Proc. Indian Acad. Sci. (Math. Sci.) Vol. 118, No. 1, February 2008, pp. 81–98.

Torsionfree sheaves over a nodal curve of arithmetic genus one

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MS received 20 July 2006; revised 6 September 2006

Abstract. We classify all isomorphism classes of stable torsionfree sheaves on an irreducible nodal curve of arithmetic genus one defined over $\mathbb{C}$. Let X be a nodal curve of arithmetic genus one defined over $\mathbb{R}$, with exactly one node, such that X does not have any real points apart from the node. We classify all isomorphism classes of stable real algebraic torsionfree sheaves over X of even rank. We also classify all isomorphism classes of real algebraic torsionfree sheaves over X of rank one.

Keywords. Torsionfree sheaves; nodal curve; stability.

1. Introduction

The isomorphism classes of algebraic vector bundles over a smooth elliptic curve defined over $\mathbb{C}$ were classified by Atiyah [At]. His classification extends to vector bundles over a smooth curve of genus one defined over $\mathbb{R}$ which admits a real point. In [BB], stable vector bundles over a Klein bottle were classified. A Klein bottle is a smooth curve of genus one defined over $\mathbb{R}$ which does not have any real points.

Our aim here is to consider stable vector bundles over a singular curve of genus one defined over $\mathbb{R}$ or $\mathbb{C}$.

Let Y be an irreducible nodal curve of arithmetic genus one defined over $\mathbb{C}$. Fix a positive integer n and an arbitrary integer d. Let $U_Y(n, d)$ denote the moduli space of semistable torsionfree sheaves of rank n and degree d on Y. Let $U^s_Y(n, d) \subset U_Y(n, d)$ be the open subvariety parametrizing the stable sheaves.

We prove the following:

Theorem 1.1.

(1) *If n and d are coprime, then the moduli space $U_Y(n, d)$ is isomorphic to Y.*
(2) *If n and d are not coprime, then the moduli space $U^s_Y(n, d)$ is empty.*

Let X be a geometrically irreducible nodal curve of arithmetic genus one defined over $\mathbb{R}$. We assume that X does not have any other real points apart from the node. Let $U_X(n, d)$ denote the moduli space of semistable torsionfree sheaves of rank n and degree d on X, and let $U'_X(n, d) \subset U_X(n, d)$ be the open subvariety parametrizing the locally free sheaves.

Let σ be the fixed point free antiholomorphic involution on $\mathbb{P}^1_{\mathbb{C}}$ defined by $\sigma(x : y) = (\bar{y} : -\bar{x})$. The pair $(\mathbb{P}^1_{\mathbb{C}}, \sigma)$ gives a nondegenerate anisotropic conic C defined over $\mathbb{R}$. Fix a point $x_0 \in \mathbb{P}^1_{\mathbb{C}}$. Identifying x_0 with $\sigma(x_0)$ we have a complex nodal curve Y of arithmetic genus 1. The involution σ induces an antiholomorphic involution on Y which we again denote by σ. Then the pair (Y, σ) gives an algebraic curve X of arithmetic genus one, defined over $\mathbb{R}$, with a single node, and $X \times_{\mathbb{R}} \mathbb{C} = Y$. Let $\sigma_{n,d}$ be the antiholomorphic involution of $U_Y(n, d)$ defined by $E \longmapsto \sigma^* \bar{E}$.

We have the following description of rank one torsionfree sheaves (note that any rank one torsionfree sheaf is stable):

PROPOSITION 1.2

Take integers d and d'.

(1) *The pair $(U_Y(1, d), \sigma_{1,d})$ is isomorphic to the pair $(U_Y(1, d + 2d'), \sigma_{1,d+2d'})$.*
(2) *The set of real points of $U_Y(1, 2d + 1)$ is a singleton. This point corresponds to the real torsionfree sheaf $\pi_*(T_C^{\otimes d})$ on X, where T_C is the tangent line bundle of C and π is the quotient map $C \longrightarrow X$.*
(3) *We have*

$$U_X(1, 2d) = U'_X(1, 2d) \cong S^1.$$

The set of real points of $U_Y(1, 2d)$ is the disjoint union

$$U'_X(1, 2d) \coprod \{p\},$$

where $\{p\} \in U_Y(1, 2d)$ corresponds to the non-real nonlocally free torsionfree sheaf $\widehat{\pi}_* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2d - 1)$ on Y.

It follows from the above proposition that there are no real vector bundles of odd degree on X. We show that there are no stable real vector bundles of rank r and degree d on X if the greatest common divisor of r and d is greater than 2. When $\gcd(r, d) = 2$, we prove the following theorem.

Theorem 1.3. *Let $r := 2r'$ and $d := 2d'$ be integers such that r' is a positive integer coprime to d'. Let*

$$\mathcal{U} \subset U_Y(r', d')$$

be the subset defined by all stable torsionfree sheaves which are not of the form $F \otimes_{\mathbb{R}} \mathbb{C}$, where F is some real algebraic torsionfree sheaf over X. Then the following two hold:

(1) *The set of isomorphism classes of stable real algebraic torsionfree sheaves over X of rank r and degree d is canonically identified with the quotient space $\mathcal{U}/(\mathbb{Z}/2\mathbb{Z})$ for the involution of $U_Y(r', d')$ defined by $W \longrightarrow \sigma^* \bar{W}$.*
(2) *Moreover, if d' is odd, then $\mathcal{U} = U_Y(r', d')$.*

2. Stable sheaves on a complex nodal curve of genus one

2.1 *Notations*

Let Y be a reduced irreducible projective curve of arithmetic genus one, with one ordinary node, defined over $\mathbb{C}$ (or an algebraically closed field of characteristic zero). Let I_k denote the trivial algebraic vector bundle over Y of rank k. For a torsionfree coherent sheaf E on Y, let $r(E)$ and $d(E)$ denote respectively the rank of E and the degree of E. The slope $d(E)/r(E)$ will be denoted by $\mu(E)$. For torsionfree sheaves E and F, let $Hom(E, F)$ denote the torsionfree sheaf of homomorphisms, and let $\mathrm{Hom}(E, F) := H^0(X, Hom(E, F))$ denote the space of all global homomorphisms from E to F. Similarly, $\mathrm{Ext}^i(E, F)$ and $Ext^i(E, F)$ will denote the Ext group and the Ext sheaf respectively. Define $E^* := Hom(E, \mathcal{O}_X)$ and $E^{**} := (E^*)^*$.

Let $y \in Y$ be the node. Let A and m be the local ring and the maximal ideal respectively at y. The stalk E_y of a torsionfree sheaf E at the node y is isomorphic to $a(E)A \oplus b(E)m$ where $a(E)$ and $b(E)$ are nonnegative integers with $a(E) + b(E) = r(E)$ (Proposition 2, p. 164 of [Se2]). We will say that E is of *local type* $a(E)A \oplus b(E)m$.

For any irreducible complex projective curve Z, we shall denote by $U_Z(n, d)$ the moduli space of semistable torsionfree sheaves of rank n and degree d on Z. The open subvariety of $U_Z(n, d)$ corresponding to the locally free sheaves will be denoted by $U'_Z(n, d)$. We will denote by $U^s_Z(n, d)$ the open subvariety of $U_Z(n, d)$ corresponding to the stable sheaves. We will employ the following convention: The superscript s will always denote the subset corresponding to stable sheaves. For a nonnegative integer p, denote by $U^s_p(n, d)$ the subset of $U^s_Y(n, d)$ consisting of stable sheaves E with $a(E) = p$. Therefore, $U^s_Y(n, d)$ is a disjoint union of $U^s_p(n, d)$ with $p = 0, \ldots, n$.

