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Entire functions sharing one polynomial with their derivatives

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Abstract. In this paper, we study the growth of solutions of a k-th order linear differential equation and that of a $k+1$-th order linear differential equation. From this we affirmatively answer a uniqueness question concerning a conjecture given by Brück in 1996 under the restriction of the hyper order less than $1/2$, and obtain some uniqueness theorems of a nonconstant entire function and its derivative sharing a finite nonzero complex number CM. The results in this paper also improve some known results. Some examples are provided to show that the results in this paper are best possible.

Keywords. Entire function; order of growth; shared value; uniqueness.

1. Introduction and main results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [6, 8, 13]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any nonconstant meromorphic function $h(z)$, we denote by $S(r, h)$ any quantity satisfying

$$S(r, h) = o(T(r, h)), \quad r \to \infty, r \notin E.$$

Let f and g be two nonconstant meromorphic functions, and let P be a polynomial or a finite complex number. We say that f and g share P CM, provided that $f - P$ and $g - P$ have the same zeros with the same multiplicities. Similarly, we say that f and g share P IM, provided that $f - P$ and $g - P$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM (see [14]). In this paper, we also need the following two definitions.

DEFINITION 1.1

Let f be a nonconstant entire function, the order of f is defined by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} = \limsup_{r\to\infty} = \frac{\log\log M(r, f)}{\log r},$$

where, and in the sequel, $M(r, f) = \max_{|z|=r}\{|f(z)|\}$.

DEFINITION 1.2

Let f be a nonconstant meromorphic function, the hyper order of f is defined by

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r}.$$

Rubel and Yang proved that if an entire function f share two distinct finite complex numbers CM with its derivative f', then $f \equiv f'$ (see [12]). How is the relation between f and f', if an entire function f share one finite complex number CM with its derivative f'? Brück made the conjecture that if f is a nonconstant entire function satisfying $\sigma_2(f) < \infty$, where $\sigma_2(f)$ is not a positive integer, and if f and f' share one finite complex number a CM, then $f - a \equiv c(f' - a)$ for some finite complex number $c \neq 0$ (see [1]). For the case $a = 0$, the above conjecture had been proved by Brück (see [1]). Brück proved the above conjecture is true, provided that $a \neq 0$ and $N(r, 1/f') = S(r, f)$ (see [1]). In 1998, Gundersen and Yang proved that the conjecture is true, provided that f satisfies the additional assumption $\sigma(f) < \infty$ (see [5]). Regarding the above conjecture, a natural question is:

Question 1.1 (see Question 1 of [15]). What can be said when a nonconstant entire function f shares one finite complex number with one of its derivatives $f^{(k)}$ $(k \geq 1)$?

Dealing with Question 1.1, Yang [15] proved the following result in 1999.

Theorem A (see Theorem 1 of [15]). *Let $Q(z)$ be a nonconstant polynomial. If F is a solution of the differential equation*

$$F^{(k)} - \mathrm{e}^{Q(z)} F = 1, \tag{1.1}$$

where, and in the sequel, $k(\geq 1)$ is a positive integer, then $\sigma(F) = \infty$.

In this paper, we will prove the following result, which improves Theorem A.

Theorem 1.1. *Let Q_1 and Q_2 be two nonzero polynomials, and let P be a polynomial. If f is a nonconstant solution of the equation*

$$f^{(k)} - Q_1 = \mathrm{e}^{P}(f - Q_2), \tag{1.2}$$

then $\sigma_2(f) = \gamma_P$, where, and in the sequel, γ_P denotes the degree of P.

From Theorem 1.1 we can get the following result on the growth of a nonconstant solution of a $k + 1$-th order linear differential equation.

COROLLARY 1.1

Let Q $(\not\equiv 0)$ and P be two polynomials. If f is a nonconstant solution of the differential equation

$$f^{(k+1)} - \mathrm{e}^{P} \cdot f' - P'\mathrm{e}^{P} \cdot f + (QP' + Q') \cdot \mathrm{e}^{P} - Q' = 0, \tag{1.3}$$

then $\sigma_2(f) = \gamma_P$.

