Zassenhaus conjecture for A_6

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Dedicated to the memory of I. S. Luthar[†]

Abstract. For the alternating group A_6 of degree 6, Zassenhaus' conjecture about rational conjugacy of torsion units in integral group rings is confirmed.

1. Introduction

It has proven exceedingly difficult to achieve progress in a number of areas revolving around units in integral group rings of finite groups, and this applies in particular to the conjecture of the title:

(ZC1) For a finite group G, every torsion unit in its integral group ring $\mathbb{Z}G$ is conjugate to an element of $\pm G$ in the units of the rational group algebra $\mathbb{Q}G$.

It remains not only unsolved but also lacking in plausible means of either proof or counter-example. Only a few non-solvable groups G are known to satisfy (ZC1); in this note, the conjecture is verified for the alternating group A_6. In 1988, Luthar and Passi [17] started investigations in this field when dealing with A_5, thereby introducing what is now called the 'Luthar–Passi method'. A little later, Luthar and Trama [20] established (ZC1) for the symmetric group S_5, with a large part of the calculations being done inside the integral group ring $\mathbb{Z}S_5$, or rather a Wedderburn embedding itself. Mentioning further that the covering group of A_5 can be dealt with some additional arguments [9], that accurately describes the state of knowledge at the turn of the century. The limitations of the original version of the Luthar–Passi method were shown in [2]. Recently, a modular version, especially suited for application to non-solvable groups, was given in [13], and used to verify (ZC1) for the groups $\mathrm{PSL}(2, p)$, $p = 7, 11, 13$. (At that point, the reader may wish to recall that $A_6 \cong \mathrm{PSL}(2, 9)$.) A brief review of the method is given in §2. It was also used in [7] to verify (ZC1) for a covering group of S_5 and for $\mathrm{GL}(2, 5)$. The method makes use of the (ordinary) character table and of modular character tables, in a process suited for being done on a computer, the result being that rational conjugacy of torsion units of a given order to group elements is either proven or not, and if not, at least some information about partial augmentations is obtained (see [3–6]).

[†]Professor Luthar died at the age of 74 in December 2006.

It should be remarked that from the very beginning, investigations on (ZC1) concentrated on solvable groups. Here we only mention that Luthar, in collaboration with others, initiated the study of metacyclic groups [16,1,19,18]. For the whole class of metacyclic groups, (ZC1) has been confirmed only very recently in [12]. Other, more recent, results can be found in [14] and [15].

We briefly recall some of the necessary background. Let G be a finite group. For a group ring element $u = \sum_{g \in G} a_g g$ (all a_g in $\mathbb{Z}$), its partial augmentation $\varepsilon_x(u)$ with respect to an element x of G, or rather its conjugacy class x^G in G, is the sum $\sum_{g \in x^G} a_g$. The augmentation of u is the sum $\sum_{g \in G} a_g$ of all of its partial augmentations. Of course, we need to consider only units of augmentation one in $\mathbb{Z}G$, which form a group which we denote by $V(\mathbb{Z}G)$.

A criterion for a torsion unit u in $V(\mathbb{Z}G)$ to be conjugate to an element of G in the units of $\mathbb{Q}G$, which is especially suited for computational purposes, is that all but one of the partial augmentations of every power of u vanish (see Theorem 2.5 of [21]). The partial augmentations of u one has to turn attention to are limited by the following remark.

Remark 1. Let u be a torsion unit in $V(\mathbb{Z}G)$. Then $g \in G$ and $\varepsilon_g(u) \neq 0$ implies that the order of g divides the order of u. Indeed, it is well-known that then prime divisors of the order of g divide the order of u (see Theorem 2.7 of [21], as well as Lemma 2.8 of [14] for an alternative proof). Further, it was observed in Proposition 2.2 of [13] that the orders of the p-parts of g cannot exceed those of u.

It is known [8] that the order of a torsion unit u in $V(\mathbb{Z}G)$ is a divisor of the exponent of G. However, it is not known whether some element of G has the same order as u. For example, the more interesting part we will be concerned with when proving (ZC1) for A_6 is about the non-existence of torsion units of order 6 in $V(\mathbb{Z}A_6)$, despite the fact that there are no elements of order 6 in A_6 (see §4). Using the Luthar–Passi method alone is not sufficient. In this situation, we are lucky that only in a rough sense 'ties' between two integral representations (affording the 5-dimensional irreducible characters) need to be considered.