2.2 *Indecomposable torsionfree sheaves*

Atiyah [At], in his pioneering work had determined indecomposable vector bundles (and hence all vector bundles) on an elliptic curve. On an elliptic curve, it follows from Serre duality that $\mathrm{Ext}^1(L_2, L_1)$ is nonzero if and only if there is a nontrivial homomorphism from L_1 to L_2. As a consequence of this, the Harder–Narasimhan filtration of a non-semistable vector bundle on an elliptic curve splits into a direct sum of vector bundles. Therefore indecomposable vector bundles on an elliptic curve are semistable. However, on the nodal curve Y there exist non-semistable indecomposable vector bundles as shown by the following example.

If L_1 and L_2 are torsionfree sheaves on Y which are not locally free, then the group $\mathrm{Ext}^1(L_2, L_1)$ is nonzero (see Lemma 2.5 of [B2]).

Lemma 2.1. *Let L_1 and L_2 be rank one torsionfree sheaves over Y, which are not locally free, with $d(L_1) = 2$ and $d(L_2) = 0$. Let E be a torsionfree sheaf given by a nontrivial extension*

$$0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_2 \longrightarrow 0. \tag{2.1}$$

Then E is indecomposable, but E is not semistable.

Proof. As $\mathrm{Ext}^1(L_2, L_1) \neq 0$ (Lemma 2.5 of [B2]), there is a nontrivial extension of L_2 by L_1. Since $d(L_1) > \mu(E) = 1$, the vector bundle E is not semistable. Suppose that E is decomposable. Let $E = N_1 \oplus N_2$ with $d(N_2) \leq d(N_1) \geq 1$.

Since both L_1 and L_2 have degrees less than three, for any $N \subset E$ we have $d(N) \leq 2$. If $d(N_1) = 2$, then $N_1 \cong L_1$ giving a splitting of the exact sequence (2.1). If $d(N_2) = 1$, then $d(N_1) = 1$, and hence neither N_1 or N_2 have nonzero homomorphisms to L_2. Thus both N_1 and N_2 are contained in L_1, contradicting the assumption that $E = N_1 \oplus N_2$. Therefore, E is indecomposable. □

2.3 *Classification of stable torsionfree sheaves*

A key lemma (p. 422, Lemma 6′ of [At]) of Atiyah says that every indecomposable vector bundle E over a complex elliptic curve C admitting a section has a filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{r(E)-1} \subset F_{r(E)} = E$$

such that each successive quotient $L_i := F_i/F_{i-1}$, $1 \leq i \leq r(E)$, is a line bundle with $L_i \supset L_1 \supset \mathcal{O}_C$, and furthermore, L_1 is a maximal degree line subbundle. This lemma (even after replacing line bundles by rank one torsionfree sheaves) and rest of Atiyah's proofs are false for the nodal curve Y. The reason for this is that for torsionfree sheaves L_1 and L_2 on Y which are not locally free, the group $\mathrm{Ext}^1(L_2, L_1)$ is always nonzero (see Lemma 2.5 of [B2]). Hence the method of Atiyah [At] and Tu [Tu] to classify stable vector bundles on an elliptic curve fails in case of a nodal curve of arithmetic genus one. Our strategy is to use a degeneration method coupled with results of Atiyah and Tu on elliptic curves to determine $U_Y(n, d)$ and $U_Y^s(n, d)$.

Lemma 2.2. *If n and d are noncoprime, then there is no stable torsionfree sheaf of rank n and degree d over Y.*

Proof. Let $\varphi\colon Z \longrightarrow T$ be a flat family of irreducible complex projective curves of arithmetic genus 1 parametrized by a smooth curve T, and let $t_0 \in T$ be a base point such that for all $t \neq t_0$, the fiber Z_t is an elliptic curve and $Z_{t_0} \cong Y$. For any point $t \in T$, the fiber $\varphi^{-1}(t)$ will be denoted by Z_t. Let $M \longrightarrow T$ be the relative moduli variety flat over T such that $M_t \cong U_{Z_t}(n, d)$ for all $t \in T$; see [Se1] for the existence of M. In particular $M_{t_0} \cong U_Y(n, d)$.

Let $M^s \subset M$ denote the subset corresponding to stable torsionfree sheaves. By openness of the stability condition, M^s is an open subset of M (p. 635, Theorem 2.8 of [Ma]). Assume that there is a stable sheaf on Y of rank n and degree d. Then $M^s \cap U_{Z_{t_0}}(n, d)$ is nonempty, hence M^s is nonempty. Since M is irreducible [Re], $M^s \cap (M \setminus M_{t_0})$ is a nonempty open subset of M. Consequently, M_t is nonempty for some $t \neq t_0$. This contradicts the fact that there are no stable vector bundles of rank n and degree d, $(n, d) \neq 1$, on an elliptic curve (Fact, p. 20 of [Tu]). This completes the proof of the lemma. □

Remark 2.3. For a vector bundle $E \in U_X^s(n, d)$ one has $h^0(End(E)) = 1$. Hence $\dim \mathrm{Ext}^1(E, E) = h^1(End(E)) = 1$. On the other hand, if $E \in U_X^s(n, d)$ is not locally free, then $\dim \mathrm{Ext}^1(E, E) \geq \dim \mathrm{Ext}^1(E_y, E_y) \geq 2$ (Lemma 2.5 of [B2]). For any $E \in U_X^s(n, d)$, since $\mathrm{Ext}^1(E, E)$ gives the Zariski tangent space at E, and the moduli space $U_X^s(n, d)$ is irreducible, we conclude the following. A vector bundle $E \in U_X^s(n, d)$ corresponds to a nonsingular point of $U_X^s(n, d)$, and a nonlocally free sheaf $E \in U_X^s(n, d)$ corresponds to a singular point of $U_X^s(n, d)$.

Lemma 2.4. *Let $\widetilde{U_Y(n, d)}$ be the normalization of $U_Y(n, d)$. For any singular point $F \in U_{n-1}(n, d)$, there are exactly two points of $\widetilde{U_Y(n, d)}$ over F.*

Proof. The normalization $\widetilde{U_Y(n, d)}$ of $U_Y(n, d)$ is the moduli space $P(n, d)$ of semistable generalized parabolic bundles (GPBs, in short) on the normalization $\mathbb{P}^1_{\mathbb{C}}$ of Y [Su]; see [B1] for generalized parabolic bundles. Let (E, V) be a GPB, where E is a vector bundle of rank n and degree d over $\mathbb{P}^1_{\mathbb{C}}$, and $V \subset E_x \oplus E_z$ is a vector subspace of dimension n.