Proof. From the condition that f is a nonconstant solution of (1.3), we can deduce that there exists some finite complex number c, such that

$$f^{(k)} - (Q + c) = \mathrm{e}^{P}(f - Q). \tag{1.4}$$

From (1.4) and Theorem 1.1 we get the conclusion of Corollary 1.1.

From Theorem 1.1 and Lemma 2.1 in §2 of this paper we can get the following corollary.

COROLLARY 1.2

Let Q_1 and Q_2 be two nonzero polynomials, and let P be a polynomial. If f is a solution of the differential equation (1.2), *such that $\sigma(f) = \infty$, then P is a nonconstant polynomial such that $\sigma_2(f) = \gamma_P$.*

From Theorem 1.1 we also get the following corollary.

COROLLARY 1.3

Let Q be a nonzero polynomial, and let P be a polynomial. If f is a nonconstant solution of the differential equation

$$f^{(k)} - Q = \mathrm{e}^{P}(f - Q), \tag{1.5}$$

such that $\sigma_2(f)$ is not a positive integer, then P must be a constant.

From Corollary 1.3 we can get the following result.

COROLLARY 1.4

Let f be a nonconstant entire function such that $\sigma_2(f) < 1$, and let Q be a nonzero polynomial. If f is a nonconstant solution of the differential equation (1.5), *then P must be a constant.*

From Corollary 1.4 we can get the following result.

COROLLARY 1.5

Let $P(z)$ be a polynomial, and let a $(\neq 0)$ be a finite complex number. If f is a nonconstant solution of the differential equation

$$f^{(k)} - a = \mathrm{e}^{P}(f - a), \tag{1.6}$$

such that $\sigma_2(f) < 1$, then P must be a constant.

From Corollary 1.5 we can get the following result.

COROLLARY 1.6

Let f be a nonconstant entire function of finite order, and let a $(\neq 0)$ be a finite complex number. If f and $f^{(k)}$ share a CM, *then $f^{(k)} - a \equiv c(f - a)$ for some finite nonzero complex number c.*

Now we give the following two examples.

Example 1.1 [5]. Let f be a solution of the differential equation $f' - 1 = \mathrm{e}^{z^n}(f - 1)$, where n is a positive integer. Then f and f' share 1 CM. Moreover, from Lemma 2.6 in §2 of this paper we can see that f is a nonconstant entire function, and from Theorem 1.1 we have $\sigma_2(f) = \sigma(\mathrm{e}^{z^n}) = n$. This example shows that the conclusions of Theorem 1.1 and Corollary 1.2 can occur. This example also shows that the condition '$\sigma_2(f)$ is not a positive integer' in Corollary 1.3 is necessary.

Example 1.2 [5]. Let $f(z) = (2e^z+z+1)/(e^z+1)$. Then we can see that f is a nonconstant meromorphic function, but not an entire function. Moreover, we can verify that $\sigma(f) = 1$ and $\sigma_2(f) = 0$, and that f and f' share 1 CM. But $(f'(z)-1)/(f(z)-1) = -e^z/(e^z+1)$. This example shows that the conclusions of Corollaries 1.3–1.6 are invalid, if f is not an entire function.

From Corollary 1.3 we also get the following three corollaries, which improve some results in [15].

COROLLARY 1.7

Let $P(z)$ be a polynomial, and let a $(\neq 0)$ be a finite complex number. If f is a nonconstant solution of the differential equation (1.6), *such that $\sigma_2(f)$ is not a positive integer*, *and if there exists one point z_0 such that $f^{(k)}(z_0) = f(z_0) \neq a$, then $f \equiv f^{(k)}$.*

COROLLARY 1.8

Let $P(z)$ be a polynomial, and let a $(\neq 0)$ be a finite complex number. If f is a nonconstant solution of the differential equation (1.6), *such that $\sigma_2(f)$ is not a positive integer*, *and if f and $f^{(k)}$ share b* IM, *where b $(\neq a)$ is a finite complex number*, *then $f \equiv f^{(k)}$.*

Proof. We discuss the following two cases.