In §3, we apply the Luthar–Passi method to torsion units in $V(\mathbb{Z}A_6)$, with the result that all torsion units not of order 6 are conjugate to elements of A_6 in the units of $\mathbb{Q}A_6$, and if there should exist a unit of order 6, its partial augmentations are essentially unique and can be specified. That is a matter of routine, and was already done by Salim [22] who calculated the coefficients of the linear inequalities arising from eq. (2) below, and then their solutions. We shall proceed in a straightforward way, making use of only inequalities arising from eq. (1).

2. The Luthar–Passi method

We briefly survey, in an informal way, the Luthar–Passi method. Still, let u be a torsion unit in $V(\mathbb{Z}G)$, of order n (say). Then, conjecture (ZC1) states that there is $g \in G$ such that $\chi(u^i) = \chi(g^i)$ for all irreducible characters χ of G and integers i. This is because complex representations are determined (up to equivalence) by their characters, and conjugacy in the units of $\mathbb{C}G$ can be shown to be equivalent to conjugacy in the units of $\mathbb{Q}G$ ('rational conjugacy'). It is perhaps even more natural to take the eigenvalues of representing matrices into consideration: If an irreducible character χ is afforded by a complex representation D of G, write $\mu(\xi, u, \chi)$ for the multiplicity of an n-th root of unity ξ as an eigenvalue

of the matrix $D(u)$. Then u is rationally conjugate to the group element g if and only if $\mu(\xi, u, \chi) = \mu(\xi, g, \chi)$ for all χ and ξ.

The same applies, mutatis mutandis, to Brauer characters. Suppose that u is a p-regular torsion unit for some prime p dividing the order of G (that is, p does not divide the order n of u). A Brauer character φ (relative to p) is a complex-valued function associated with a modular representation of G (in characteristic p), defined on the set of p-regular elements of G. The way Brauer defined these functions actually shows that the domain of φ can be extended to the set of p-regular torsion units of $V(\mathbb{Z}G)$, and the decomposition matrix gives the relation between the values of irreducible Brauer characters and ordinary characters on these units. Multiplicities $\mu(\xi, u, \varphi)$ are defined in the obvious way. Then one can formulate, in complete analogy, that u is rationally conjugate to the group element g if and only if $\varphi(u^i) = \varphi(g^i)$ for all irreducible Brauer characters φ and integers i, or $\mu(\xi, u, \varphi) = \mu(\xi, g, \varphi)$ for all φ and ξ.

Let ψ be either an irreducible ordinary character, or an irreducible Brauer character of G relative to p when u is p-regular. The multiplicities $\mu(\xi, u, \psi)$ can be computed from the values $\psi(u^i)$ by Fourier inversion, and taking the Galois action into account, the formula reads—with ζ denoting a primitive n-th root of unity:

$$\mu(\xi, u, \psi) = \frac{1}{n} \sum_{d|n} \mathrm{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\psi(u^d)\xi^{-d}).$$

Partial augmentations come into play by means of

$$\psi(u) = \sum_{x^G} \varepsilon_x(u)\psi(x), \tag{1}$$

where it is understood that a Brauer character vanishes off the p-regular elements. This is obvious if ψ is an ordinary character, and can be seen as a consequence of the non-singularity of the Cartan matrix if ψ is a Brauer character, remembering the fact that the partial augmentations of the p-regular torsion unit u at p-singular group elements are zero. Combining both equations, one obtains

$$\mu(\xi, u, \psi) = \frac{1}{n} \sum_{x^G} \sum_{d|n} \varepsilon_x(u^d)\mathrm{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\psi(x)\xi^{-d}). \tag{2}$$

This should be seen as a linear system of equations with rational coefficients in the unknown partial augmentations. Note that the $\mu(\xi, u, \psi)$ are non-negative integers, bounded above by $\psi(1)$.

3. Application of the Luthar–Passi method

Let u be a torsion unit in $V(\mathbb{Z}A_6)$. The order of u is a divisor of the exponent of A_6, which is $2^2 \cdot 3 \cdot 5$. The character table of A_6 is shown in table 1 in the form obtained by requiring `CharacterTable("A6")` in GAP [10] (dots indicate zeros). So $k\mathbf{x}$ is the '$\mathbf{x}$'-th class of elements of order k, and $\varepsilon_{k\mathbf{x}}$ will denote the partial augmentation of u with respect to this class. We will also make use of the Brauer character tables of A_6 for the primes 2 and 3, as shown in table 2 (which can be accessed from the GAP library through the `mod` command if the character table is required in the above form). We assume that $u \neq 1$, so $\varepsilon_{1\mathbf{a}} = 0$ by the familiar Berman–Higman result. We shall show that all but one

Table 1. Character table of A_6.