Let x and z be the two points in the normalization $\mathbb{P}^1_{\mathbb{C}}$ of Y that lie over the node y. Let p_x and p_z be the projections of V to E_x and E_z respectively. There are divisors D_x and D_z on $P(n, d)$ defined by

$$D_x := \{(E, V) | \text{rank } \ p_x = n, \ \ \text{rank } \ p_z < n\}$$

and

$$D_z := \{(E, V) | \text{rank } \ p_z = n, \ \ \text{rank } \ p_x < n\}.$$

Both D_x and D_z are normalizations of the complement $U_Y(n, d) \backslash U'_Y(n, d)$; the normalization maps are isomorphisms over the subset $U_{n-1}(n, d) \subset U_Y(n, d) \backslash U'_Y(n, d)$ [Su]. Hence for any $F \in U_{n-1}(n, d)$, there are exactly two points in $P(n, d)$ lying over F, one in D_x and one in D_z. □

Theorem 2.5. *Let Y and $U_Y(n, d)$ be as above.*

(1) *For $(n, d) = 1$, the moduli space $U_Y(n, d)$ is isomorphic to Y.*
(2) *For $(n, d) > 1$, the moduli space $U^s_Y(n, d)$ is empty.*

Proof. Part (2) of the theorem is a restatement of Lemma 2.2, we need to prove only part (1).

First assume that $(n, d) = 1$. Let $Z \longrightarrow T$ be a flat family of irreducible projective curves of arithmetic genus one parametrized by a smooth irreducible curve T such that there is a point $t_0 \in T$ with the following property: for $t \neq t_0$, the fiber Z_t is an elliptic curve, and $Z_{t_0} = Y$. Let $M \rightarrow T$ be the relative moduli variety as in the proof of Lemma 2.2. It follows from the work of Atiyah [At], that $M_t \cong Z_t$ for all $t \neq t_0$. Hence M_{t_0} is an irreducible projective curve of arithmetic genus 1.

By Remark 2.3, the nonsingular points of M_{t_0} correspond to vector bundles over Y. By (Proposition 2.7 of [B2]), if $U^s_p(n, d)$ is nonempty, then

$$\dim \ U^s_p(n, d) \leq n^2(g-1) + 1 - (n-p)^2 = p + 1 - n.$$

Hence for $n - p \geq 2$, we have $\dim U^s_p(n, d) < 0$. This proves that the variety $U_Y(n, d)$ consists of locally free sheaves and sheaves of local type $(n-1)A \oplus m$.

Therefore, there exists a (nonzero) determinant morphism det: $U_Y(n, d) \longrightarrow U_Y(1, d)$ [B1]. This morphism sends vector bundles to line bundles, and it sends nonlocally free sheaves to rank one nonlocally free sheaves. Since $U_Y(n, d)$ is projective, and $U_Y(1, d)$ is an irreducible variety of dimension one, we conclude that $\det(U_Y(n, d)) = U_Y(1, d)$. If $U_Y(n, d)$ consisted of vector bundles only, then $\det(U_Y(n, d)) \subset U'_Y(1, d)$, which is a contradiction. This proves that there exists a stable torsionfree sheaf of rank n and degree d on Y of local type $(n-1)A \oplus m$. Therefore, by Remark 2.3, the moduli space M_{t_0} is singular.

If $\widetilde{M}_{t_0}$ is the normalization of M_{t_0}, then

$$p_a(\widetilde{M}_{t_0}) = p_a(M_{t_0}) - \sum_{i=1}^{r} \frac{e_i(e_i - 1)}{2},$$

where $p_a \geq 0$ denotes the arithmetic genus and $e_i \geq 0$ are the multiplicities of the infinitesimally near points of the singular points of M_{t_0} (see Chapter V, Corollary 3.7 and Proposition 3.8 in pp. 389–390 of [Ha]). Since $p_a(M_{t_0}) = 1$, it follows that $p_a(\widetilde{M}_{t_0}) = 0$, $r = 1$ and $e_i = 2$. Thus the desingularization of M_{t_0} is $\mathbb{P}^1_{\mathbb{C}}$ and M_{t_0} has a unique ordinary double point, i.e., a node or a cusp. Lemma 2.4 implies that the unique singular point is a node. Hence $M_{t_0} \cong Y$. □

Remark 2.6. Recently we have learnt that vector bundles on the nodal curve Y have been studied by many authors by different methods like matrix methods, Fourier–Mukai transforms, and using line bundles on étale coverings of Y [BBDG], [FM], [Bu]. In these works, explicit descriptions of indecomposable vector bundles have been obtained. Our Lemma 2.2 was proved by these methods (cf. Theorem 21 of [BBDG] for comparison). The fact that for (n, d) coprime, the moduli space $U'_Y(n, d)$ is isomorphic to the affine line can be proved by many methods. But it has not been shown that $U_Y(n, d)$ is isomorphic to Y.

3. Torsionfree sheaves on a real nodal curve of genus one

3.1 *Notation*

Let σ be the fixed point free antiholomorphic involution on $\mathbb{P}^1_{\mathbb{C}}$ defined by

$$\sigma(x : y) = (\bar{y} : -\bar{x}).$$

The pair $(\mathbb{P}^1_{\mathbb{C}}, \sigma)$ gives a nondegenerate anisotropic conic C defined over the field of real numbers such that $C \times_{\mathbb{R}} \mathbb{C} \cong \mathbb{P}^1_{\mathbb{C}}$. Fix a point $x_0 \in \mathbb{P}^1_{\mathbb{C}}$. Identifying x_0 and $\sigma(x_0)$ gives a complex nodal curve Y of arithmetic genus 1. The involution σ induces an antiholomorphic involution on Y which we again denote by σ. Then the pair (Y, σ) gives an algebraic curve X of arithmetic genus one, defined over $\mathbb{R}$, with a single node, and one has $X \times_{\mathbb{R}} \mathbb{C} = Y$.

The desingularization of X is the anisotropic conic C; the conic C has no real points. The only real point of X is the node $y \in X$. Let

$$\pi : C \longrightarrow X \tag{3.1}$$

be the normalization map.

3.2 *Torsionfree sheaves on X*

Our aim is to study algebraic torsionfree sheaves on X. We shall do this by relating them to those on Y and C.

Lemma 3.1. *Let E_1 and E_2 be two real algebraic torsionfree sheaves over X. Let $V_i := E_i \otimes_{\mathbb{R}} \mathbb{C}$, $i = 1, 2$, be the corresponding complex algebraic torsionfree sheaves over Y. If the two sheaves V_1 and V_2 are isomorphic, then E_1 is isomorphic to E_2.*

Proof. This can be proved exactly as done for Lemma 2.1 of [BB] by changing $V_1^* \otimes V_2$ in the proof to $Hom(V_1, V_2)$. □

DEFINITION 3.2

A real torsionfree sheaf E over X is called *stable* (respectively, *semistable*) if for all real subsheaves $F \subset E$ with $0 < \text{rank}(F) < \text{rank}(E)$, the inequality

$$\frac{\text{degree}(F)}{\text{rank}(F)} < \frac{\text{degree}(E)}{\text{rank}(E)}$$

(respectively, $\frac{\text{degree}(F)}{\text{rank}(F)} \leq \frac{\text{degree}(E)}{\text{rank}(E)}$) holds.

A semistable torsionfree sheaf over X is called *polystable* if it is a direct sum of stable torsionfree sheaves.

We denote by $U_X(n,d)$ the moduli space of real algebraic semistable torsionfree sheaves of rank n and degree d on X, and denote by $U'_X(n,d) \subset U_X(n,d)$ the open subvariety corresponding to locally free sheaves. As before, $U^s_X(n,d) \subset U_X(n,d)$ will be the subvariety defined by stable sheaves.

