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Realizations of the canonical representation

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Abstract. A characterisation of the maximal abelian subalgebras of the bounded operators on Hilbert space that are normalised by the canonical representation of the Heisenberg group is given. This is used to classify the perfect realizations of the canonical representation.

Keywords. Heisenberg group; inductive algebra; canonical commutation relation.

1. Introduction

DEFINITION 1.1

The Heisenberg group G is the set of triples

$$\{(x, y, z) | x, y \in \mathbb{R}^n, z \in \mathbb{C}, |z| = 1\}$$

with multiplication defined by

$$(x, y, z)(x', y', z') = (x + x', y + y', zz' \mathrm{e}^{\pi i(x \cdot y' - y \cdot x')}).$$

The fundamental fact about the Heisenberg group is the Stone–von Neumann theorem.

Theorem 1.2 (Stone–von Neumann). *There is a unique irreducible unitary representation ρ of G such that*

$$\rho(0, 0, z) = zI.$$

This representation is called the *canonical representation*. The terminology comes from Quantum Mechanics where the derived (Lie algebra) representation is known as the 'canonical commutation relation'.

Traditionally, the canonical representation is realized on the Hilbert space $L^2(\mathbb{R}^n)$ by the action

$$(\rho(x, y, z) f)(t) = z\mathrm{e}^{\pi i(x \cdot y - 2y \cdot t)} f(t - x). \tag{1}$$

Based in part on the author's doctoral thesis (University of Chicago), written under the direction of Professor Tim Steger.

However, there are many other different looking ways of realizing this representation. They come from inducing a character from a maximal abelian subgroup. To describe these realizations, we need some terminology. For details and proofs, see [2].

Fix a decomposition $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ once for all. Define $e\colon \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{C}$ by

$$e((x, y), (x', y')) = \mathrm{e}^{2\pi i(x\cdot y' - y\cdot x')}.$$

DEFINITION 1.3

An additive subgroup $M \subseteq \mathbb{R}^{2n}$ is *isotropic* if $e|_{M\times M} \equiv 1$. If M is maximal with respect to this property then M is *maximal isotropic*.

Note that the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ acts on $\mathbb{R}^{2n}$ preserving e. So it preserves the set of maximal isotropic subgroups. For $k = 0, \ldots, n$, the subgroup $M = (\mathbb{R}^k \times \mathbb{Z}^{n-k}) \times \mathbb{Z}^{n-k}$ is a maximal isotropic subgroup. The general maximal isotropic subgroup is got by moving these prototypes under the action of $\mathrm{Sp}(2n, \mathbb{R})$.

Theorem 1.4 (Stone–von Neumann). *For any maximal isotropic subgroup $M \subseteq \mathbb{R}^{2n}$, ρ may be realized on the Hilbert space*

$$\mathcal{H}_0 = \left\{ f\colon \mathbb{R}^{2n} \to \mathbb{C} \,\middle|\, \begin{array}{l} f \text{ is measurable;} \\ f(w+m) = e\left(\frac{m}{2}, w\right)^{-1} f(w), \quad \forall m \in M; \\ \int_{\mathbb{R}^{2n}/M} |f|^2 < \infty; \end{array} \right\}$$

by the action

$$(\rho(x, y, z) f)(w) = z e\left(\frac{w}{2}, (x, y)\right) f(w + (x, y)).$$

The aim of this work is to prove that in an appropriate sense (to be made precise in §2), these are the only perfect realizations of the canonical representation. Given a perfect realization, we consider a certain von Neumann algebra that is canonically attached to it. In §6, we show that it must be isomorphic to one which comes from a realization of the type described in Theorem 1.4. We actually prove something stronger, viz., that any *maximal inductive* algebra is a von Neumann algebra coming from a realization of the type described in Theorem 1.4. The proof of this involves the study of spectral synthesis for a certain slice of the inverse group Fourier transform on the Heisenberg group. The necessary analysis is developed in §§3, 4 and 5.

2. Systems of imprimitivity and perfect realizations

The results in this section are part of mathematical folklore and so we do not give proofs.

For the material on fields of Hilbert spaces we refer to Dixmier [1]. In this section, we will work in the generality of locally compact groups. Let G be a second countable locally compact group, and ρ, an irreducible unitary representation of G on a separable Hilbert space $\mathcal{H}$. All concepts in this section refer to ρ.

DEFINITION 2.1

An *inductive* algebra is an abelian subalgebra $\mathcal{A} \subseteq L(\mathcal{H})$ such that for all $g \in G$,

$$\rho(g)\mathcal{A}\rho(g^{-1}) = \mathcal{A}.$$

Observe that an inductive algebra comes equipped with a G-action.

DEFINITION 2.2

An inductive algebra $\mathcal{A}$ is *maximal* if for all inductive algebras $\tilde{\mathcal{A}}$,

$$\mathcal{A} \subseteq \tilde{\mathcal{A}} \implies \mathcal{A} = \tilde{\mathcal{A}}.$$

Observe that a maximal inductive algebra is weakly closed.

DEFINITION 2.3

A *system of imprimitivity* is an inductive $*$-algebra.

Let B be a standard Borel space. Let μ be a σ-finite measure on B. Recall that a *field of Hilbert spaces* over B is a mapping $b \mapsto E(b)$ such that for every $b \in B$, $E(b)$ is a Hilbert space. Let $\Gamma(B, E)$ denote the space of all sections of E.

DEFINITION 2.4

We say that E is a *measurable field of Hilbert spaces* over (B, μ) if there is given a linear subspace $\mathcal{E} \subseteq \Gamma(B, E)$ possessing the following properties:

(1) For every $s \in \mathcal{E}$, the function $b \mapsto \|s(b)\|$ is measurable;
(2) If $t \in \Gamma(B, E)$ is such that, for every $s \in \mathcal{E}$, the function $b \mapsto \langle s(b), t(b)\rangle$ is measurable, then $t \in \mathcal{E}$;
(3) There exists a sequence $(s_1, s_2, \dots)$ in $\mathcal{E}$ such that, for every $b \in B$, the sequence $(s_1(b), s_2(b), \dots)$ is total in $E(b)$.

The space $\mathcal{E}$ is called the *space of measurable sections* of E.

Suppose G acts on B by Borel automorphisms preserving the class of μ, and acts on $\bigcup_{b\in B} E(b)$ such that $gE(b) \subseteq E(gb)$ and the map g: $E(b) \to E(gb)$ is an isometry. Then we get an action of G on the space $\Gamma(B, E)$ of all sections by

$$(gs)(b) = \sqrt{\frac{\mathrm{d}g_*\mu}{\mathrm{d}\mu}}\, gs(g^{-1}b).$$

DEFINITION 2.5

We say that E is a *measurable G-field of Hilbert spaces* over (B, μ) if G preserves $\mathcal{E}$.