	1a	2a	3a	3b	4a	5a	5b
χ_{1a}	1	1	1	1	1	1	1
χ_{5a}	5	1	2	-1	-1	.	.
χ_{5b}	5	1	-1	2	-1	.	.
χ_{8a}	8	.	-1	-1	.	α_1	α_2
χ_{8b}	8	.	-1	-1	.	α_2	α_1
χ_{9a}	9	1	.	.	1	-1	-1
χ_{10a}	10	-2	1	1	.	.	.

Irrational entries: $\alpha_1 = (1 + \sqrt{5})/2 = 1 + \nu + \nu^4$ and

$\alpha_2 = (1 - \sqrt{5})/2 = 1 + \nu^2 + \nu^3$,

where $\nu = \exp(2\pi i/5)$.

Table 2. Brauer character tables of A_6 mod 2 and mod 3 .

	1a	3a	3b	5a	5b
$\varphi_{2,1a}$	1	1	1	1	1
$\varphi_{2,4a}$	4	1	-2	-1	-1
$\varphi_{2,4b}$	4	-2	1	-1	-1
$\varphi_{2,8a}$	8	-1	-1	α_1	α_2
$\varphi_{2,8b}$	8	-1	-1	α_2	α_1

	1a	2a	4a	5a	5b
$\varphi_{3,1a}$	1	1	1	1	1
$\varphi_{3,3a}$	3	-1	1	α_1	α_2
$\varphi_{3,3b}$	3	-1	1	α_2	α_1
$\varphi_{3,4a}$	4	.	-2	-1	-1
$\varphi_{3,9b}$	9	1	1	-1	-1

of the partial augmentations of u vanish if its order is not 6. When u is assumed to have order 6, the Luthar–Passi method is not strong enough to exclude the possibility of having $(\varepsilon_{2a}, \varepsilon_{3a}, \varepsilon_{3b})$ equal to $(-2, 1, 2)$ or $(-2, 2, 1)$. Note that we have to consider the cases when u has order 2, 4, 3, 5 (then vanishing of all but one of the partial augmentations will be shown), when u has order 10 or 15 (a contradiction must be reached), and when u has order 6 (with the final contradiction to be obtained in the next section). Remark 1 on the vanishing of certain partial augmentations will be used without further mention. Finally, we remark that A_6 has automorphisms which permute either the classes 3a and 3b or the classes 5a and 5b (leaving the remaining classes fixed), which helps from repeating steps.

A subscript $*$ will always stand as a placeholder for all possible subscripts. We denote by Θ_* a representation of G, over a sufficiently large field, which affords the Brauer character φ_*, and write $\Theta_*(u) \sim \mathrm{diag}(\lambda_1, \ldots, \lambda_{\varphi_*(1)})$ to indicate that $\Theta_*(u)$ has eigenvalues $\lambda_1, \ldots, \lambda_{\varphi_*(1)}$ (multiplicities taken into account), which will, in the sense of Brauer, be identified with complex roots of unity.

When u has order 2.   There is only one class of elements of order 2, so there is nothing to do.

When u has order 4.   Taking augmentation gives $\varepsilon_{2a} + \varepsilon_{4a} = 1$, so $|-2\varepsilon_{2a} + 1| = |-\varepsilon_{2a} + \varepsilon_{4a}| = |\varphi_{3,3a}(u)| < 3$ (strict inequality as $|\varphi_{3,3a}(u^2)| = |\varphi_{3,3a}(2a)| \neq 3$). Thus $\varepsilon_{2a} \in \{0, 1\}$, and one of ε_{2a} and ε_{4a} is zero (of course, this must be ε_{2a}).

When u has order 3.   Taking augmentation gives $\varepsilon_{3a} + \varepsilon_{3b} = 1$. Thus $|3\varepsilon_{3a} - 2| = |\varepsilon_{3a} - 2\varepsilon_{3b}| = |\varphi_{2,4a}(u)| \leq 4$ and $|-3\varepsilon_{3a} + 1| = |-2\varepsilon_{3a} + \varepsilon_{3b}| = |\varphi_{2,4b}(u)| \leq 4$. So $\varepsilon_{3a} \in \{0, 1, 2\} \cap \{-1, 0, 1\} = \{0, 1\}$, and we are done.