Case 1. Suppose that $b \neq 0$. From the condition that f and $f^{(k)}$ share b IM and by Hayman's inequality (see Theorem 3.5 of [6]) we can see that there exists one point z_0 such that $f^{(k)}(z_0) = f(z_0) = b \neq a$. Combining Corollary 1.7 we can get the conclusion of Corollary 1.8.

Case 2. Suppose that $b = 0$. First, from Corollary 1.3 we have

$$f^{(k)} - a \equiv c(f - a), \tag{1.7}$$

where $c \neq 0$ is some finite complex number. We discuss the following two subcases.

Subcase 2.1. Suppose that there exists one point z_0 such that $f(z_0) = f^{(k)}(z_0) = 0$. Then from (1.7) we can get $c = 1$, and so the conclusion of Corollary 1.8 is valid.

Subcase 2.2. Suppose that 0 is a Picard exceptional value of f. Let

$$f = e^{\beta_1}, \tag{1.8}$$

where β_1 is a nonconstant entire function. From (1.8) and by induction we can deduce

$$f^{(k)} = ((\beta_1')^k + P[\beta_1']) \cdot e^{\beta_1}, \tag{1.9}$$

where $P[\beta_1']$ is a differential polynomial in β_1', and the degree of $P[\beta_1']$ is not greater than $k-1$. Substituting (1.8) and (1.9) into (1.7) we can get

$$\frac{((\beta_1')^k + P[\beta_1']) \cdot e^{\beta_1} - a}{e^{\beta_1} - a} \equiv c. \tag{1.10}$$

If $(\beta_1')^k + P[\beta_1'] - c \not\equiv 0$, then from (1.8) and (1.10) we deduce $T(r, f) = S(r, f)$, this is impossible. Thus $(\beta_1')^k + P[\beta_1'] - c \equiv 0$. From this and (1.10) we deduce $c = 1$. Combining (1.7) we get the conclusion of Corollary 1.8.

Remark 1.2. The conditions of Corollary 1.8 imply that f and $f^{(k)}$ share the finite value a CM and b IM. The conclusion of Corollary 1.8 is still true under a weaker condition 'f and $f^{(k)}$ share two values IM.' This is Frank's conjecture which was proved by Li and Yang in [9].

COROLLARY 1.9

Let $P(z)$ be a polynomial, let $a\ (\neq 0)$ be a finite complex number, and let n be a positive integer. If f is a nonconstant solution of the differential equation (1.6), *such that $\sigma_2(f)$ is not a positive integer, and if there exists one point z_0 such that $f^{(n)}(z_0) = f^{(n+k)}(z_0) \neq 0$, then $f \equiv f^{(k)}$.*

In 1999, Yang proved the following result.

Theorem B (see Theorem 2 of [15]). *Let f be a nonconstant entire function of finite order, and let $a\ (\neq 0)$ be a finite complex number. If f and $f^{(k)}$ share a* CM, *then $f - a \equiv c(f^{(k)} - a)$ for some finite complex number $c \neq 0$.*

In this paper, we will prove the following result, which improves Theorem B.

Theorem 1.2. *Let f be a nonconstant entire function of hyper order $\sigma_2(f) < 1/2$, and let Q_1 and Q_2 be two nonzero polynomials. If $f - Q_1$ and $f^{(k)} - Q_2$ share* 0 CM, *then $f^{(k)} - Q_1 \equiv c(f - Q_2)$ for some finite complex number $c \neq 0$.*

From Theorem 1.2 we can get the following two corollaries.

COROLLARY 1.10

Let f be a nonconstant entire function of hyper order $\sigma_2(f) < 1/2$, and let Q be a nonzero polynomial. If $f - Q$ and $f^{(k)} - Q$ share 0 CM, *then $f^{(k)} - Q \equiv c(f - Q)$ for some finite complex number $c \neq 0$.*

COROLLARY 1.11

Let f be a nonconstant entire function of hyper order $\sigma_2(f) < 1/2$, and let $a\ (\neq 0)$ be a finite complex number. If f and $f^{(k)}$ share a CM, *then $f^{(k)} - a \equiv c(f - a)$ for some finite complex number $c \neq 0$.*

Yi and Yang [14] posed the following question.