We will omit explicit reference to the measure μ when there is no danger of confusion. Note that in this situation, G also acts on $L^2(B, E)$ giving rise to a unitary representation.

DEFINITION 2.6

A *perfect realization* over B is a measurable G-field E of Hilbert spaces over B together with a unitary equivalence between ρ and $L^2(B, E)$.

DEFINITION 2.7

Let E be a perfect realization over (B, μ). Let B' be a standard G-space. Suppose f: $B \to B'$ is a Borel G-map. Set $\mu' = f_*\mu$. Then μ' is G-quasi-invariant. Suppose μ' is σ-finite. We get a G-quasi-equivariant disintegration $\lambda_{b'}$ of μ over μ'. Set

$E'(b') = L^2(f^{-1}(b'), E, \lambda_{b'})$ and let G act in the obvious way on $\bigcup_{b' \in B'} E'(b)$. If $s \in \Gamma(B', E')$, define $f^*s \in \Gamma(B, E)$ by $f^*s(b) = s(f(b))(b)$. Set

$$\mathcal{E}' = \{s \in \Gamma(B', E') | f^*s \in \mathcal{E}\}.$$

Then E' is a perfect realization over (B', μ'). We call E' the *push forward* of E by f.

Remark 2.8. For the proof of existence of a disintegration, see Theorem 7.1 of [3].

DEFINITION 2.9

A perfect realization E is minimal if whenever E is isomorphic to a push forward by a map f. Then f is a Borel equivalence of co-null sets.

Perfect realizations and systems of imprimitivity are essentially equivalent as explained in Theorems 2.10 and 2.11. Observe that a perfect realization gives rise to a system of imprimitivity. In fact, we take $\mathcal{A} = \{m_f | f \in L^\infty(B)\}$ where $m_f \colon L^2(B, E) \to L^2(B, E)$ is given by $(m_f s)(b) = f(b)s(b)$. Conversely, every system of imprimitivity arises in this way. There is a uniqueness theorem in this context.

Theorem 2.10. *For $j = 1, 2$, let $\iota_j \colon \mathcal{H} \to L^2(B_j, E_j)$ be a perfect realization of ρ on a G-field E_j of Hilbert spaces over a standard Borel measure space (B_j, μ_j). Consider $L^\infty(B_j)$ as an algebra of multiplier operators on $L^2(B_j, E_j)$. Let $\mathcal{A}_j = \iota_j^{-1} L^\infty(B_j) \iota_j$. Suppose that $\mathcal{A}_1 = \mathcal{A}_2$. Then there exist G-maps $F_1 \colon E_1 \to E_2$ and $F_2 \colon E_2 \to E_1$ such that*

(1) *F_1 covers a G-map $f_1 \colon B_1 \to B_2$ and likewise for F_2.*
(2) *F_j is bounded and linear on each fibre.*
(3) *F_j takes measurable sections to measurable sections.*
(4) *F_1 induces a unitary map from $L^2(B_1, E_1)$ to $L^2(B_2, E_2)$ which coincides with $\iota_2 \iota_1^{-1}$; likewise for F_2.*

Moreover, one has

(1) *f_1 and f_2 are measurable maps which invert one another almost everywhere.*
(2) *$(f_1)_* \mu_1$ is equivalent to μ_2 and likewise for f_2.*
(3) *The maps*

$$\left(\frac{\mathrm{d}(f_1)_* \mu_1}{\mathrm{d}\mu_2} \right)^{1/2} F_1$$

$$\left(\frac{\mathrm{d}(f_2)_* \mu_2}{\mathrm{d}\mu_1} \right)^{1/2} F_2$$

are unitary and invert one another on almost every fibre.

Finally, the maps F_1 and F_2 satisfying these conditions are unique up to adjustment on sets of measure zero in B_1 and B_2.

Theorem 2.11. *A system of imprimitivity is maximal if and only if it corresponds to a minimal perfect realization.*

Remark 2.12. Note that in Theorem 1.4, $\mathcal{H}_0$ is essentially $L^2(\mathbb{R}^{2n}/M, E)$ where E is a certain family of Hilbert spaces (in fact, a line bundle) over $\mathbb{R}^{2n}/M$. It is not hard to see that E is a minimal perfect realization of the canonical representation in our sense.

3. Background

3.1 *The symplectic Fourier transform*

DEFINITION 3.1

If $f \in L^1(\mathbb{R}^{2n})$, we define its *symplectic Fourier transform* $\check{f}$ by

$$\check{f}(w') = \int_{\mathbb{R}^{2n}} e(w', w) f(w)\, \mathrm{d}w.$$

Observe that if $\hat{f}$ is the usual Fourier transform of f, then $\check{f}(w') = \hat{f}(Jw')$ where J is the linear transformation defined by $J(x', y') = (-y', x')$. All the usual theorems of Fourier analysis are valid for the symplectic Fourier transform with the exception that it has order 2 instead of 4.

3.2 *Trace class operators*

Let $\mathcal{H}$ be a Hilbert space. Let $S^1(\mathcal{H})$ be the Banach space of trace class operators on $\mathcal{H}$ with the trace norm $\|\cdot\|_{S^1}$.

DEFINITION 3.2

Let $\varphi, \psi \in \mathcal{H}$. The *rank one operator* $\varphi \otimes \bar{\psi} \in L(\mathcal{H})$ is defined by

$$(\varphi \otimes \bar{\psi}) f = \langle f, \psi \rangle \varphi, \quad f \in \mathcal{H}.$$

Recall the *singular value decomposition* for compact operators on $\mathcal{H}$.

Theorem 3.3. *Let X be a compact operator on $\mathcal{H}$. Then there exist orthonormal sets $\{\psi_k\}_{k=1}^{\infty}$ and $\{\varphi_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ and non-negative real numbers $\{\lambda_k\}_{k=1}^{\infty}$ with $\lambda_k \to 0$ so that*

$$X = \sum_{k=1}^{\infty} \lambda_k \varphi_k \otimes \overline{\psi_k} \quad (\textit{in the operator norm}).$$

Proof. See [4]. □

COROLLARY 3.4

Let $X \in S^1(\mathcal{H})$ and let

$$X = \sum_{k=1}^{\infty} \lambda_k \varphi_k \otimes \overline{\psi_k}$$

be its singular value decomposition. Let

$$F_n = \sum_{k=1}^{n} \lambda_k \varphi_k \otimes \overline{\psi_k}.$$

Then

(1) $F_n \to X \quad (in\ S^1(\mathcal{H}))$;
(2) $\|X\|_{S^1} = \sum_{k=1}^{\infty} \lambda_k$; *and*
(3) $\mathrm{tr}(X) = \sum_{k=1}^{\infty} \lambda_k \langle \varphi_k, \psi_k \rangle$.