3.3 *The involution on $U_Y(n,d)$ induced by σ*

Let E be a complex algebraic vector bundle over Y. Let $\bar{E}$ be the C^∞ complex vector bundle over Y whose underlying real vector bundle is same as that of E, but the complex structure of each fiber of $\bar{E}$ is the conjugate of the complex structure of the fibers of E. The vector bundle $\bar{E}$ does not have a natural holomorphic structure. However, the vector bundle $\sigma^*\bar{E}$ has a natural holomorphic – hence algebraic – structure. It is easy to see that $\sigma^*\bar{E}$ is the complex algebraic vector bundle given by the complex algebraic vector bundle E using the automorphism of the field $\mathbb{C}$ defined by $z \longmapsto \bar{z}$. If $\{U_i\}_i$ is an open cover of Y, then $\{\sigma(U_i)\}_i$ is also an open cover of Y. If g_{U_i,U_j} are the transition functions of E on the open set $U_i \cap U_j$, then for the open cover $\{\sigma(U_i)\}_i$, the transition functions $f_{\sigma(U_i),\sigma(U_j)}$ of $\sigma^*\bar{E}$ are given by $f_{\sigma(U_i),\sigma(U_j)}(t) = \overline{g_{U_i,U_j}(\sigma^{-1}(t))}$ for $t \in \sigma(U_i) \cap \sigma(U_j)$.

A torsionfree coherent sheaf E on Y is locally free except possibly at the node. Let Y' be the nonsingular subset of Y; so Y' is the complement of the node. As explained above, $\sigma^*\bar{E}|_{Y'}$ is an algebraic vector bundle. Locally in a neighborhood U of the node, there is an injection

$$i\colon E \hookrightarrow U \times \mathbb{C}^r$$

for some r. On the image of i, take the complex structure induced from $U \times \bar{\mathbb{C}}^r$. As i is holomorphic, this complex structure $\bar{E}$ is independent of the choice of the injection i. This shows that $\sigma^*\bar{E}$ is a complex algebraic sheaf on Y. As before, $\sigma^*\bar{E}$ is the coherent sheaf given by the coherent sheaf E using the automorphism of the field $\mathbb{C}$ defined by $z \longmapsto \bar{z}$.

Let V be a real algebraic torsionfree sheaf over X. Let

$$V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$$

be the corresponding complex algebraic torsionfree sheaf over Y. The involution σ lifts to an algebraic isomorphism

$$\delta\colon V_{\mathbb{C}} \longrightarrow \sigma^*\overline{V_{\mathbb{C}}} \tag{3.2}$$

such that the composition

$$V_{\mathbb{C}} \xrightarrow{\delta} \sigma^*\overline{V_{\mathbb{C}}} \xrightarrow{\sigma^*\bar{\delta}} \sigma^*\overline{\sigma^*\overline{V_{\mathbb{C}}}} = V_{\mathbb{C}} \tag{3.3}$$

is the identity map of $V_{\mathbb{C}}$. Note that since $\sigma^2 = \mathrm{Id}_Y$, and $\bar{\bar{F}} = F$ for any complex torsionfree sheaf F, it follows that $\sigma^*\overline{\sigma^*\overline{V_{\mathbb{C}}}}$ is canonically identified with $V_{\mathbb{C}}$. This also follows from the fact that the automorphism of the field $\mathbb{C}$ defined by $z \longmapsto \bar{z}$ is an involution.

Remark 3.3.

(1) A torsionfree sheaf E on Y descends to X if and only if there exists an isomorphism

$$\delta\colon E \longrightarrow \sigma^* \bar{E}$$

such that the composition $\sigma^*\bar{\delta}\circ\delta = \mathrm{Id}_E$. If E descends to X and $F \subset E$ is a subsheaf, then F descends to X if and only if $\delta(F) = \sigma^*\bar{F} \subset \sigma^*\bar{E}$.

(2) The self-map of $U_Y(n, d)$ defined by

$$\sigma_{n,d}\colon E \longmapsto \sigma^*\bar{E} \tag{3.4}$$

is an antiholomorphic involution. The real points of $U_Y(n, d)$ correspond to E such that $\sigma^*\bar{E} \cong E$.

Take any complex algebraic torsionfree sheaf E of rank r over Y. Consider the algebraic torsionfree sheaf

$$S(E) := E \oplus \sigma^*\bar{E} \tag{3.5}$$

of rank $2r$ over Y. For $S(E)$ we have

$$\sigma^*\overline{S(E)} = \sigma^*\overline{E \oplus \sigma^*\bar{E}} = \sigma^*\bar{E} \oplus E.$$

Therefore, there is a canonical isomorphism

$$\sigma_E\colon S(E) \longrightarrow \sigma^*\overline{S(E)} \tag{3.6}$$

defined by $(v_1, v_2) \longmapsto (v_2, v_1)$.

It is easy to check that the composition $(\sigma^*\overline{\sigma_E}) \circ \sigma_E$ is the identity automorphism of $S(E)$. Therefore, the pair $(S(E), \sigma_E)$ gives a real algebraic torsionfree sheaf $V(E)$ over X of rank r. We shall call $V(E)$ the torsionfree sheaf on X defined by $E \oplus \sigma^*\bar{E}$ on Y.

Lemma 3.4. *Let E be a real torsionfree sheaf over X. Let $E_{\mathbb{C}} = E \otimes_{\mathbb{R}} \mathbb{C}$ be the corresponding torsionfree sheaf over Y.*

(1) *The torsionfree sheaf E is semistable if and only if $E_{\mathbb{C}}$ over Y is semistable.*
(2) *The torsionfree sheaf E is polystable if and only if $E_{\mathbb{C}}$ is polystable.*
(3) *There is a real algebraic morphism $U_X(n, d) \to U_Y(n, d)$ whose image is contained in the set of real points of $U_Y(n, d)$. (Here the complex variety $U_Y(n, d)$ is considered as a real variety using the inclusion of $\mathbb{R}$ in $\mathbb{C}$.)*
(4) *If $(n, d) = 1$, then $U_X(n, d)$ is contained in the set of real points of $U_Y(n, d)$.*

Proof. Statements (1) and (2) can be proved exactly as in Lemma 4.1 of [BB] using the invariance of Harder–Narasimhan filtration and the Socle respectively (see p. 18, Corollary 1.3.8 of [HL]).

To prove (3), in view of (1), there is a morphism

$$U_X(n, d) \longmapsto U_Y(n, d)$$

defined by

$$E \mapsto E_{\mathbb{C}}.$$

Clearly, the image of this morphism is contained in the real points of $U_Y(n, d)$.

To prove (4), take coprime integers n and d. A semistable torsionfree sheaf of rank n and degree d is stable. We know that two stable torsionfree sheaves define same point of the moduli space if and only if they are isomorphic. The result now follows from statement (3) and Lemma 3.1. □

We remark that, unlike the case smooth curves of genus one, a torsionfree sheaf E over X may not split into a direct sum of semistable torsionfree sheaves. Examples similar to that in Lemma 2.1 can be constructed.

4. Stable torsionfree sheaves on *X*

We continue with the notation of the previous section.

We first determine $U_X(1, d)$ and the real points of $U_Y(1, d)$ for all integers d.