Question 1.2 (see pp. 458 of [14]). Let f be a nonconstant meromorphic function, and let a be a finite nonzero complex number. If f, $f^{(n)}$ and $f^{(m)}$ share the value a CM, where n and m $(n < m)$ are distinct positive integers not all even or odd, then can we get the result $f \equiv f^{(n)}$?

Li and Yang [10] proved the following two theorems, which dealt with Question 1.2.

Theorem C (see Theorem 1 of [10]). *Let $f(z)$ be a nonconstant entire function, let a be a finite nonzero complex number, and let n be a positive integer. If f, $f^{(n)}$ and $f^{(n+1)}$ share the value a* CM, *then $f = f'$.*

Theorem D (see Theorem 2 of [10]). *Let $f(z)$ be a nonconstant entire function, let a be a finite nonzero complex number, and let n (≥ 2) be a positive integer. If f, f' and $f^{(n)}$ share a* CM, *then f assumes the form*

$$f(z) = b\mathrm{e}^{cz} - \frac{a(1-c)}{c},$$

where b, c are finite nonzero complex numbers and $c^{n-1} = 1$.

In this paper, we will prove the following result, which is an improvement and supplement of Theorem C and Theorem D.

Theorem 1.3. *Let f be a nonconstant entire function of hyper order $\sigma_2(f) < 1/2$, let $a(\neq 0)$ be a finite complex number, and let n and k be two positive integers. If f, $f^{(n)}$ and $f^{(n+k)}$ share a* CM, *then there exist finite complex numbers $\lambda_j (\neq 0)$ $(1 \leq j \leq k)$, c $(\neq 0)$ and b_0 satisfying*

$$\lambda_j{}^n = \lambda_j{}^{n+k} = c \, (1 \leq j \leq k) \tag{1.11}$$

and

$$cb_0 + (1-c)a = 0, \tag{1.12}$$

such that

$$f(z) = \sum_{j=1}^{k} \frac{\gamma_j}{c} \mathrm{e}^{\lambda_j z} + b_0, \tag{1.13}$$

where γ_j $(1 \leq j \leq k)$ are certain finite complex numbers.

2. Some lemmas

Lemma 2.1 (*see pp.* 36–37 *of* [7] *or Theorem* 3.1 *of* [8]). *If f is an entire function of order $\sigma(f)$, then*

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log \nu(r, f)}{\log r},$$

where, and in the sequel, $\nu(r, f)$ denotes the central-index of $f(z)$.

Lemma 2.2 (*see Lemma* 2 of [2] *or Lemma* 4 *of* [3]). *If f is a transcendental entire function of hyper order $\sigma_2(f)$, then*

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log \nu(r, f)}{\log r}.$$

Lemma 2.3 (*see Lemma* 1.1.1 of [8]). *Let g: $(0, +\infty) \longrightarrow R$, h: $(0, +\infty) \longrightarrow R$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E of finite linear measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.*

Lemma 2.4 (*see Lemma* 13 of [4]). *Let $f(z)$ be an entire function of order $\sigma(f)=\sigma<1/2$, and denote $A(r)=\inf_{|z|=r}\log|f(z)|$, $B(r)=\sup_{|z|=r}\log|f(z)|$. If $\sigma<\alpha<1$, then*

$$\underline{\log\text{dens}}\{r\colon A(r)>(\cos\pi\alpha)B(r)\}\geq 1-\frac{\sigma}{\alpha},$$

where

$$\underline{\log\text{dens}}(H)=\liminf_{r\to\infty}\frac{\int_1^r\frac{\chi_H(t)}{t}\mathrm{d}t}{\log r}$$

and

$$\overline{\log\text{dens}}(H)=\limsup_{r\to\infty}\frac{\int_1^r\frac{\chi_H(t)}{t}\mathrm{d}t}{\log r},$$

where $\chi_H(t)$ is the characteristic function of a set H.