Proof. See [4]. □

DEFINITION 3.5

$\mathbb{R}^{2n}$ acts on $S^1(\mathcal{H})$ by

$$(x_1, y_1) \cdot X = \rho(x_1, y_1, 1) X \rho(x_1, y_1, 1)^{-1}.$$

DEFINITION 3.6

If $q \in L^1(\mathbb{R}^{2n})$, then

$$q \cdot X = \iint q(x_1, y_1)((x_1, y_1) \cdot X)\, \mathrm{d}x_1\, \mathrm{d}y_1.$$

Theorem 3.7. *With this action,* $S^1(\mathcal{H})$ *becomes a continuous* $L^1(\mathbb{R}^{2n})$*-module with an approximate identity.*

Proof. The module properties are purely formal. If $q \in L^1(\mathbb{R}^{2n})$ and $X \in S^1(\mathcal{H})$, then

$$\begin{aligned} \|q \cdot X\|_{S^1} &= \left\| \iint q(x_1, y_1)(x_1, y_1) \cdot X\, \mathrm{d}x\, \mathrm{d}y \right\|_{S^1} \\ &\leq \iint |q(x_1, y_1)| \|(x_1, y_1) \cdot X\|_{S^1}\, \mathrm{d}x\, \mathrm{d}y \\ &= \|q\|_1 \|X\|_{S^1}. \end{aligned}$$

This shows the continuity.

Let K_N be the Fejer kernel. We claim that K_N is an approximate identity for $S^1(\mathcal{H})$. Let $\varphi, \psi \in \mathcal{H}$. Let $\varepsilon > 0$. By the strong continuity of ρ, there exists $\delta > 0$ such that $\|(x_1, y_1)\| < \delta$ implies that

$$\|\varphi - \rho(x_1, y_1, 1)\varphi\|_2 < \frac{\varepsilon}{4\|\psi\|_2}$$

and

$$\|\psi - \rho(x_1, y_1, 1)\psi\|_2 < \frac{\varepsilon}{4\|\varphi\|_2}.$$

By standard properties of the Fejer kernel, N can be chosen large enough such that

$$\iint_{\mathbb{R}^{2n} \setminus B_\delta(0)} K_N < \frac{\varepsilon}{2}.$$

So

$$
\begin{aligned}
&\varphi \otimes \bar{\psi} - K_N \cdot (\varphi \otimes \bar{\psi}) \\
&\quad = \iint K_N(x_1, y_1)(\varphi \otimes \bar{\psi} - (x_1, y_1) \cdot (\varphi \otimes \bar{\psi}))\, \mathrm{d}x_1\, \mathrm{d}y_1 \\
&\quad = \iint K_N(x_1, y_1)(\varphi \otimes \bar{\psi} - \rho(x_1, y_1, 1)(\varphi \otimes \bar{\psi})\rho(x_1, y_1, 1)^{-1})\, \mathrm{d}x_1\, \mathrm{d}y_1 \\
&\quad = \iint K_N(x_1, y_1)(\varphi \otimes \bar{\psi} - (\rho(x_1, y_1, 1)\varphi) \otimes \overline{(\rho(x_1, y_1, 1)\psi)})\, \mathrm{d}x_1\, \mathrm{d}y_1 \\
&\quad = \iint K_N(x_1, y_1) \Big((\varphi - \rho(x_1, y_1, 1)\varphi) \otimes \bar{\psi} \\
&\qquad + (\rho(x_1, y_1, 1)\varphi) \otimes \overline{(\psi - \rho(x_1, y_1, 1)\psi)}\Big)\, \mathrm{d}x_1\, \mathrm{d}y_1.
\end{aligned}
$$

Therefore, for N large enough,

$$
\begin{aligned}
\|\varphi \otimes \bar{\psi} - K_N \cdot (\varphi \otimes \bar{\psi})\|_{S^1} &\leq \iint K_N(x_1, y_1)(\|(\varphi - \rho(x_1, y_1, 1)\varphi) \otimes \bar{\psi}\|_{S^1} \\
&\quad + \|(\rho(x_1, y_1, 1)\varphi) \otimes \overline{(\psi - \rho(x_1, y_1, 1)\psi)}\|_{S^1})\, \mathrm{d}x_1\, \mathrm{d}y_1 \\
&= \iint K_N(x_1, y_1)(\|\varphi - \rho(x_1, y_1, 1)\varphi\|_2 \|\psi\|_2 \\
&\quad + \|\varphi\|_2 \|\psi - \rho(x_1, y_1, 1)\psi\|_2)\, \mathrm{d}x_1\, \mathrm{d}y_1 \\
&= \iint_{B_\delta(0)} K_N(x_1, y_1)(\|\varphi - \rho(x_1, y_1, 1)\varphi\|_2 \|\psi\|_2 \\
&\quad + \|\varphi\|_2 \|\psi - \rho(x_1, y_1, 1)\psi\|_2)\, \mathrm{d}x_1\, \mathrm{d}y_1 \\
&\quad + \iint_{\mathbb{R}^{2n} \setminus B_\delta(0)} K_N(x_1, y_1)(\|\varphi - \rho(x_1, y_1, 1)\varphi\|_2 \|\psi\|_2 \\
&\quad + \|\varphi\|_2 \|\psi - \rho(x_1, y_1, 1)\psi\|_2)\, \mathrm{d}x_1\, \mathrm{d}y_1 \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

So if F is a finite rank operator and $\varepsilon > 0$, N can be chosen large enough such that

$$\|F - K_N \cdot F\|_{S^1} < \varepsilon.$$

Now, if $X \in S^1(\mathcal{H})$ and $\varepsilon > 0$, we can find a finite rank operator F such that $\|X - F\|_{S^1} < \frac{\varepsilon}{3}$. By the previous remark, N can be chosen large enough such that

$$\|F - K_N \cdot F\|_{S^1} < \frac{\varepsilon}{3}.$$

So

$$\begin{aligned}\|X - K_N \cdot X\|_{S^1} &= \|(X - F) + (F - K_N \cdot F) + K_N \cdot (F - X)\|_{S^1}\\ &\le \|X - F\|_{S^1} + \|F - K_N \cdot F\|_{S^1} + \|K_N\|_1 \|F - X\|_{S^1}\\ &< \varepsilon. \qquad \square\end{aligned}$$

3.3 *The alpha transform*

Let $X \in S^1(\mathcal{H})$. For $(x, y) \in \mathbb{R}^{2n}$, set

$$\alpha(X)(x, y) = \mathrm{tr}(\rho(x, y, 1)X).$$

Note that if $\varphi, \psi \in \mathcal{H}$, and $X = \varphi \otimes \bar{\psi}$, then $\alpha(X)(x, y)$ is just the matrix coefficient $\langle \rho(x, y, 1)\varphi, \psi\rangle$. However, in order to suggest its analogy with the Fourier transform (which will be developed below), we make the following definition.