PROPOSITION 4.1

Take integers d and d'.

(1) *The pair $(U_Y(1, d), \sigma_{1,d})$ is isomorphic to the pair $(U_Y(1, d + 2d'), \sigma_{1,d+2d'})$.*
(2) *The set of real points of $U_Y(1, 2d + 1)$ is a singleton. This point corresponds to the real torsionfree sheaf $\pi_*(T_C^{\otimes d})$ on X, where T_C is the tangent line bundle of C, and π is the projection in* (3.1).
(3) *We have*

$$U_X(1, 2d) = U'_X(1, 2d) \cong S^1.$$

The set of real points of $U_Y(1, 2d)$ is the disjoint union

$$U'_X(1, 2d) \coprod \{p\},$$

where $\{p\} \in U_Y(1, 2d)$ corresponds to the non-real nonlocally free torsionfree sheaf $\widehat{\pi}_*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2d - 1)$ on Y; here $\widehat{\pi}$ is the quotient map $\mathbb{P}^1_{\mathbb{C}} \longrightarrow Y$.

Proof. To prove (1), consider a divisor $D = x + \sigma(x)$, where $x \in Y$. One has

$$\sigma^*\overline{\mathcal{O}_Y(D)} = \mathcal{O}_Y(\sigma(D)) = \mathcal{O}_Y(D).$$

Furthermore the tautological isomorphism $\delta\colon \mathcal{O}_Y(D) \longrightarrow \sigma^*\overline{\mathcal{O}_Y(D)}$ satisfies the condition that $(\sigma^*\overline{\delta}) \circ \delta = \mathrm{Id}_{\mathcal{O}_Y(D)}$. Therefore the line bundle $\mathcal{O}_Y(D)$ over Y corresponds to a real algebraic line bundle over X. The map

$$L \longmapsto L \otimes \mathcal{O}_Y(d'D)$$

defines the required isomorphism in statement (1).

To prove (2) we need to show that the involution $\sigma_{1,2d+1}$ of $U_Y(1, 2d+1)$ has a unique fixed point. In view of statement (1), it suffices to show that $\sigma_{1,-1}$ on $U_Y(1, -1)$ has a unique fixed point. Note that $(U_Y(1, -1), \sigma_{1,-1})$ is canonically identified with (Y, σ) by sending any point $x \in Y$ to the maximal ideal in $\mathcal{O}_Y$ for x. Since the node is the only fixed point of the involution σ of Y, it follows that the only fixed point for the involution $\sigma_{1,-1}$ is $\widehat{\pi}_* K_{\mathbb{P}^1_{\mathbb{C}}}$ (the map $\widehat{\pi}$ is defined in statement (4) of the proposition). This sheaf $\widehat{\pi}_* K_{\mathbb{P}^1_{\mathbb{C}}}$ descends to X, and the descended sheaf is identified with the real torsionfree sheaf $\pi_* K_C$. Hence the unique real point of $U_Y(1, 2d+1)$ descends to X as $\pi_* L$, where L is a line bundle of degree $2d$ on C. The only line bundle on C of degree $2d$ is $T_C^{\otimes d}$ (see [BN]), hence statement (2) follows.

To prove (3), the variety $U_Y(1, 2d)$ has a unique point corresponding to a nonlocally free sheaf, viz. that corresponding to $\widehat{\pi}_* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2d-1)$. Since $\sigma_{1,2d}$ maps nonlocally free sheaves to nonlocally free sheaves, this point must be a real point. If the corresponding sheaf is real then it comes from a real torsionfree sheaf L on X. Then $L = \pi_* N$ with N being a real line bundle on C of degree $2d-1$. Since C has no real line bundles of odd degree [BN], it follows that X has no real nonlocally free torsionfree sheaves of even degree. Thus $U_X(1, 2d) = U'_X(1, 2d)$.

Statement (1) in the proposition says that $U'_X(1, 2d) \cong U'_X(1, 0)$. A line bundle on Y of degree 0 is obtained by identifying the fiber of the trivial line bundle $L = \mathbb{P}^1_{\mathbb{C}} \times \mathbb{C}$ over the point y_1 with the fiber of L over y_2, where y_1 and y_2 are the two points lying over the node y. For any $\lambda \in \mathbb{C}^*$, let L_λ denote the line bundle on Y of degree 0 obtained by identifying the fiber $L_{y_1} = \mathbb{C}$ with $L_{y_2} = \mathbb{C}$ using multiplication with λ.

One has an isomorphism

$$\delta\colon L \longrightarrow \sigma^* \bar{L},$$

defined by

$$(y_1, c) \longmapsto (y_2, \bar{c}), \quad c \in \mathbb{C},$$

where $\bar{c}$ is the complex conjugate of c. Clearly, $\sigma^* \bar{\delta} \circ \delta = \mathrm{Id}_L$. Then δ induces a morphism

$$\delta'\colon L_\lambda \longrightarrow \sigma^* \bar{L}_\lambda,$$

where L_λ is the line bundle over Y defined above, if and only if

$$\lambda \circ \delta_{y_2} \circ \lambda = \delta_{y_1}.$$

The last equation is equivalent to $\lambda\bar{\lambda} = 1$, i.e., $\lambda \in S^1$. Note that $\sigma^* \bar{\delta}' \circ \delta' = \mathrm{Id}_{L_\lambda}$ as $\sigma^* \bar{\delta} \circ \delta = \mathrm{Id}_L$. The statement (3) now follows as the morphism $\mathbb{C}^* \to U'(1, 0)$ defined by $\lambda \to L_\lambda$ is an isomorphism ([B1], [Se2]). □

Henceforth we will assume that $n \geq 2$.

PROPOSITION 4.2

Let n and d be integers with $n \geq 2$.

(1) *There are no real vector bundles of odd degree on X.*
(2) *If $(n, d) = 1$ with n and d both odd, then there exists a unique real point of $U_Y(n, d)$. The unique point corresponds to the unique stable nonlocally free torsionfree sheaf on X of rank n and degree d.*

(3) *If* $(n, d) = 1$ *with d even, then*

$$U_X(n, d) = U'_X(n, d),$$

i.e., there is no stable real nonlocally free torsionfree sheaf on X.

(4) *For* $d = 1$ *with n arbitrary, there exists a unique stable real nonlocally free torsionfree sheaf of rank n and degree* 1 *on X.*

Proof. If E is a real vector bundle of degree d on X, then its determinant $\det E := \wedge^n(E)$ is a real line bundle of degree d. By Proposition 4.1(2), there are no real line bundles of odd degree on X. Hence statement (1) in the proposition follows.

To prove (2) first note that since d is odd, by Part (1), $U'_X(n, d)$ is empty. Theorem 2.5 says that $U_Y(n, d)$ has a unique point corresponding to a stable nonlocally free torsionfree sheaf E. This point, being unique, is invariant under $\sigma_{n,d}$. Hence it is a real point, proving statement (2).

