is an $\mathfrak{a}$-filter regular sequence on M. (Note that the existence of such a sequence is explained in the beginning of the previous section.) Put $S_0 := M$ and $S_i := H^i_{(x_1,\dots,x_i)}(M)$ for $i = 1, \dots, n+1$. Now, using Remarks 2.4(i) in the light of Proposition 2.1, for each $i = 0, 1, \dots, n$, we have the following isomorphisms:

$$H^0_{(x_{i+1})}(S_i) \cong H^0_{(x_1,\dots,x_{i+1})}(S_i) \cong H^i_{\mathfrak{a}}(M)$$

and

$$H^1_{(x_{i+1})}(S_i) \cong H^{i+1}_{(x_1,\dots,x_{i+1})}(M) = S_{i+1}.$$

Summing up, by Remarks 2.4(ii), for each $i = 0, 1, \dots, n$, we obtain the exact sequence

$$0 \longrightarrow H^i_{\mathfrak{a}}(M) \longrightarrow S_i \longrightarrow (S_i)_{x_{i+1}} \longrightarrow S_{i+1} \longrightarrow 0$$

which in turn, yields the following exact sequence:

$$0 \longrightarrow D_R(S_{i+1}) \longrightarrow D_R((S_i)_{x_{i+1}}) \xrightarrow{f_i} D_R(S_i) \longrightarrow D_R(H^i_{\mathfrak{a}}(M)) \longrightarrow 0.$$

By breaking the above exact sequence in two exact sequences

$$0 \longrightarrow \operatorname{Im} f_i \longrightarrow D_R(S_i) \longrightarrow D_R(H^i_{\mathfrak{a}}(M)) \longrightarrow 0$$

and

$$0 \longrightarrow D_R(S_{i+1}) \longrightarrow D_R((S_i)_{x_{i+1}}) \longrightarrow \operatorname{Im} f_i \longrightarrow 0,$$

and applying the functor $\operatorname{Hom}_R(R/\mathfrak{a}, -)$ on them, one can deduce the long exact sequence

$$\begin{aligned}
0 &\longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Im} f_i) \longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, D_R(S_i)) \\
&\longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, D_R(H^i_{\mathfrak{a}}(M))) \\
&\longrightarrow \cdots \\
&\longrightarrow \operatorname{Ext}^j_R(R/\mathfrak{a}, \operatorname{Im} f_i) \longrightarrow \operatorname{Ext}^j_R(R/\mathfrak{a}, D_R(S_i)) \\
&\longrightarrow \operatorname{Ext}^j_R(R/\mathfrak{a}, D_R(H^i_{\mathfrak{a}}(M))) \\
&\longrightarrow \operatorname{Ext}^{j+1}_R(R/\mathfrak{a}, \operatorname{Im} f_i) \longrightarrow \cdots
\end{aligned}$$

and the isomorphism

$$\operatorname{Ext}_R^{j+1}(R/\mathfrak{a}, D_R(S_{i+1})) \cong \operatorname{Ext}_R^{j}(R/\mathfrak{a}, \operatorname{Im} f_i) \text{ for all } j \in \mathbb{N}_0,$$

because the multiplication by $x_{i+1} \in \mathfrak{a}$ provides an automorphism on $(S_i)_{x_{i+1}}$. Therefore, we obtain the following long exact sequence

$$\begin{aligned}
0 &\longrightarrow \operatorname{Ext}_R^1(R/\mathfrak{a}, D_R(S_{i+1})) \longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, D_R(S_i)) \\
&\longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, D_R(H_{\mathfrak{a}}^i(M))) \\
&\longrightarrow \cdots \\
&\longrightarrow \operatorname{Ext}_R^{j+1}(R/\mathfrak{a}, D_R(S_{i+1})) \longrightarrow \operatorname{Ext}_R^{j}(R/\mathfrak{a}, D_R(S_i)) \\
&\longrightarrow \operatorname{Ext}_R^{j}(R/\mathfrak{a}, D_R(H_{\mathfrak{a}}^i(M))) \\
&\longrightarrow \operatorname{Ext}_R^{j+2}(R/\mathfrak{a}, D_R(S_{i+1})) \longrightarrow \cdots .
\end{aligned} \tag{3.1}$$

We will also need the following lemmas.