DEFINITION 3.8

Let $X \in S^1(\mathcal{H})$. Then the *alpha transform* of X is the function on $\mathbb{R}^{2n}$ defined by

$$\alpha(X)(x, y) = \mathrm{tr}(\rho(x, y, 1)X).$$

Lemma 3.9. $\alpha((x_1, y_1) \cdot X)(x, y) = e((x_1, y_1), (x, y))\alpha(X)(x, y)$.

Proof.

$$\begin{aligned}\alpha((x_1, y_1) \cdot X) &= \alpha(\rho(x_1, y_1, 1)X\rho(x_1, y_1, 1)^{-1})\\ &= \mathrm{tr}(\rho(x, y, 1)\rho(x_1, y_1, 1)X\rho(x_1, y_1, 1)^{-1})\\ &= \mathrm{tr}(\rho((x, y, 1)(x_1, y_1, 1))X\rho(x_1, y_1, 1)^{-1})\\ &= e((x_1, y_1), (x, y))\mathrm{tr}(\rho((x_1, y_1, 1)(x, y, 1))X\rho(x_1, y_1, 1)^{-1})\\ &= e((x_1, y_1), (x, y))\mathrm{tr}(\rho(x_1, y_1, 1)\rho(x, y, 1)X\rho(x_1, y_1, 1)^{-1})\\ &= e((x_1, y_1), (x, y))\mathrm{tr}(\rho(x, y, 1)X)\\ &= e((x_1, y_1), (x, y))\alpha(X)(x, y). \qquad \square\end{aligned}$$

Lemma 3.10. $\alpha(q \cdot X)(x, y) = \check{q}(x, y)\alpha(X)(x, y)$.

Proof. This follows immediately from Lemma 3.9. □

Lemma 3.11. *Let* $\varphi, \psi \in \mathcal{H}$. *Then* $\alpha(\varphi \otimes \bar{\psi}) = \langle \rho(x, y, 1)\varphi, \psi\rangle$.

Proof. Note that $\rho(x, y, 1)(\varphi \otimes \bar{\psi}) = (\rho(x, y, 1)\varphi) \otimes \bar{\psi}$. Let $\psi_1 = \frac{\psi}{\|\psi\|}$ and extend $\{\psi_1\}$ to an orthonormal basis $\{\psi_1, \psi_2, \ldots\}$ of $\mathcal{H}$. Then

$$\begin{aligned}\alpha(\varphi \otimes \bar{\psi}) &= \mathrm{tr}(\rho(x, y, 1)(\varphi \otimes \bar{\psi}))\\ &= \mathrm{tr}(\rho(x, y, 1)\varphi \otimes \bar{\psi})\end{aligned}$$

$$= \sum_{n=1}^{\infty} \langle (\rho(x, y, 1)\varphi \otimes \bar{\psi})\psi_n, \psi_n \rangle$$

$$= \sum_{n=1}^{\infty} \langle \psi_n, \psi \rangle \langle \rho(x, y, 1)\varphi, \psi_n \rangle$$

$$= \langle \rho(x, y, 1)\varphi, \psi \rangle. \qquad \square$$

Let $C_0(\mathbb{R}^{2n})$ denote the space of all continuous functions on $\mathbb{R}^{2n}$ which vanish at infinity. There is an analogue of the Riemann–Lebesgue lemma for the alpha transform.

Lemma 3.12. *If* $X \in S^1(\mathcal{H})$, *then* $\|\alpha(X)\|_\infty \leq \|X\|_{S^1}$ *and* $\alpha(X) \in C_0(\mathbb{R}^{2n})$.

Proof. For any $Z \in S^1(\mathcal{H})$, we have $|\mathrm{tr}(Z)| \leq \mathrm{tr}(|Z|)$ by the spectral theorem. So for any $X \in S^1(\mathcal{H})$,

$$\begin{aligned} \|\alpha(X)\|_\infty &= \sup_{(x,y)\in\mathbb{R}^{2n}} |\mathrm{tr}(\rho(x, y, 1)X)| \\ &\leq \sup_{(x,y)\in\mathbb{R}^{2n}} \mathrm{tr}(|\rho(x, y, 1)X|) \\ &= \mathrm{tr}(|X|) \quad \text{(since } \rho \text{ is unitary)} \\ &= \|X\|_{S^1}. \end{aligned} \tag{2}$$

Assume for a moment that ρ is realized on $L^2(\mathbb{R}^n)$ as in (1). Then for compactly supported continuous $\varphi', \psi' \in L^2(\mathbb{R}^n)$,

$$\begin{aligned} \alpha(\varphi' \otimes \bar{\psi}') &= \langle \rho(x, y, 1)\varphi', \psi' \rangle \\ &= \int \mathrm{e}^{-2\pi i y t} \varphi' \left(t - \frac{x}{2}\right) \bar{\psi}' \left(t + \frac{x}{2}\right) \mathrm{d}t, \end{aligned}$$

and so by the classical Riemann–Lebesgue lemma, $\alpha(\varphi' \otimes \bar{\psi}') \in C_0(\mathbb{R}^{2n})$. But, if $\varphi, \psi \in L^2(\mathbb{R}^n)$, and $\varepsilon > 0$, we can find compactly supported continuous φ, ψ such that

$$\|\varphi - \varphi'\|_2 < \frac{\varepsilon}{2\|\psi\|_2}$$

and

$$\|\psi - \psi'\|_2 < \frac{\varepsilon}{2\|\varphi'\|_2}.$$

So

$$\begin{aligned} \|\alpha(\varphi \otimes \bar{\psi}) - \alpha(\varphi' \otimes \bar{\psi}')\|_\infty &= \|\alpha(\varphi \otimes \bar{\psi} - \varphi' \otimes \bar{\psi}')\|_\infty \\ &\leq \|\varphi \otimes \bar{\psi} - \varphi' \otimes \bar{\psi}'\|_{S^1} \quad \text{(by (2))} \\ &= \|(\varphi - \varphi') \otimes \bar{\psi} + \varphi' \otimes \overline{(\psi - \psi')}\|_{S^1} \\ &\leq \|(\varphi - \varphi') \otimes \bar{\psi}\|_{S^1} + \|\varphi' \otimes (\psi - \psi')\|_{S^1} \\ &= \|\varphi - \varphi'\|_2 \|\psi\|_2 + \|\varphi'\|_2 \|\psi - \psi'\|_2 \\ &< \varepsilon. \end{aligned}$$