To prove (3), Lemma 3.4 says that $U_X(n, d)$ is contained in the set of real points of $U_Y(n, d)$. As in the proof of statement (2), the point corresponding to the unique nonlocally free torsionfree sheaf E_0 is a real point. Hence there exists an involution

$$\delta\colon E_0 \longrightarrow \sigma^* \bar{E}_0.$$

Note that E_0 has local type $(n-1)A \oplus m$ and hence its determinant $\det E_0$ is a torsionfree sheaf of rank 1 and degree d. The above involution δ induces an involution $\det \delta$ on $\det E_0$. Hence $\det E_0$ is a real point of $U_Y(1, d)$. Moreover, if E_0 comes from a real torsionfree sheaf on X, then $\det E_0$ comes from a real torsionfree sheaf on X. If d is even, then by Proposition 4.1(3), there is no real nonlocally free torsionfree sheaf of rank 1 on X. It follows that E_0 does not come from a real nonlocally free torsionfree sheaf. Thus $U_X(n, d) = U'_X(n, d)$.

We shall prove statement (4) in the proposition by induction on n. For $n = 1$, the direct image $\pi_* \mathcal{O}_C \in U_X(1, 1)$ is the nonlocally free sheaf required. Suppose that E' is a stable nonlocally free torsionfree sheaf of rank $n - 1 \geq 1$ and degree 1. Let I denote the trivial line bundle. Since I is locally free, $\mathrm{Ext}^1(E', I) \cong H^1(E'^*)$ (Lemma 2.5(B) of [B2]). Since E'^* is a stable torsionfree sheaf of negative slope, we have $h^0(E'^*) = 0$ and $h^1(E') = 1$ by Riemann–Roch theorem. Hence there exists a unique nonsplit exact sequence

$$0 \to I \to E \to E' \to 0.$$

We shall prove that E is stable. Let $N \subset E$ be a saturated subsheaf of rank $r < n$ and degree d. Let N' be the image of N in E'. The rank and degree of N' will be denoted by r' and d' respectively. Let M be the kernel of the homomorphism $N \to N'$. Since $M \subset I$ and $N' \subset E'$, one has $d' \geq d$, and furthermore, either $r = r'$ or $r = r' + 1$. We note that if $r = r' + 1$, then $r' \leq n - 1$. If $d \leq 0$, then N does not contradict the stability condition of E, so we may, and we will, assume that $d \geq 1$. Then

$$d'/r' \geq d/r' \geq 1/r' \geq 1/(n-1).$$

This gives a contradiction to the stability condition of E' if $r' < n - 1$. Hence we must have $r' = n - 1$ and $d' \geq 1$. Then $d' = 1$ and $N' = E'$, giving a splitting of the extension which contradicts our assumption. Thus E is stable.

Let E' denote any stable real torsionfree sheaf of rank n and degree one which is not locally free. Then $E'_{\mathbb{C}}$ is polystable by Lemma 3.4. Since $E'_{\mathbb{C}}$ has degree 1, it then follows that $E'_{\mathbb{C}}$ is a stable torsionfree sheaf on Y which is not locally free and hence it is unique (up to an isomorphism). The uniqueness of E' now follows from Lemma 3.1. This completes the proof of the proposition. □

PROPOSITION 4.3

The moduli space $U'_X(2n+1, 2)$ is nonempty for all n.

Proof. The case of $n = 0$ follows from Proposition 4.1(3). We will assume that $n \geq 1$. Take any $E'' \in U'_X(2n-1, 2)$. Since E''^* is semistable of negative degree, we have $h^0(E''^*) = 0$ and hence by Riemann–Roch theorem, $h^1(E''^*) = 2$. Therefore there exists a nonsplit exact sequence

$$0 \to I_2 \to E \to E'' \to 0, \tag{4.1}$$

where I_2 is the trivial vector bundle of rank two. We shall show that a general extension of type (4.1) gives a stable vector bundle E.

Suppose that E is not semistable. Let N be a stable subsheaf of E with

$$\mu(N) > 2/(2n+1).$$

In particular, $d(N) := \text{degree}(N) \geq 1$. It follows that N cannot be contained in I_2. Hence the composite $N \hookrightarrow E \to E''$ is nonzero. Let N'' and N' denote respectively the image and kernel of this composition homomorphism. Let r', r, r'' and t', t, t'' be respectively the ranks and degrees of N', N, N''.

Case of $N'' = E''$. If $N' = 0$, then $N \cong N'' = E''$ giving a splitting of the exact sequence in (4.1). Hence $N' \neq 0$, which implies that $r(N') = 1$ and $t' \leq 0$ as $N' \subset I_2$. Then $r = 2n$, and

$$d(N) > \frac{4n}{2n+1} = 2 - \frac{2}{2n+1}.$$

Since $n \geq 1$, this implies that $t \geq 2$, and $t' = t - t'' \geq 0$. As $N' \subset I_2$, we then have $t' = 0$ and $N' = I_1 = \mathcal{O}_X$. Thus we have an exact sequence

$$0 \to I \to N \to E'' \to 0.$$

This means that the elements $e_1, e_2 \in H^1(E''^*)$ giving the extension in (4.1) are not linearly independent. Let $U_1 \subset \mathbb{P}(H^1(E''^* \otimes I_2))$ be the subset defined by

$$U_1 := \{(e_1 \colon e_2) \mid e_1, e_2 \in H^1(E''^*) \text{ are linearly independent}\} \subset \mathbb{P}(H^1(E''^* \otimes I_2)).$$

If we choose $e := (e_1 \colon e_2) \in U_1$, then the case $N'' = E''$ does not occur.

Case of $N'' \neq E''$. Using the fact that N'' is a quotient of the stable torsionfree sheaf N and a proper subsheaf of the stable vector bundle E'', we have

$$0 < \frac{2}{2n+1} < \frac{t}{r} \leq \frac{t''}{r''} < \frac{2}{2n-1}. \tag{4.2}$$

Since $r'' \leq 2n-1$, this implies that

$$0 < \frac{2r''}{2n+1} < t'' < \frac{2r''}{2n-1} \leq 2. \tag{4.3}$$

Thus $t'' = 1$, and $0 < t = t'' + t' \leq t'' = 1$ (recall that $N' \subset I_2$, hence $t' \leq 0$). Hence, we have $t = 1$. From (4.2) we have

$$0 < \frac{2}{2n+1} < \frac{1}{r} \leq \frac{1}{r''} < \frac{2}{2n-1},$$

so that

$$0 < \frac{2n-1}{2} < r'' \leq r < \frac{2n+1}{2}.$$

Hence

$$r = r'' = n, t = t'' = 1. \tag{4.4}$$

This implies that $N' = 0$ and $N \cong N''$. Note that $N \cong N''$ is the unique (up to an isomorphism) nonlocally free torsionfree sheaf of rank n and degree 1 (Proposition 4.2(4)). Set $Q := E''/N''$. We have an exact sequence

$$0 \to N \to E'' \to Q \to 0 \tag{4.5}$$

(recall that $N = N''$).

We claim that $\mathrm{Hom}(N, E'')$ is of dimension ≤ 2. Note that the tensor product of two stable torsionfree sheaves (or vector bundles) on a nodal curve may not be stable, hence we cannot prove the claim directly. To prove the above claim, we first check that Q is stable. Assume that Q is not stable. Let $Q' \subset Q$ be a subsheaf with $\mu(Q') \geq \mu(Q)$. Since $\mu(Q) = 1/(n-1)$, one has

$$d(Q') := \mathrm{degree}(Q') > d(Q) := \mathrm{degree}(Q) > 0,$$

i.e., $d(Q') \geq 2$. Let F be the inverse image of Q' in E''. Then $d(F) := \mathrm{degree}(F) = d(Q') + 1 \geq 2$ and $r(F) = m + r(Q') < 2n - 1$. Hence $\mu(F) > \frac{2}{2n-1} = \mu(E'')$ contradicting the stability condition of E''. Therefore, Q is stable.