Lemma 3.2. *For any $\mathfrak{a}$-filter regular sequence $x_1, \ldots, x_n$ $(n \geqslant 1)$ on M,*

(i) $\operatorname{Hom}_R(R/\mathfrak{a}, D_R(H^n_{(x_1,\ldots,x_n)}(M))) = 0$, *and*
(ii) $\operatorname{Ext}^1_R(R/\mathfrak{a}, D_R(H^n_{(x_1,\ldots,x_n)}(M))) = 0$ *whenever* $n \geqslant 2$.

Proof. Let $x_1, \ldots, x_n$ be an $\mathfrak{a}$-filter regular sequence on M. By using the notations as in Remark 3.1, we obtain the exact sequence

$$0 \longrightarrow D_R(S_n) \longrightarrow D_R((S_{n-1})_{x_n}) \longrightarrow \operatorname{Im} f_{n-1} \longrightarrow 0.$$

Since the multiplication by $x_n \in \mathfrak{a}$ provides an automorphism on $(S_{n-1})_{x_n}$, by applying the functor $\operatorname{Hom}_R(R/\mathfrak{a}, -)$ on the above exact sequence, one can deduce that $\operatorname{Hom}_R(R/\mathfrak{a}, D_R(S_n)) = 0$. Now, by applying the exact sequence (3.1) with $i = n-1$, the second assertion follows immediately from (i). $\square$

Lemma 3.3. *Let M be a finite dimensional, finitely generated R-module and n be a positive integer such that $\operatorname{Ext}^j_R(R/\mathfrak{a}, D_R(H^t_{\mathfrak{a}}(M)))$ is finitely generated for all $t > n$ and all $j \in \mathbb{N}_0$. Then, for any $\mathfrak{a}$-filter regular sequence $x_1, \ldots, x_i$ on M with $i > n$,*

$$\operatorname{Ext}^j_R(R/\mathfrak{a}, D_R(H^i_{(x_1,\ldots,x_i)}(M)))$$

is finitely generated for all $j \in \mathbb{N}_0$.

Proof. Let M be a finitely generated R-module of dimension d. By Theorem 6.1.2 of [3], we only need to prove the claim for $n < i \leqslant d$. Let j be an arbitrary non-negative integer and $x_1, \ldots, x_{d+1}$ be an $\mathfrak{a}$-filter regular sequence on M. We use the notations as in Remark 3.1. By using the telescoping method, it is enough for us to show that $\operatorname{Ext}^j_R(R/\mathfrak{a}, D_R(S_i))$ is finitely generated if and only if $\operatorname{Ext}^{j+1}_R(R/\mathfrak{a}, D_R(S_{i+1}))$ is finitely generated, for all i with $n < i \leqslant d$. (Note that, by Theorem 6.1.2 of [3],

$S_i = 0$ for all $i > d$.) When $j = 0$, the result is an immediate consequence from Lemma 3.2. So we may assume that $j > 0$. Now since, by our assumption, the R-modules $\mathrm{Ext}_R^{j-1}(R/\mathfrak{a}, D_R(H_{\mathfrak{a}}^i(M)))$ and $\mathrm{Ext}_R^{j}(R/\mathfrak{a}, D_R(H_{\mathfrak{a}}^i(M)))$ are finitely generated, the claim follows from (3.1). □

Now, we prove the main result of this section.

Theorem 3.4. *Let R be a complete semi-local ring with respect to the topology defined by its Jacobson radical. Let n be a positive integer such that* $\mathrm{Ext}_R^j(R/\mathfrak{a}, D_R(H_{\mathfrak{a}}^t(M)))$ *is finitely generated for all $t > n$ and all $j \in \mathbb{N}_0$. Then*

$$\mathrm{Hom}_R(R/\mathfrak{a}, D_R(H_{\mathfrak{a}}^n(M)))$$

is also finitely generated and so $V(\mathfrak{a}) \cap \mathrm{Ass}_R D_R(H_{\mathfrak{a}}^n(M))$ is finite.