When u has order 5. Taking augmentation gives $\varepsilon_{5a} + \varepsilon_{5b} = 1$, so $\varphi_{3,3a}(u) = (1 + \nu + \nu^4)\varepsilon_{5a} + (1 + \nu^2 + \nu^3)\varepsilon_{5b} = 1 + (1 - \varepsilon_{5a})(\nu^2 + \nu^3) + \varepsilon_{5a}(\nu + \nu^4)$. Since $\varphi_{3,3a}(u)$ is a sum of three fifth root of unity, it follows that $\varepsilon_{5a} \in \{0, 1\}$, as desired.

By slight abuse of notation, we shall also write $k\mathrm{x}$ for a representative of the class $k\mathrm{x}$.

When u has order 10. Taking augmentation gives $\varepsilon_{2a} + \varepsilon_{5a} + \varepsilon_{5b} = 1$. Thus $\varphi_{3,3a}(u) = -1 + \varepsilon_{5a}(3 + \sqrt{5})/2 + \varepsilon_{5b}(3 - \sqrt{5})/2$. We can assume without loss of generality that u^6 is rationally conjugate to 5a. Also u^5 is rationally conjugate to 2a. We have $\Theta_{3,3a}(5\mathrm{a}) \sim \mathrm{diag}(1, \nu, \nu^4)$ and $\Theta_{3,3a}(2\mathrm{a}) \sim \mathrm{diag}(1, -1, -1)$. The eigenvalues of $\Theta_{3,3a}(u)$ are products of the diagonal entries of the diagonal matrices just depicted, always taken one from each matrix, in such a way that all entries are involved. Since $\varphi_{3,3a}(u)$ is real, it follows that $\Theta_{3,3a}(u) \sim \mathrm{diag}(1, -\nu, -\nu^4)$ and $\varphi_{3,3a}(u) = (3 - \sqrt{5})/2$. Comparing the given values for $\varphi_{3,3a}(u)$ results in an equation which has no integral solution (reduce modulo 3), a contradiction.

When u has order 15. Then we can assume without loss of generality that u^{10} is rationally conjugate to 3a and that u^6 is rationally conjugate to 5a. Setting $\zeta = \exp(2\pi i/3)$, we have $\Theta_{2,4a}(3\mathrm{a}) \sim \mathrm{diag}(1, 1, \zeta, \zeta^2)$ and $\Theta_{2,4a}(5\mathrm{a}) \sim \mathrm{diag}(\nu, \nu^2, \nu^3, \nu^4)$. The eigenvalues of $\Theta_{2,4a}(u)$ are the products of the diagonal entries of these diagonal matrices, taken in a suitable order. But there is no possibility to choose an ordering such that the sum of the products becomes rational, a contradiction.

When u has order 6. Taking augmentation gives $\varepsilon_{2a} + \varepsilon_{3a} + \varepsilon_{3b} = 1$. Thus $\chi_{5a}(u) = 1 + \varepsilon_{3a} - 2\varepsilon_{3b}$ and $\chi_{5b}(u) = 1 - 2\varepsilon_{3a} + \varepsilon_{3b}$. Now, u^3 is rationally conjugate to 2a, and we can assume without loss of generality that u^2, and hence also u^4, is rationally conjugate to 3b. Write D_{5a} and D_{5b} for ordinary representations affording χ_{5a} and χ_{5b}, respectively. We have $D_{5a}(2\mathrm{a}) \sim \mathrm{diag}(1, 1, 1, -1, -1)$ and $D_{5a}(3\mathrm{b}) \sim \mathrm{diag}(1, \zeta, \zeta^2, \zeta, \zeta^2)$. The eigenvalues of $D_{5a}(u)$ are products of the diagonal entries of these diagonal matrices, suitably taken. Their sum $\chi_{5a}(u)$ is rational, which forces