Therefore,

$$\alpha(\varphi \otimes \bar{\psi}) \in C_0(\mathbb{R}^{2n}).$$

It follows that for any finite rank operator F, $\alpha(F) \in C_0(\mathbb{R}^{2n})$. But if $\varepsilon > 0$, we can find a finite rank operator F such that $\|X - F\|_{S^1} < \varepsilon$. Therefore,

$$\|\alpha(X) - \alpha(F)\|_\infty < \varepsilon.$$

Therefore,

$$\alpha(X) \in C_0(\mathbb{R}^{2n}). \qquad \square$$

4. The main estimate

For calculations, it is convenient to take the realization (1) for ρ. Thus, in this section $\mathcal{H} = L^2(\mathbb{R}^n)$.

In view of Lemma 3.12, one might hope that $\alpha(S^1(\mathcal{H})) = (L^1(\mathbb{R}^{2n}))\check{}$. The following example shows that this is not the case:

Example 4.1. Take $\varphi = \psi = \chi_{[-1,1]}$, the characteristic function of the interval $[-1, 1]$. Then $\alpha(\varphi \otimes \psi)$ is not the symplectic Fourier transform of an L^1 function.

Proof. Let

$$\gamma(x_2, y_2) = \int \mathrm{e}^{-2\pi i y_2 t} \varphi\left(x_2 - \frac{t}{2}\right) \overline{\psi\left(x_2 + \frac{t}{2}\right)} \mathrm{d}t.$$

Then a short calculation shows that $\check{\gamma} = \alpha(\varphi \otimes \bar{\psi})$ as tempered distributions. But $\gamma \notin L^1(\mathbb{R}^{2n})$. $\square$

However, alpha transforms are *locally* symplectic Fourier transforms. To see this, we need to introduce an auxiliary function.

DEFINITION 4.2

Let $X \in S^1(\mathcal{H})$. Then

$$\beta(X)(x, y) = \mathrm{e}^{-\frac{\pi}{2}(|x|^2+|y|^2)} \alpha(X)(x, y).$$

Theorem 4.3. *If $X \in S^1(\mathcal{H})$, then*

$$\|\check{\beta}(X)\|_1 \leq \|X\|_{S^1}.$$

Proof. We will first get the estimate for a rank one operator $\varphi \otimes \bar{\psi}$ with $\varphi, \psi \in L^2(\mathbb{R}^n)$. If f is a complex-valued function defined on $\mathbb{R}^n$ and $v \in \mathbb{R}^n$, define the functions $\tilde{f}$ and f_v by

$$\tilde{f}(x) = f(-x)$$

and

$$f_v(x) = \mathrm{e}^{-\pi(x-v)^2} f(x).$$

Then

$$
\begin{aligned}
&\breve{\beta}(\varphi \otimes \bar{\psi})(x_2, y_2) \\
&= \iint e((x_2, y_2), (x, y))\beta(X)(x, y)\, \mathrm{d}x\, \mathrm{d}y \\
&= \iint \mathrm{e}^{2\pi i(x_2 y - y_2 x)} \mathrm{e}^{-\frac{\pi}{2}(x^2+y^2)} \int \mathrm{e}^{-2\pi i y t} \varphi\left(t - \frac{x}{2}\right) \bar{\psi}\left(t + \frac{x}{2}\right) \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}y \\
&= \iint \mathrm{e}^{-2\pi i y_2 x} \mathrm{e}^{-\frac{\pi}{2}x^2} \int \mathrm{e}^{-2\pi i y(t - x_2)} \mathrm{e}^{-\frac{\pi}{2}y^2}\, \mathrm{d}y \varphi\left(t - \frac{x}{2}\right) \bar{\psi}\left(t + \frac{x}{2}\right) \mathrm{d}t\, \mathrm{d}x \\
&= \sqrt{2} \iint \mathrm{e}^{-2\pi i y_2 x} \mathrm{e}^{-2\pi(t - x_2)^2} \mathrm{e}^{-\frac{\pi}{2}x^2} \varphi\left(t - \frac{x}{2}\right) \bar{\psi}\left(t + \frac{x}{2}\right) \mathrm{d}t\, \mathrm{d}x \\
&= \sqrt{2} \iint \mathrm{e}^{-2\pi i y_2 x} \left(\mathrm{e}^{-\pi\left(t - \frac{x}{2} - x_2\right)^2} \varphi\left(t - \frac{x}{2}\right)\right) \\
&\quad \times \left(\mathrm{e}^{-\pi\left(t + \frac{x}{2} - x_2\right)^2} \bar{\psi}\left(t + \frac{x}{2}\right)\right) \mathrm{d}t\, \mathrm{d}x \\
&= \sqrt{2} \iint \mathrm{e}^{-2\pi i y_2 x} \varphi_{x_2}\left(t - \frac{x}{2}\right) \bar{\psi}_{x_2}\left(t + \frac{x}{2}\right) \mathrm{d}t\, \mathrm{d}x \\
&= \sqrt{2} \int \mathrm{e}^{-2\pi i y_2 x} (\tilde{\varphi}_{x_2} * \bar{\psi}_{x_2})(x)\, \mathrm{d}x \\
&= \sqrt{2} \check{\varphi}_{x_2}(y_2) \hat{\bar{\psi}}_{x_2}(y_2).
\end{aligned}
$$