Lemma 2.5 (*see Lemma* 4 *of* [11]). *Let $f_1, f_2, \ldots, f_n$ be nonconstant meromorphic functions satisfying*

$$\overline{N}(r,f_j)+\overline{N}\left(r,\frac{1}{f_j}\right)=S(r)\quad(j=1,2,\ldots,n)$$

and

$$T(r,f_j)\neq S(r),\qquad T\left(r,\frac{f_i}{f_j}\right)\neq S(r)\quad(j\neq i,\ j,i=1,2,\ldots,n).$$

Let $a_0, a_1, \ldots, a_m$ $(m\leq n)$ be meromorphic functions satisfying $T(r,a_j)=S(r)$ $(j=0,1,\ldots,m)$. If

$$\sum_{j=1}^{m}a_jf_j\equiv a_0,$$

then $a_0\equiv a_1\equiv\cdots\equiv a_m\equiv 0$, where $S(r)=o(T(r))$, as $r\longrightarrow\infty$ and $r\notin E$, and $T(r)=\sum_{j=1}^n T(r,f_j)$.

Lemma 2.6 (*see Proposition* 8.1 *of* [8]). *Suppose that all the coefficients $a_0(z)(\equiv 0)$, $a_1(z), a_2(z), \ldots, a_{n-1}(z)$ and $g(z)(\not\equiv 0)$ of the nonhomogeneous linear differential equation*

$$f^{(n)}+a_{n-1}(z)f^{(n-1)}+\cdots+a_1(z)f'+a_0(z)f=g(z)\tag{2.1}$$

are entire functions. Then all solutions of (2.1) *are entire functions.*

3. Proof of Theorems

Proof of Theorem 1.1. Suppose that f is a polynomial, then from (1.2) we can see that there exists a certain finite nonzero complex number c, such that $\mathrm{e}^P\equiv c$, and so $\sigma_2(f)=\gamma_P=0$, this reveals the conclusion of Theorem 1.1. Next we suppose that f is a transcendental entire function. We discuss the following two cases.

Case 1. Suppose that

$$\sigma(f) = \infty. \tag{3.1}$$

From (3.1) and Lemma 2.1 we have

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log \nu(r, f)}{\log r} = \infty. \tag{3.2}$$

If P is a constant, from (1.2), Lemma 2.1 and Theorem 3.2 in [8] we can deduce $\sigma(f) < \infty$, which contradicts (3.1). Thus P is a nonconstant polynomial. Let

$$P(z) = p_n z^n + p_{n-1} z^{n-1} + \cdots + p_1 z + p_0, \tag{3.3}$$

where $p_n (\neq 0)$, $p_{n-1}, \ldots, p_1$ and p_0 are finite complex numbers. From (3.3) we have

$$\lim_{|z|\to\infty} \frac{|P(z)|}{|p_n z^n|} = 1. \tag{3.4}$$

From (3.4) we can see that there exists a sufficiently large positive number r_0, such that

$$\frac{|P(z)|}{|p_n z^n} > \frac{1}{e} \quad (|z| > r_0). \tag{3.5}$$

From (1.2) and (3.5) we deduce

$$\begin{aligned} n \log r + \log|p_n| - 1 &\leq \log|P(z)| = \log|\log \mathrm{e}^{P(z)}| \leq |\log\log \mathrm{e}^{P(z)}| \\ &= \left|\log\log \frac{f^{(k)} - Q_1}{f - Q_2}\right| \quad (|z| > r_0). \end{aligned} \tag{3.6}$$

From the condition that f is a transcendental entire function, we have

$$M(r, f) \longrightarrow \infty, \tag{3.7}$$

as $r \longrightarrow \infty$. Let

$$M(r, f) = |f(z_r)|, \tag{3.8}$$

where $z_r = r\mathrm{e}^{i\theta(r)}$, and $\theta(r) \in [0, 2\pi)$ is some nonnegative real number. From (3.8) and the Wiman–Valiron theory (see Theorem 3.2 of [8]), we can see that there exists a subset $E_1 \subset (1, \infty)$ with finite logarithmic measure, i.e., $\int_{E_1} \frac{\mathrm{d}t}{t} < \infty$, such that for some point $z_r = r\mathrm{e}^{i\theta(r)}$ ($\theta(r) \in [0, 2\pi)$) satisfying $|z_r| = r \notin E_1$ and $M(r, f) = |f(z_r)|$, we have