Dualizing (4.5) and tensoring with the vector bundle E'' gives the short exact sequence

$$0 \to Q^* \otimes E'' \to E''^* \otimes E'' \to N^* \otimes E'' \to 0 \tag{4.6}$$

(Lemma 2.2(2) of [B2]). The long exact sequence of cohomologies associated to this short sequence is

$$0 \longrightarrow H^0(Q^* \otimes E'') \longrightarrow H^0(E''^* \otimes E'') \longrightarrow H^0(N^* \otimes E'') \longrightarrow H^1(Q^* \otimes E''). \tag{4.7}$$

Since E'' is locally free and Q^* is torsionfree, $Q^* \otimes E''$ is torsionfree and $\mathrm{Hom}\,(Q, E'') = H^0(Q^* \otimes E'')$. Since both Q and E'' are stable and $\mu(Q) > \mu(E'')$, one has $h^0(Q^* \otimes E'') = 0$ and $h^0(E''^* \otimes E'') = 1$. By Riemann–Roch theorem, $h^1(Q^* \otimes E'') = 1$. The exact sequence (4.7) shows that $\mathrm{Hom}(N, E'')$ is of dimension ≤ 2 so that the variety S of stable subsheaves $N'' \cong N$ of E'' of rank n and degree one has dimension at most one.

Since I_2 is locally free, one has $H^1(N^* \otimes I_2) = \mathrm{Ext}^1(N, I_2)$ (Lemma 2.5(B) of [B2]). A homomorphism $i\colon N \to E''$ lifts to a homomorphism $N \to E$ if and only if $e \in H^1(E''^* \otimes I_2)$ defining the extension (4.1) lies in the kernel of the linear map

$$f_i\colon H^1(E''^* \otimes I_2) \to H^1(N^* \otimes I_2).$$

The long exact sequence of cohomologies associated to the dual of the exact sequence in (4.5) gives

$$H^0(N^*) \to H^1(Q^*) \to H^1(E''^*) \to H^1(N^*).$$

Since N^* and Q^* are stable and of negative degree, $h^0(N^*) = 0$, $h^0(Q^*) = 0$ and $h^1(Q^*) = 1$. Hence the kernel K_i of the map $f_i\colon H^1(E''^* \otimes I_2) \to H^1(N^* \otimes I_2)$ is of dimension two. Let $P_N = \mathbb{P}(K_i) \subset \mathbb{P}(H^1(E''^* \otimes I_2))$, then P_N has dimension one. As N vary over S, the linear subspaces P_N sweep out a closed subset of $\mathbb{P}(H^1(E''^* \otimes I_2)) \cong \mathbb{P}^3_{\mathbb{R}}$ of dimension at most two (recall that the dimension of S is at most one). Let U_2 be the complement of this closed proper subset of $\mathbb{P}(H^1(E''^* \otimes I_2))$.

Choose any $e \in U_1 \cap U_2 \subset \mathbb{P}(H^1(E''^* \otimes I_2))$. Then the extension (4.1) gives a stable vector bundle E. Note that the group $\mathrm{GL}(2, \mathbb{R})$ acts on I_2 and hence on the sequence (4.1). It acts transitively on bases (e_1, e_2) of $H^1(E''^*)$. Hence all $e \in U_1$ give isomorphic E. Thus, for a given $E'' \in U'_X(2n-1, 2)$, the above construction gives a unique $E \in U'_X(2n+1, 2)$.

If we start with $E'' = E_0 \in U'_X(1, 2)$, then we get $E \in U'_X(3, 2)$, denote this E by E_1. Repeating this process j times, we get a sequence of stable vector bundles

$$E_j,\ j \geq 0,\ E_j \in U'_X(2j+1, 2),\ E_{j+1}/I_2 \cong E_j.$$

This completes the proof of the proposition. □

Conversely, let $E \in U'_X(2n+1, 2)$, where $n \geq 1$. The stability condition of E implies that $h^1(E) = 0$ and $h^0(E) = d(E) = 2$. If the evaluation map $H^0(E) \otimes I \longrightarrow E$ is not injective, then there will be a section of E generating a subsheaf of degree ≥ 1 contradicting the stability condition of E. Thus we have $I_2 \subset E$. Set $E'' := E/I_2$. It is easy to see that if E has a subsheaf N of rank $n+1$ and degree one containing I_2, then E'' is not stable, and furthermore, any $N'' \subset E''$ with $\mu(N'') \geq \mu(E'')$ has to be of the form N/I_2. Therefore, the set theoretic map

$$h_n\colon U'_X(1, 2) \longrightarrow U'_X(2n+1, 2) \tag{4.8}$$

defined by $E_0 \longmapsto E_n$ may not be surjective.

PROPOSITION 4.4

Let $n > 0$ be an integer.

(1) *There is a surjective rational map*

$$h'_n\colon U_X(2n+1, 2) \to U_X(1, 2) \cong S^1.$$

(2)

$$h'_1\colon U_X(3, 2) \to U_X(1, 2) \cong S^1$$

is a bijective morphism.

(3) *There are surjective rational maps*

$$U_X(2n+1, 2(2n+1)+2) = U'_X(2n+1, 2(2n+1)+2) \to S^1,$$

$$U_X(2n+1, 2(2n+1)-2) = U'_X(2n+1, 2(2n+1)-2) \to S^1.$$

(4) *For each $d \in 2\mathbb{Z}$ with $d \notin 3\mathbb{Z}$, there is a bijective morphism*

$$U_X(3, d) = U'_X(3, d) \to S^1.$$

The moduli space $U_X(3, d)$ is empty otherwise.

Proof. Let h_n be the map defined in (4.8). Set $Z_n := h_n(U'_X(1, 2))$. The inverse map

$$h'_n := h_n^{-1}\colon Z_n \longrightarrow U'_X(1, 2) \tag{4.9}$$

is a morphism, being the composition of the maps $Z_n \to Z_{n-1}$ defined by

$$E_n \longmapsto E_n/(H^0(E_n) \otimes I).$$

By Proposition 4.1(3) we have $U_X(1, 2) \cong S^1$, and by Proposition 4.2(3) we have $U_X(2n+1, 2) = U'_X(2n + 1, 2)$. Hence statement (1) in the proposition is proved.

To prove statement (2), note that $E'' = E/I_2$ is automatically stable because it is of rank 1. Hence we have $Z_1 = U_X(3, 2)$.

There exist line bundles of all even degrees on X (see Proposition 4.1(3)). It is easy to see that if E is a stable torsionfree sheaf, and L a line bundle, then $E \otimes L$ is stable. Thus we have $U(j, 2) \cong U(j, 2t + 2)$ for all integers $j > 0$ and t. A torsionfree sheaf E on X is stable if and only if E^* is so (Lemma 2.6 of [B2]), so that $U(j, d) \cong U(j, -d)$ for all integers $j > 0$ and d. Hence statements (3) and (4) follow from statements (1) and (2). Note that it follows from Lemma 2.2 and Proposition 4.2 that, if d is odd or d is a multiple of 3, then there are no real stable torsionfree sheaves of rank 3 and degree d on X. □

We will now classify real algebraic stable torsionfree sheaves on X of rank n and degree d with $(n, d) = 2$.