Proof. In view of Theorems 6.1.2 and 7.1.6 of [3] and Proposition 2.3, we may assume that $n < \dim M$. Let $x_1, \ldots, x_{n+1}$ be an $\mathfrak{a}$-filter regular sequence on M. By using the notations as in Remark 3.1 and (3.1), we have the following exact sequence:

$$\begin{aligned} \mathrm{Hom}_R(R/\mathfrak{a}, D_R(S_n)) &\longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, D_R(H_{\mathfrak{a}}^n(M))) \\ &\longrightarrow \mathrm{Ext}_R^2(R/\mathfrak{a}, D_R(S_{n+1})). \end{aligned}$$

Now, by Lemmas 3.2 and 3.3, $\mathrm{Hom}_R(R/\mathfrak{a}, D_R(H_{\mathfrak{a}}^n(M)))$ is finitely generated. Hence

$$\mathrm{Ass}_R\mathrm{Hom}_R(R/\mathfrak{a}, D_R(H_{\mathfrak{a}}^n(M))) = V(\mathfrak{a}) \cap \mathrm{Ass}_R D_R(H_{\mathfrak{a}}^n(M))$$

is finite. □

DEFINITION 3.5 (see [30])

Let $(R, \mathfrak{m})$ be a local ring, M an R-module and $E := E_R(R/\mathfrak{m})$ the injective hull of $R/\mathfrak{m}$. We define a prime ideal $\mathfrak{p}$ of R to be a coassociated prime of M if $\mathfrak{p}$ is an associated prime of $\mathrm{Hom}_R(M, E)$. We denote the set of coassociated primes of M by $\mathrm{Coass}_R M$ (or simply $\mathrm{Coass} M$, if there is no ambiguity about the underlying ring).

Note that $\mathrm{Coass} M = \emptyset$ if and only if $M = 0$. For basic theory of coassociated primes the reader is referred to [30].

As mentioned in the Introduction, Delfino and Marley, in Lemma 3 of [6], determined the set of coassociated primes of $H_{\mathfrak{a}}^d(M)$ in the case that $(R, \mathfrak{m})$ is a complete local ring and M is a finitely generated R-module of dimension d. In the following Corollary, we study the coassociated primes of local cohomology modules in a certain case. This is a dual of the main results of [2] and [18] 'in some sense'.

COROLLARY 3.6

Let $(R, \mathfrak{m})$ be a complete local ring and n be a positive integer such that $H_{\mathfrak{a}}^t(M)$ is Artinian for all $t > n$. Then $\mathrm{Coass} H_{\mathfrak{a}}^n(M) \cap V(\mathfrak{a})$ is a finite set.

Proof. The result follows from Proposition 2.3, Theorem 3.4 and the fact that $\mathrm{Ass}_R\mathrm{Hom}_R(R/\mathfrak{a}, D_R(H_{\mathfrak{a}}^n(M))) = \mathrm{Coass}_R H_{\mathfrak{a}}^n(M) \cap V(\mathfrak{a})$. □

4. *F*-depth and Matlis duals of local cohomology modules

In this section we study the finiteness properties of $H^r_{\mathfrak{a}}(M)$, where r is the least integer such that $H^r_{\mathfrak{a}}(M)$ is not Artinian. To do this, we recall the notion of filter-depth (see [24]).

Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a}$ an ideal of R and M a finitely generated R-module such that $\mathrm{Supp}_R(M/\mathfrak{a}M) \nsubseteq \{\mathfrak{m}\}$. We define the filter-depth of M with respect to $\mathfrak{a}$, denoted by f-depth$(\mathfrak{a}, M)$, as

$$f\text{-depth}(\mathfrak{a}, M) := \min\{\mathrm{depth}_{\mathfrak{a}R_{\mathfrak{p}}} M_{\mathfrak{p}} \colon \mathfrak{p} \in \mathrm{Supp}_R(M/\mathfrak{a}M)\backslash\{\mathfrak{m}\}\}.$$

Lü and Tang [19] defined the f-depth $(\mathfrak{a}, M)$ as the length of a maximal $\mathfrak{m}$-filter regular sequence on M in $\mathfrak{a}$. Then it turns out that their f-depth of $\mathfrak{a}$ on M is the least integer r such that $H^r_{\mathfrak{a}}(M)$ is not Artinian and, in the case $\mathrm{Supp}_R(M/\mathfrak{a}M) \nsubseteq \{\mathfrak{m}\}$, their f-depth coincides with one of Melkersson [24].

The next theorem is the main aim of this section.

Theorem 4.1. *Let $(R, \mathfrak{m})$ be a complete local ring and $\mathfrak{a}$ a proper ideal of R such that*

(i) $n := f$*-depth*$(\mathfrak{a}, R) > 0$ *and*
(ii) *there exists an $\mathfrak{a}$-filter regular sequence $x_1, \dots, x_{n+1}$ on R such that*

$$\mathrm{Ext}^j_R(R/\mathfrak{a}, D_R(H^{n+1}_{(x_1,\dots,x_{n+1})}(R)))$$

is finitely generated for all $j \in \mathbb{N}_0$.

Then $\mathrm{Ext}^j_R(R/\mathfrak{a}, D_R(H^n_{\mathfrak{a}}(R)))$ is finitely generated for all $j \in \mathbb{N}_0$ with $j \neq n$.

Proof. Let $x_1, \dots, x_{n+1}$ be an $\mathfrak{a}$-filter regular sequence on R such that

$$\mathrm{Ext}^j_R(R/\mathfrak{a}, D_R(H^{n+1}_{(x_1,\dots,x_{n+1})}(R)))$$

is finitely generated for all $j \in \mathbb{N}_0$. Then, in view of Remark 3.1, we can obtain the exact sequence

$$\mathrm{Ext}^j_R(R/\mathfrak{a}, D_R(S_n)) \longrightarrow \mathrm{Ext}^j_R(R/\mathfrak{a}, D_R(H^n_{\mathfrak{a}}(R))) \longrightarrow \mathrm{Ext}^{j+2}_R(R/\mathfrak{a}, D_R(S_{n+1})),$$

where $S_i := H^i_{(x_1,\dots,x_i)}(R)$ for $i = n, n+1$. Hence we only need to show that $\mathrm{Ext}^j_R(R/\mathfrak{a}, D_R(S_n))$ is finitely generated for all $j \in \mathbb{N}_0$ with $j \neq n$. When $j = 0$ or $j = 1$, by Lemma 3.2, there is nothing to prove. So, we can assume that $j > 1$. Now, in view of the long exact sequence (3.1) in Remark 3.1, we get the exact sequence

$$\mathrm{Ext}^{j-2}_R(R/\mathfrak{a}, D_R(H^{n-1}_{\mathfrak{a}}(R))) \longrightarrow \mathrm{Ext}^j_R(R/\mathfrak{a}, D_R(S_n)) \longrightarrow \mathrm{Ext}^{j-1}_R(R/\mathfrak{a}, D_R(S_{n-1})).$$

Since f-depth $(\mathfrak{a}, R) = n$, $H^{n-1}_{\mathfrak{a}}(R)$ is Artinian and so, in view of Proposition 2.3, it is enough for us to show that $\mathrm{Ext}^{j-1}_R(R/\mathfrak{a}, D_R(S_{n-1}))$ is finitely generated. Two cases now arise, depending on whether $j - n$ is positive or not.