$$\frac{f^{(k)}(z_r)}{f(z_r)} = \left(\frac{\nu(r, f)}{z_r}\right)^k (1 + o(1)), \tag{3.9}$$

as $r \longrightarrow \infty$. Since f is a transcendental entire function, Q_1 and Q_2 are two nonzero polynomials, from (3.1) and (3.8) we deduce

$$\lim_{r\to\infty} \frac{|Q_j(z_r)|}{|f(z_r)|} = \lim_{r\to\infty} \frac{|Q_j(z_r)|}{M(r, f)} = 0 \quad (j = 1, 2). \tag{3.10}$$

Since

$$\frac{f^{(k)}-Q_1}{f-Q_2}=\frac{\frac{f^{(k)}}{f}-\frac{Q_1}{f}}{1-\frac{Q_2}{f}}, \tag{3.11}$$

from (3.2) and (3.6)–(3.11) we deduce

$$n\log|z_r|+\log|p_n|-1\le\left|\log\log\left(\left(\frac{\nu(r,f)}{z_r}\right)^k(1+o(1))\right)\right| \tag{3.12}$$

and

$$\log\left(\left(\frac{\nu(r,f)}{z_r}\right)^k(1+o(1))\right)$$
$$=k\left(1-\frac{\log r}{\log\nu(r,f)}-\frac{i\theta(r)}{\log\nu(r,f)}\right)\log\nu(r,f)+o(1), \tag{3.13}$$

as $r\longrightarrow\infty$. From (3.2), (3.13), Lemma 2.2 and $\theta(r)\in[0,2\pi)$, we get

$$\limsup_{r\to\infty}\frac{\left|\log\log\left(\left(\frac{\nu(r,f)}{z_r}\right)^k(1+o(1)(1))\right)\right|}{\log r}$$
$$\le\limsup_{r\to\infty}\frac{\log\log\nu(r,f)}{\log r}$$
$$+\limsup_{r\to\infty}\frac{\left|\log\left(k\left(1-\frac{\log r}{\log\nu(r,f)}-\frac{i\theta(r)}{\log\nu(r,f)}\right)\right)\right|}{\log r}$$
$$+\lim_{r\to\infty}\frac{\log 2}{\log r}+\lim_{r\to\infty}\frac{2k_1\pi}{\log r}$$
$$=\limsup_{r\to\infty}\frac{\log\log\nu(r,f)}{\log r}=\sigma_2(f), \tag{3.14}$$

where k_1 is some nonnegative integer. From (3.12), (3.14) and $|z_r|=r$, we deduce

$$n\le\limsup_{r\to\infty}\frac{\log\log\nu(r,f)}{\log r}=\sigma_2(f). \tag{3.15}$$

On the other hand, from (3.3) and Theorem 1.45 in [14] we have

$$\sigma(\mathrm{e}^P)=\gamma_P=n. \tag{3.16}$$

From (3.15) and (3.16) we have

$$\sigma(\mathrm{e}^P)\le\sigma_2(f). \tag{3.17}$$

On the other hand, from (1.2) and (3.9)–(3.11) we deduce

$$\left(\frac{\nu(r,f)}{z_r}\right)^k(1+o(1))=\mathrm{e}^{P(z_r)} \tag{3.18}$$

as $r \longrightarrow \infty$. From (3.2) and (3.18) we get

$$\begin{aligned}\limsup_{r\to\infty} \frac{\log\log \nu(r,f)}{\log r} &= \limsup_{r\to\infty} \frac{\log\log\left(\frac{\nu(r,f)}{z_r}\right)^k}{\log r}\\ &\le \limsup_{r\to\infty} \frac{\log\log\left(\left(\frac{\nu(r,f)}{|z_r|}\right)^k \cdot |1+o(1)|\right)}{\log r}\\ &\le \limsup_{r\to\infty} \frac{\log\log M(r, \mathrm{e}^{P(z)})}{\log r}. \end{aligned} \tag{3.19}$$

From (3.19), Definition 1.1 and Lemma 2.2 we get

$$\sigma_2(f) \le \sigma(\mathrm{e}^P). \tag{3.20}$$

From (3.16), (3.17) and (3.20) we get the conclusion of Theorem 1.1.