It follows from the Cauchy–Schwartz inequality and the Plancherel theorem that

$$
\begin{aligned}
\|\breve{\beta}(\varphi \otimes \bar{\psi})\|_1 &= \sqrt{2} \iint |\check{\varphi}_{x_2}(y_2) \hat{\bar{\psi}}_{x_2}(y_2)|\, \mathrm{d}x_2\, \mathrm{d}y_2 \\
&\leq \sqrt{2} \int \left(\int |\check{\varphi}_{x_2}(y_2)|^2\, \mathrm{d}y_2\right)^{1/2} \left(\int |\hat{\bar{\psi}}_{x_2}(y_2)|^2\, \mathrm{d}y_2\right)^{1/2} \mathrm{d}x_2 \\
&= \sqrt{2} \int \left(\int |\varphi_{x_2}(y)|^2\, \mathrm{d}y\right)^{1/2} \left(\int |\bar{\psi}_{x_2}(y)|^2\, \mathrm{d}y\right)^{1/2} \mathrm{d}x_2 \\
&\leq \sqrt{2} \left(\iint |\varphi_{x_2}(y)|^2\, \mathrm{d}y\, \mathrm{d}x_2\right)^{1/2} \left(\iint |\bar{\psi}_{x_2}(y)|^2\, \mathrm{d}y\, \mathrm{d}x_2\right)^{1/2} \\
&= \left(\int |\varphi(y)|^2\, \mathrm{d}y\right)^{1/2} \left(\int |\psi(y)|^2\, \mathrm{d}y\right)^{1/2} \\
&= \|\varphi\|_2\, \|\psi\|_2 \\
&= \|\varphi \otimes \bar{\psi}\|_{S^1}.
\end{aligned}
$$

Let

$$X = \sum_{k=1}^{\infty} \lambda_k \varphi_k \otimes \overline{\psi_k}$$

be the singular value decomposition of X and set

$$F_n = \sum_{k=1}^{n} \lambda_k \varphi_k \otimes \overline{\psi_k}.$$

Then

$$\begin{aligned}
\|\breve{\beta}(F_n)\|_1 &\le \sum_{k=1}^{\infty} \lambda_k \quad \text{(by the previous calculation)}\\
&\le \|X\|_{S^1} \quad \text{(by Corollary 3.4).}
\end{aligned}$$

Note that for any $Z \in S^1(\mathcal{H})$,

$$\begin{aligned}
\|\breve{\beta}(Z)\|_\infty &\le \|\beta(Z)\|_1\\
&= \iint |\mathrm{e}^{-\frac{\pi}{2}(x^2+y^2)} \alpha(Z)(x, y)| \,\mathrm{d}x \,\mathrm{d}y\\
&\le \|\alpha(Z)\|_\infty \iint \mathrm{e}^{-\frac{\pi}{2}(x^2+y^2)} \,\mathrm{d}x \,\mathrm{d}y\\
&\le \|Z\|_{S^1}.
\end{aligned}$$

Since $F_n \to X$ in $S^1(\mathcal{H})$,

$$\begin{aligned}
\|\breve{\beta}(X) - \breve{\beta}(F_n)\|_\infty &= \|\breve{\beta}(X - F_n)\|_\infty\\
&\le \|X - F_n\|_{S^1}\\
&\to 0.
\end{aligned}$$

Therefore, $\breve{\beta}(F_n) \to \breve{\beta}(X)$ almost everywhere. Therefore,

$$\begin{aligned}
\|\breve{\beta}(X)\|_1 &= \iint |\breve{\beta}(X)(x, y)| \,\mathrm{d}x \,\mathrm{d}y\\
&= \iint \liminf_{n\to\infty} |\breve{\beta}(F_n)(x, y)| \,\mathrm{d}x \,\mathrm{d}y\\
&\le \liminf_{n\to\infty} \iint |\breve{\beta}(F_n)(x, y)| \,\mathrm{d}x \,\mathrm{d}y \quad \text{(by Fatou's lemma)}\\
&\le \liminf_{n\to\infty} \|\breve{\beta}(F_n)\|_1\\
&\le \|X\|_{S^1} \quad \text{(by the previous calculation.)}
\end{aligned}$$

□

COROLLARY 4.4

If $X \in S^1(\mathcal{H})$ and $\eta \in C_c^\infty(\mathbb{R}^{2n})$, then $\alpha(X)\eta$ is the symplectic Fourier transform of an L^1-function.

Proof.

$$\alpha(X)\eta = (\alpha(X)\mathrm{e}^{-\frac{\pi}{2}(x^2+y^2)})(\eta\mathrm{e}^{\frac{\pi}{2}(x^2+y^2)}).$$

By Theorem 4.3, the first factor is the symplectic Fourier transform of an L^1 function. Since the second factor is C_c^∞, it is the symplectic Fourier transform of an L^1 function. So their product is the symplectic Fourier transform of an L^1 function. □

COROLLARY 4.5

If $X \in S^1(\mathcal{H})$ and $\alpha(X)$ is compactly supported, then $\alpha(X)$ is the symplectic Fourier transform of an L^1-function.

Proof. Let $\eta \in C_c^\infty(\mathbb{R}^{2n})$ be identically 1 on $\mathrm{supp}(\alpha(X))$. Then by the previous result, $\alpha(X)\eta$ is the symplectic Fourier transform of an L^1-function. But $\alpha(X) = \alpha(X)\eta$. □

5. Spectral synthesis for the alpha transform

Theorem 5.1. *Let $g \in L^1(\mathbb{R}^{2n})$ with $\breve{g}$ compactly supported. Let $f \in L^1(\mathbb{R}^{2n})$ with $\breve{f}(w') \neq 0$ for all $w' \in \mathrm{supp}(\breve{g})$. Then there exists $F \in L^1(\mathbb{R}^{2n})$ such that*

$$F * f * g = g.$$

Proof. See the proof of Theorem 9.3 of [6]. □

Remark 5.2. By a submodule, we will always mean a closed submodule.

DEFINITION 5.3

Let $\mathcal{B}$ be a submodule of $S^1(\mathcal{H})$. Then the cospectrum of $\mathcal{B}$ is

$$\mathrm{V}(\mathcal{B}) = \{(x,y) \in \mathbb{R}^{2n} | \alpha(X)(x,y) = 0 \quad \forall X \in \mathcal{B}\}.$$

DEFINITION 5.4

Let $M \subset \mathbb{R}^{2n}$. Then the module of M is

$$I(M) = \{Y \in S^1 | \alpha(Y) = 0 \text{ on } M\}.$$

Lemma 5.5. *Let $\mathcal{B}$ be a submodule of $\mathcal{H}$. Let K be a compact subset of $\mathbb{R}^{2n} \backslash V(\mathcal{B})$. Then there exists $T_K \in \mathcal{B}$ such that $\alpha(T_K)$ is compactly supported and $\alpha(T_K) \equiv 1$ on K.*