When $j - n > 0$, by repeating the above argument, we eventually get to the exact sequence

$$\begin{aligned}\mathrm{Ext}_R^{j-n-1}(R/\mathfrak{a}, D_R(H_\mathfrak{a}^0(R))) &\longrightarrow \mathrm{Ext}_R^{j-n+1}(R/\mathfrak{a}, D_R(S_1))\\ &\longrightarrow \mathrm{Ext}_R^{j-n}(R/\mathfrak{a}, D_R(R)).\end{aligned}$$

Since, by Proposition 2.3, $D_R(H_\mathfrak{a}^0(R))$ is finitely generated, we only need to show that $\mathrm{Ext}_R^{j-n}(R/\mathfrak{a}, D_R(R))$ is finitely generated. But this is zero, because $D_R(R)$ is injective.

When $j - n < 0$, again by repeating the above argument, we eventually get to the exact sequence

$$\begin{aligned}\mathrm{Ext}_R^{0}(R/\mathfrak{a}, D_R(H_\mathfrak{a}^{n-j+1}(R))) &\longrightarrow \mathrm{Ext}_R^{2}(R/\mathfrak{a}, D_R(S_{n-j+2}))\\ &\longrightarrow \mathrm{Ext}_R^{1}(R/\mathfrak{a}, D_R(S_{n-j+1}(R))).\end{aligned}$$

Now since $D_R(H_\mathfrak{a}^{n-j+1}(R))$ is finitely generated, we must show that

$$\mathrm{Ext}_R^{1}(R/\mathfrak{a}, D_R(S_{n-j+1}(R)))$$

is finitely generated. This follows from Lemma 3.2(ii).

This completes the proof. □

COROLLARY 4.2

Let $(R, \mathfrak{m})$ *be a complete local ring and* $x_1, \ldots, x_n$ *be an* $\mathfrak{m}$*-filter regular sequence on* R*. Set* $\mathfrak{b} := (x_1, \ldots, x_n)$*. Then* $\mathrm{Ext}_R^j(R/\mathfrak{b}, D_R(H_\mathfrak{b}^n(R)))$ *is finitely generated for all* $j \in \mathbb{N}_0$ *with* $j \neq n$*.*

Proof. If $H_\mathfrak{b}^n(R)$ is Artinian, then there is nothing to prove. So we may assume that $H_\mathfrak{b}^n(R)$ is not Artinian. Now, since $x_1, \ldots, x_n$ is an $\mathfrak{m}$-filter regular sequence on R, by Definition 3.3 and Theorem 3.10 of [19], we can conclude that f-depth $(\mathfrak{b}, R) = n$. Also, note that $x_1, \ldots, x_n$ is a $\mathfrak{b}$-filter regular sequence on R. Now, by the first paragraph in the second section, there exists $y \in \mathfrak{b}$ such that $x_1, \ldots, x_n, y$ is a $\mathfrak{b}$-filter regular sequence on R. Therefore, by Corollary 3.3.3 of [3], $H_{(x_1,\ldots,x_n,y)}^{n+1}(R) = 0$. The claim now follows from Theorem 4.1. □

COROLLARY 4.3

Let $(R, \mathfrak{m})$ *be a* d*-dimensional complete local ring with* $d \geqslant 2$ *and* $\mathfrak{a}$ *be an ideal of* R *such that* f*-depth*$(\mathfrak{a}, R) = d - 1$*. Then* $\mathrm{Ext}_R^j(R/\mathfrak{a}, D_R(H_\mathfrak{a}^{d-1}(R)))$ *is finitely generated for all* $j \in \mathbb{N}_0$ *with* $j \neq d - 1$*.*

Proof. The result follows from Theorem 7.1.6 of [3], Proposition 2.3 and Theorem 4.1. □

Remark 4.4. Let $\mathfrak{b}$ be an ideal of R containing $\mathfrak{a}$. So, by employing the methods of proofs which are similar to those used in this paper, one can establish the results about finiteness properties of $\mathrm{Ext}_R^j(R/\mathfrak{b}, D_R(H_\mathfrak{a}^i(M)))$. Hence, in the case that $\mathfrak{b}$ is a maximal ideal $\mathfrak{m}$ of R, we can prepare some conditions which ensure that $\mathrm{Ext}_R^j(R/\mathfrak{m}, D_R(H_\mathfrak{a}^i(M)))$ has finite length.

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