Case 2. Suppose that

$$\sigma(f) < \infty. \tag{3.21}$$

First, from (3.21) we deduce

$$\sigma_2(f) = 0. \tag{3.22}$$

On the other hand, from (1.2), (3.7)–(3.11), (3.21), Lemma 2.1 and $\theta(r) \in [0, 2\pi)$, we deduce

$$\begin{aligned}|P(z_r)| = |\log \mathrm{e}^{P(z_r)}| &= |k(\log \nu(r,f) - \log(r\mathrm{e}^{i\theta(r)})) + o(1)|\\ &= |k(\log \nu(r,f) - \log r - i\theta(r)) + o(1)|\\ &\le O(\log r), \end{aligned} \tag{3.23}$$

as $r \longrightarrow \infty$. From (3.23) and the condition that $P(z)$ is a polynomial, we deduce that $P(z)$ is a constant, and so $\gamma_P = 0$. This and (3.22) reveal the conclusion of Theorem 1.1. Theorem 1.1 is thus completely proved.

Proof of Theorem 3.2. From the hypothesis of Theorem 1.2 we have

$$f^{(k)} - Q_1 = \mathrm{e}^{\beta_2}(f - Q_2), \tag{3.24}$$

where β_2 is an entire function. First, we suppose that f is a polynomial. From (3.24) we have $\mathrm{e}^{\beta_2} \equiv c$, where $c \ne 0$ is a finite complex number, and so the conclusion of Theorem 1.2 is valid. Next we suppose that f is a transcendental entire function. First, from (3.24) and the lemma of logarithmic derivative (see Corollary 2.3.4 of [8] or Lemma 1.4 of [14]), we deduce

$$T(r, \mathrm{e}^{\beta_2}) \le 2T(r,f) + O(\log rT(r,f)) \quad (r \notin E),$$

from which and Lemma 2.3 we deduce $\sigma_2(\mathrm{e}^{\beta_2}) \le \sigma_2(f)$. Combining (3.24), Theorem 1.45 in [14] and the condition $\sigma_2(f) < 1/2$, we have

$$\sigma(\beta_2) = \sigma_2(\mathrm{e}^{\beta_2}) \le \sigma_2(f) < 1/2. \tag{3.25}$$

From (3.25) and Lemma 2.4 we can see that there exists a certain positive number α satisfying $\sigma < \alpha < 1/2$, such that

$$\underline{\log \text{dens}}\{r\colon A(r) > (\cos \pi\alpha)B(r)\} \geq 1 - \frac{\sigma}{\alpha}, \tag{3.26}$$

where

$$A(r) = \inf_{|z|=r} \log|\beta_2(z)| \tag{3.27}$$

and

$$B(r) = \sup_{|z|=r} \log|\beta_2(z)|. \tag{3.28}$$

On the other hand, from (3.26) and the Wiman–Valiron theory (see Theorem 3.2 of [8]), we can see that at most there exists a subset $E_2 \subset \{r\colon A(r) > (\cos \pi\alpha)B(r)\} \subseteq (1, \infty)$ with finite logarithmic measure, i.e., $\int_{E_2} \frac{\mathrm{d}t}{t} < \infty$, such that (3.9) holds for some point $z_r = r\mathrm{e}^{i\theta(r)}$ $(\theta(r) \in [0, 2\pi))$ satisfying $|z_r| = r \in I =\colon \{r\colon A(r) > (\cos \pi\alpha)B(r)\} \setminus E_2$ and $M(r, f) = |f(z_r)|$. Next in the same manner as in the proof of Theorem 1.1 we can get (3.7), (3.8), (3.10) and (3.11). We discuss the following two cases.