Proof. For each $w \in K$, there exists $T_w \in \mathcal{B}$ such that $\alpha(T_w) \neq 0$ on a neighborhood of w. By replacing T_w with $\breve{\eta} \cdot T_w$ where $\eta \in C_0^\infty(\mathbb{R}^{2n})$ and is identically 1 on K, we may assume that $\alpha(T_w)$ is also compactly supported. So by Corollary 4.5 there exists $f_w \in L^1(\mathbb{R}^{2n})$ such that $\breve{f}_w = \alpha(T_w)$. By Lemma 9.2 of [6], there exist a compact neighborhood U_w of w and $h_w \in L^1(\mathbb{R}^{2n})$ such that $\|h_w\|_1 < \frac{1}{2}$ and

$$\breve{h}_w(w') = 1 - \frac{\breve{f}_w(w')}{\breve{f}_w(w)} \quad \text{on } U_w.$$

Since K is compact, there exists $w_1, \ldots, w_n \in K$ such that $U_{w_1}, \ldots U_{w_n}$ cover K. Let $\rho_1, \ldots, \rho_n$ be a smooth partition of unity subordinate to this cover. Set

$$T_K = \sum_{i=1}^{n} \left(\frac{\breve{\rho}_i}{\breve{f}_{w_i}(w_i)} * \sum_{k=0}^{\infty} h_{w_i}^{*k} \right) \cdot T_{w_i}.$$

Then $T_K \in \mathcal{B}$, and

$$\begin{aligned}
\alpha(T_K)(w) &= \sum_{i=1}^{n} \left(\frac{\breve{\rho}_i}{\breve{f}_{w_i}(w_i)} * \sum_{k=0}^{\infty} h_{w_i}^{*k} \right)^{\vee} (w) \alpha(T_{w_i})(w) \\
&= \sum_{i=1}^{n} \left(\frac{\rho_i(w)}{\breve{f}_{w_i}(w_i)} \sum_{k=0}^{\infty} \breve{h}_{w_i}^{k}(w) \right) \breve{f}_{w_i}(w) \\
&= \sum_{i=1}^{n} \left(\frac{1}{\breve{f}_{w_i}(w_i)} \frac{\rho_i(w)}{1 - \breve{h}_{w_i}(w)} \right) \breve{f}_{w_i}(w) \\
&= \sum_{i=1}^{n} \left(\frac{1}{\breve{f}_{w_i}(w_i)} \frac{\rho_i(w) \breve{f}_{w_i}(w_i)}{\breve{f}_{w_i}(w)} \right) \breve{f}_{w_i}(w) \\
&= \sum_{i=1}^{n} \rho_i(w).
\end{aligned}$$

So $\alpha(T_K)$ is compactly supported and $\alpha(T_K) \equiv 1$ on K. □

Theorem 5.6. *Let M be a closed subgroup of $\mathbb{R}^{2n}$. If $X \in I(M)$ and $\varepsilon > 0$, then there exists $Z \in I(M)$ such that* $\mathrm{supp}(\alpha(Z))$ *is a compact subset of $\mathbb{R}^{2n} \backslash M$ and $\|X - Z\|_{S^1} < \varepsilon$.*

Proof. Let K_N be the Fejer kernel. Recall that $\breve{K}_N$ is compactly supported. So for large enough N, $Y = \alpha(K_N \cdot X)$ is supported on a compact set K and

$$\|X - Y\|_{S^1} < \frac{\varepsilon}{2}.$$

So there exists $f \in L^1(\mathbb{R}^{2n})$ such that $\breve{f} = \alpha(Y)$. Moreover, $Y = f \cdot T_K$. By Calderon's theorem (Theorem 7.5.2 of [5]), there exists $g \in L^1(\mathbb{R}^{2n})$ such that $\mathrm{supp}(\breve{g})$ is a compact subset of $\mathbb{R}^{2n} \backslash M$ and

$$\|f - g * f\|_1 < \frac{\varepsilon}{2\|T_K\|_{S^1}}.$$

Set $Z = g \cdot Y$. Then $\mathrm{supp}(\alpha(Z))$ is a compact subset of $\mathbb{R}^{2n} \backslash M$ and

$$\begin{aligned}
\|Y - Z\|_{S^1} &= \|Y - g \cdot Y\|_{S^1} \\
&= \|f \cdot T_K - g \cdot (f \cdot T_K)\|_{S^1} \\
&= \|(f - g * f) \cdot T_K\|_{S^1} \\
&\leq \|T_K\|_{S^1} \, \|f - g * f\|_1 \\
&< \frac{\varepsilon}{2}.
\end{aligned}$$

Therefore,

$$\|X - Z\|_{S^1} \leq \|X - Y\|_{S^1} + \|Y - Z\|_{S^1} < \varepsilon. \qquad \square$$

For a proof of this theorem that does not appeal to Calderon's theorem, see [7].

Theorem 5.7. *Let $\mathcal{B}$ be a submodule of $S^1(\mathcal{H})$ such that $V(\mathcal{B})$ is a closed subgroup of $\mathbb{R}^{2n}$. Then*

$$I(V(\mathcal{B})) = \mathcal{B}.$$

Proof. Clearly, $\mathcal{B} \subset I(V(\mathcal{B}))$. By Theorem 5.6, the set of $Y \in S^1(\mathcal{H})$ with $\alpha(Y)$ compactly supported away from $V(\mathcal{B})$ is dense in $I(V(\mathcal{B}))$. Thus it suffices to show that each such Y is in $\mathcal{B}$. Let such a Y be given. Then by Lemma 5.5, there exists $X \in \mathcal{B}$ such that $\alpha(X)$ is compactly supported and $\alpha(X) \equiv 1$ on $\operatorname{supp}(\alpha(Y))$. By Lemma 4.5, there exists $f, g \in L^1(\mathbb{R}^{2n})$ such that

$$\check{f} = \alpha(X) \quad \text{and} \quad \check{g} = \alpha(Y).$$

By Theorem 5.1, there exists $F \in L^1(\mathbb{R}^{2n})$ with $F * f * g = g$. Then

$$\begin{aligned} \alpha((F * g) \cdot X) &= (F * g)\check{}\,\alpha(X) \\ &= \check{F}\check{f}\check{g} \\ &= (F * f * g)\check{} \\ &= \check{g} \\ &= \alpha(Y). \end{aligned}$$

Therefore, $(F * g) \cdot X = Y$. So $Y \in \mathcal{B}$. $\square$

6. Classification of inductive algebras

Remark 6.1. The pairing

$$(T, X) \mapsto \operatorname{tr}(TX), \quad T \in L(\mathcal{H}), X \in S^1(\mathcal{H})$$

identifies $L(\mathcal{H})$ as the dual of $S^1(\mathcal{H})$. Annihilators are taken with respect to this pairing. We give $L(\mathcal{H})$ the weak topology.