Lemma 4.5. *Let V be a stable real algebraic torsionfree sheaf over X. Let $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ be the corresponding complex algebraic torsionfree sheaf over Y. Then either $V_{\mathbb{C}}$ is stable, or $V_{\mathbb{C}}$ is isomorphic to $F \oplus \sigma^*\bar{F}$, where F is a stable torsionfree sheaf over Y.*

Proof. Since $V_{\mathbb{C}}$ is a pullback of a real algebraic torsionfree sheaf on X, there is an isomorphism

$$\delta\colon V_{\mathbb{C}} \longrightarrow \sigma^*\overline{V_{\mathbb{C}}}. \tag{4.10}$$

The torsionfree sheaf $V_{\mathbb{C}}$ is polystable, as V is so (see Lemma 3.4(2)).

Assume that $V_{\mathbb{C}}$ is not stable. Let

$$V_{\mathbb{C}} = \bigoplus_{i=1}^{\ell} F_i \tag{4.11}$$

be a decomposition of $V_{\mathbb{C}}$ into a direct sum of stable torsionfree sheaves. Consider the holomorphic torsionfree sheaf $\sigma^*\overline{F_1} \subset \sigma^*\overline{V_{\mathbb{C}}}$. Let

$$F' := \delta^{-1}(\sigma^*\overline{F_1}) \subset V_{\mathbb{C}},$$

where δ is the isomorphism in equation (4.10).

Let $\psi\colon F' \longrightarrow V_{\mathbb{C}}/F_1$ be the natural projection. Note that $V_{\mathbb{C}}/F_1$ is polystable, F' is stable, and $\mu(V_{\mathbb{C}}/F_1) = \mu(F')$. Hence the homomorphism ψ is either the zero homomorphism, or it is a sheaf injection with its cokernel a torsionfree sheaf. If $\psi = 0$, then $F' = F_1$ and F_1 defines a real algebraic torsionfree subsheaf $F_{\mathbb{R}}$ of V with same slope as V. This contradicts the assumption that V is stable.

Therefore, ψ is an injection with cokernel a torsionfree sheaf. This implies that $F_1 + F'$, which is the subsheaf of $V_{\mathbb{C}}$ generated by F_1 and F', has a torsionfree quotient, and also, $F_1 + F'$ is identified with $F_1 \oplus F'$. Furthermore, the isomorphism δ takes the subsheaf $F_1 \oplus F' \subset V_{\mathbb{C}}$ to the subsheaf $\sigma^*(\overline{F_1 \oplus F'}) \subset \sigma^*\overline{V_{\mathbb{C}}}$. Indeed, this follows from the fact that $(\sigma^*\bar{\delta}) \circ \delta = \mathrm{Id}_{V_{\mathbb{C}}}$. Therefore, $F_1 \oplus F'$ defines a real algebraic subsheaf W of V with

$$\frac{\mathrm{degree}(W)}{\mathrm{rank}(W)} = \frac{\mathrm{degree}(V)}{\mathrm{rank}(V)}.$$

Since V is stable, this implies that $F_1 \oplus F' = V_{\mathbb{C}}$. As $F' = \delta^{-1}(\sigma^*\overline{F_1})$ is isomorphic to $\sigma^*\overline{F_1}$, this proves the lemma. □

COROLLARY 4.6

Let E be a stable real torsionfree sheaf of rank r and degree d over X. Then either $\gcd(r, d) = 1$ *or* $\gcd(r, d) = 2$.

Proof. This follows from Theorem 2.5 and Lemma 4.5. □

PROPOSITION 4.7

Fix a positive integer r' and an odd integer d' such that r' and d' are coprime. Let $r := 2r'$ and $d := 2d'$.

(1) *For any stable torsionfree sheaf V over Y of rank r' and degree d', the real algebraic torsionfree sheaf over X defined by $V \oplus \sigma^*\bar{V}$ is stable.*
(2) *For any stable real algebraic torsionfree sheaf E over X of rank r and degree d, there is a stable torsionfree sheaf V over Y of rank r' and degree d' such that*

$$E \otimes_{\mathbb{R}} \mathbb{C} = V \oplus \sigma^*\bar{V}.$$

(3) *Let V and W be stable torsionfree sheaves of rank r' and degree d' over Y, and let $V_{\mathbb{R}}$ (respectively, $W_{\mathbb{R}}$) be the real torsionfree sheaves over X given by $V \oplus \sigma^*\bar{V}$ (respectively, $W \oplus \sigma^*\bar{W}$). Then $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$ are isomorphic if and only if either V is isomorphic to W or V is isomorphic to $\sigma^*\bar{W}$.*

Proof. The proof is exactly same as that of Proposition 6.7 of [BB]. □

PROPOSITION 4.8

Fix a positive integer r' and an even integer d' such that r' and d' are coprime. Let $\mathcal{S}(r, d)$ denote the set of isomorphism classes of stable torsionfree sheaves over Y of rank r' and degree d' which are not of the form $E \otimes_{\mathbb{R}} \mathbb{C}$, where E is some real algebraic torsionfree sheaf over X. Set $r := 2r'$ and $d := 2d'$.

(1) *For any stable torsionfree sheaf $V \in \mathcal{S}(r, d)$ over Y, the real algebraic torsionfree sheaf over X defined by $V \oplus \sigma^*\bar{V}$ is stable.*

(2) *For any stable real algebraic torsionfree sheaf E over X of rank r and degree d, there is a stable torsionfree sheaf $V \in \mathcal{S}(r, d)$ over Y such that*

$$E \otimes_{\mathbb{R}} \mathbb{C} = V \oplus \sigma^* \bar{V}.$$

(3) *Take torsionfree sheaves $V, W \in \mathcal{S}(r, d)$. Let $V_{\mathbb{R}}$ (respectively, $W_{\mathbb{R}}$) be the real algebraic torsionfree sheaf over X given by $V \oplus \sigma^* \bar{V}$ (respectively, $W \oplus \sigma^* \bar{W}$). Then $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$ are isomorphic if and only if either V is isomorphic to W or V is isomorphic to $\sigma^* \bar{W}$.*

Proof. This can be proved exactly as Proposition 6.9 of [BB] is proved. □

Theorem 4.9. *Let $r := 2r'$ and $d := 2d'$ be integers such that r' is a positive integer coprime to d'. Let*

$$\mathcal{U} \subset U_Y(r', d')$$

be the subset defined by all stable torsionfree sheaves which are not of the form $F \otimes_{\mathbb{R}} \mathbb{C}$, where F is some real algebraic torsionfree sheaf over X. Then the following two hold:

(1) *The set of isomorphism classes of stable real algebraic torsionfree sheaves over X of rank r and degree d is canonically identified with the quotient space $\mathcal{U}/(\mathbb{Z}/2\mathbb{Z})$ for the involution of $U_Y(r', d')$ defined by $W \longrightarrow \sigma^* \bar{W}$.*
(2) *Moreover, if d' is odd, then $\mathcal{U} = U_Y(r', d')$.*

Proof. The theorem follows from Propositions 4.7 and 4.8. □

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