$$D_{5a}(u) \sim \mathrm{diag}(1, \zeta, \zeta^2, -\zeta, -\zeta^2) \tag{3}$$

and $\chi_{5a}(u) = 1$. Thus $\varepsilon_{3a} = 2\varepsilon_{3b}$ and $\chi_{5b}(u) = 1 - 3\varepsilon_{3b}$. We have $D_{5b}(2\mathrm{a}) \sim \mathrm{diag}(1, 1, 1, -1, -1)$ and $D_{5b}(3\mathrm{b}) \sim \mathrm{diag}(1, 1, 1, \zeta, \zeta^2)$. In the same way as before, we obtain that either

$$D_{5b}(u) \sim \mathrm{diag}(1, -1, -1, \zeta, \zeta^2) \tag{4}$$

and $\chi_{5b}(u) = -2$, or $\chi_{5b}(u) = 4$. If $\chi_{5b}(u) = 4$, then $\varepsilon_{3b} = -1$, $\varepsilon_{3a} = -2$ and $\varepsilon_{2a} = 4$; but then $\chi_{10a}(u) = -11$, a contradiction. Hence $\chi_{5b}(u) = -2$, and $\varepsilon_{3b} = 1$, $\varepsilon_{3a} = 2$ and $\varepsilon_{2a} = -2$, one of the possibilities mentioned at the beginning.

4. Non-existence of a unit of order 6 in $V(\mathbb{Z}A_6)$

Finally, we assume that there exists a unit u of order 6 in $V(\mathbb{Z}A_6)$, and are looking for a contradiction. According to the previous section, we can assume that u has partial augmentations $(\varepsilon_{2a}, \varepsilon_{3a}, \varepsilon_{3b})$ equal to $(-2, 2, 1)$, that u^3 is rationally conjugate to 2a and that u^2 is rationally conjugate to 3b. In particular, the character values of all powers of u are known, from which the eigenvalues of representing matrices can be computed.

We shall take advantage only of representations D_{5a} and D_{5b} affording the irreducible characters χ_{5a} and χ_{5b}, for which the eigenvalues of $D_{5a}(u)$ and $D_{5b}(u)$ are already given by (3) and (4), taking into account that χ_{5a} and χ_{5b} have the same 3-modular constituents, namely the principal character and the 3-modular character of degree 4 (as can be seen from tables 1 and 2).

Set $R = \mathbb{Z}_{(3)}$, the localization of $\mathbb{Z}$ at the prime 3, and $k = R/3R$ (the prime field of characteristic 3). For an RA_6-lattice L, we let $\bar{L}$ stand for the kA_6-module $L/3L$, and we also let $x \mapsto \bar{x}$ denote the natural homomorphism from RA_6 to kA_6.

There are two doubly transitive actions of A_6 on a set with six elements, corresponding to the two conjugacy classes of subgroups of A_6 which are isomorphic to A_5, and D_{5a} and D_{5b} can be chosen to be the associated deleted permutation representations. Denote by L_a and L_b the non-trivial factor lattices of the RA_6-permutation lattices arising from these actions, so that a matrix representation D_{5*} affording χ_{5*} can be obtained with respect to an R-basis of L_*.

Set $e = (1 + u^3)/2$ and $f = 1 - e$, so $1 = e + f$ is an idempotent decomposition in RA_6. Let $w = e + fu^2 \in RA_6$. From (4) one sees that $D_{5b}(w)$ is the identity matrix. In particular, $\bar{w}$ acts as the identity on the composition factors of $\bar{L}_b$, which are also the composition factors of $\bar{L}_a$, as already noted.

Now we turn our attention to the other representation. Set $L = L_a$. Note that the simple kA_6-module $\tilde{M}$ of dimension 4 is a submodule of $\bar{L}$, namely $\tilde{M} = \bar{L}I(kA_6)$ where $I(kA_6)$ denotes the augmentation ideal of kA_6. (If we set $M = LI(RA_6)$, then M is a sublattice of L of index 3 and $\tilde{M}$ is the image of M under the map $L \to \bar{L}$.) We have $L = Le \oplus Lf$. From (3) one sees that Lf is an R-lattice of rank 2 on which w acts nontrivially by multiplication with u^2, a unit of order 3. But $\overline{Lf} = \bar{L}\bar{f} \subseteq \tilde{M}$, so $\bar{w}$ acts trivially on $\overline{Lf}$, by the preceding paragraph. So we have obtained a contradiction, as the kernel of the natural map $GL_2(R) \to GL_2(k)$ has no 3-torsion. (A particular case of a well-known lemma due to Minkowski; see, for example, the beautiful discussion in §5.2 of [11].)

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