Theorem 6.2. *Let $M \subset \mathbb{R}^2$ be a maximal isotropic subgroup. Set*

$$\mathcal{A} = \overline{\operatorname{span}\{\rho(x, y, 1) | (x, y) \in M\}}.$$

Then $\mathcal{A}$ is a maximal inductive algebra.

Proof. Realize ρ as in Theorem 1.4 and compute $\rho(x, y, 1)$ for $(x, y) \in M$:

$$\begin{aligned} (\rho(x, y, 1) f)(w) &= e(w/2, (x, y)) f(w + (x, y)) \\ &= e(w/2, (x, y))\delta(x, y)^{-1} e((x, y)/2, w)^{-1} f(w) \\ &= e(w, (x, y))\delta(x, y)^{-1} f(w). \end{aligned}$$

So $\rho(x, y, 1)$ is just multiplication by the character $e(w, (x, y))\delta(x, y)^{-1}$ and so $\rho(x, y, 1) \in L^\infty(\mathbb{R}^{2n}/M)$. Now, in view of Pontrjagin duality, it is not hard to see that *every* character of $\mathbb{R}^{2n}/M$ is of this form. From the well-known fact that the linear span of all characters is weakly dense in $L^\infty(\mathbb{R}^{2n}/M)$, it follows that $\mathcal{A} = L^\infty(\mathbb{R}^{2n}/M)$. But $L^\infty(\mathbb{R}^{2n}/M)$ is a maximal inductive algebra by Remark 2.12 and Theorem 2.11. □

Theorem 6.3. *Let $\mathcal{A}$ be a maximal inductive algebra. Then there is a maximal isotropic subgroup $M \subset \mathbb{R}^{2n}$ such that for each $(x, y) \in M$, there exist elements $f_{(x,y)} \in \mathcal{A}$ with the following properties*:

(1) $\forall (x_1, y_1, z_1) \in G$, $\rho(x_1, y_1, z_1) f_{(x,y)} \rho(x_1, y_1, z_1)^{-1} = e((x, y), (x_1, y_1)) f_{(x,y)}$.
(2) $\mathcal{A} = \overline{\operatorname{span}\{f_{(x,y)} | (x, y) \in M\}}$.

Moreover, $\mathcal{A}$ is a system of imprimitivity.

Proof. Let $\mathcal{B} = \mathcal{A}^\perp \subseteq S^1(\mathcal{H})$. Let $X \in \mathcal{B}$. If $(x_1, y_1) \in \mathbb{R}^{2n}$ and $T \in \mathcal{A}$, then

$$\begin{aligned} \operatorname{tr}(T((x_1, y_1) \cdot X)) &= \operatorname{tr}(T\rho(x_1, y_1, 1)X\rho(x_1, y_1, 1)^{-1}) \\ &= \operatorname{tr}(\rho(x_1, y_1, 1)^{-1}T\rho(x_1, y_1, 1)X) \\ &= 0 \end{aligned}$$

because $\rho(x_1, y_1, 1)^{-1}T\rho(x_1, y_1, 1) \in \mathcal{A}$. It follows that if $q \in L^1(\mathbb{R}^{2n})$ and $T \in \mathcal{A}$,

$$\operatorname{tr}(T(q \cdot X)) = 0.$$

Therefore, $\mathcal{B}$ is a submodule.

Set $M = V(\mathcal{B})$. Since $\mathcal{A}$ is weakly closed, $\mathcal{A} = \mathcal{B}^\perp$. In particular, $\mathcal{B}^\perp$ is an abelian subalgebra. If $(x, y), (x', y') \in M$, then for all $X \in \mathcal{B}$,

$$\alpha(X)(x, y) = 0 \quad \text{and} \quad \alpha(X)(x', y') = 0.$$

That is,

$$\operatorname{tr}(\rho(x, y, 1)X) = 0 \quad \text{and} \quad \operatorname{tr}(\rho(x', y', 1)X) = 0.$$

Therefore, $\rho(x, y, 1), \rho(x', y', 1) \in \mathcal{B}^\perp$. Therefore, $\rho(x, y, 1)\rho(x', y', 1) \in \mathcal{B}^\perp$. Therefore, $\rho(x + x', y + y', 1) \in \mathcal{B}^\perp$. Therefore, for all $X \in \mathcal{B}$, $\alpha(x + x', y + y')(X) = 0$ and so $(x + x', y + y') \in M$. So M is a subgroup. Moreover, since $\mathcal{B}^\perp$ is abelian,

$$e((x, y), (x', y')) = \rho(x, y, 1)\rho(x', y', 1)\rho(x, y, 1)^{-1}\rho(x', y', 1)^{-1} = 1$$

So M is isotropic.

Let $f_{(x,y)} = \rho(x, y, 1)$. Then for $g = (x_1, y_1, z_1) \in G$,

$$\rho(g) f_{(x,y)} \rho(g)^{-1} = e((x, y), (x_1, y_1)) f_{(x,y)}.$$

Moreover $I(M)^\perp = \overline{\operatorname{span}\{f_{(x,y)} | (x, y) \in M\}}$. But, by Theorem 5.7, we also have $I(M) = \mathcal{B}$ and therefore $I(M)^\perp = \mathcal{A}$. Since the set $\{f_{(x,y)} | (x, y) \in M\}$ is $*$-closed, so is $\mathcal{A}$.

Now, if M is contained in a larger isotropic subgroup M', then $\overline{\operatorname{span}\{f_{(x,y)} | (x, y) \in M'\}}$ would be an inductive algebra containing $\mathcal{A}$. Since $\mathcal{A}$ is maximal, it follows that $M' = M$. So M is maximal. □

References

[1] Dixmier Jacques, Von Neumann Algebras (Amsterdam, New York, Oxford: North-Holland Publishing Company) (1981)
[2] Mumford David, Nori Madhav and Norman Peter, Tata Lectures on Theta III (Boston: Birkhäuser) (1991)
[3] Parthasarathy K R, Probability measures on metric spaces (New York and London: Academic Press) (1967)
[4] Reed M and Simon B, Methods of Modern Mathematical Physics I: Functional Analysis (New York: Academic Press) (1972)
[5] Rudin Walter, Fourier Analysis on Groups, Interscience Publishers, a division of John Wiley & Sons, New York (1962)
[6] Rudin Walter, Functional Analysis (New York: McGraw-Hill, Inc.) (1991)
[7] Vemuri M K, A non-commutative Sobolev inequality and its application to spectral synthesis, preprint