Case 1. Suppose that (3.1) holds. Then from (3.1) we can get (3.2). From (3.2), (3.7)–(3.11), (3.24), (3.26), (3.27) and $\theta(r) \in [0, 2\pi)$, we deduce

$$\begin{aligned} A(z) &\leq \log|\beta_2(z_r)| \leq |\log \beta_2(z_r)| \\ &= \left|\log\log \frac{f^{(k)}(z_r) - Q_1(z_r)}{f(z_r) - Q_2(z_r)}\right| = \left|\log\log \frac{\frac{f^{(k)}(z_r)}{f(z_r)} - \frac{Q_1(z_r)}{f(z_r)}}{1 - \frac{Q_2(z_r)}{f(z_r)}}\right| \\ &= \left|\log\log \left(\left(\frac{\nu(r, f)}{z_r}\right)^k (1 + o(1))\right)\right| \\ &= |\log(k(\log \nu(r, f) - \log(r\mathrm{e}^{i\theta(r)})) + o(1))| \\ &= \left|\log\log \nu(r, f) + \log\left(k - \frac{\log r}{\log \nu(r, f)} - \frac{i\theta(r)}{\log \nu(f)} + o(1)\right)\right| \\ &\leq \log\log \nu(r, f) + \log(k + 1), \end{aligned} \tag{3.29}$$

as $r \in I$ and $r \longrightarrow \infty$. Again from (3.26)–(3.28) we deduce

$$\log M(r, \beta_2) = B(z) \leq \frac{1}{\cos \pi\alpha} A(z) + O(1), \quad r \in I. \tag{3.30}$$

On the other hand, from (3.25) and Lemma 2.2 we can deduce

$$\log\log \nu(r, f) \leq 2 \log r, \tag{3.31}$$

as $r \in I$ and $r \longrightarrow \infty$. Thus from (3.29)–(3.31) we deduce

$$\log M(r, \beta_2) = O(\log r), \tag{3.32}$$

as $r \in I$ and $r \longrightarrow \infty$. From (3.32) we can see that β_2 is a polynomial. Combining (3.24), (3.25) and Theorem 1.1 we get $\sigma_2(f) = \gamma_{\beta_2} < 1/2$. From this we can see that β_2 is a constant, and so from (3.24) we get the conclusion of Theorem 1.2.

Case 2. Suppose that (3.21) holds. Then from (3.21) we can get (3.22). Next in the same manner as in Case 2 in the proof of Theorem 1.1, we can verify that β_2 is a constant, and so from (3.24) we get the conclusion of Theorem 1.2. Theorem 1.2 is thus completely proved.

Proof of Theorem 3.3. From Theorem 1.2 and the assumptions of Theorem 1.3 we can see that there exist two finite nonzero complex numbers c and d, such that

$$f^{(n)} - a = c(f - a) \tag{3.33}$$

and

$$f^{(n+k)} - a = d(f^{(n)} - a). \tag{3.34}$$

Let

$$f^{(n)}(z) = g(z). \tag{3.35}$$

From (3.34) and (3.35) we get

$$g^{(k)} - a = d(g - a). \tag{3.36}$$

From (3.36) we have

$$g^{(k+1)} - dg' = 0. \tag{3.37}$$

From (3.37) we get the characteristic equation

$$\lambda^{k+1} - \mathrm{d}\lambda = 0. \tag{3.38}$$

Since the general solution of (3.37) has the form

$$f^{(n)}(z) = g(z) = \sum_{j=1}^{k} \gamma_j \mathrm{e}^{\lambda_j z} + b, \tag{3.39}$$

with certain constants γ_j $(1 \leq j \leq k)$, where $\lambda_1, \lambda_2 \ldots \lambda_k$ are the nonzero solution of (3.38), and b is a finite complex number,

$$f(z) = \sum_{j=1}^{k} \frac{\gamma_j}{\lambda_j^n} \mathrm{e}^{\lambda_j z} + \frac{bz^n}{n!} + \sum_{j=0}^{n-1} b_j z^j, \tag{3.40}$$

where $b_0, b_1, b_2, \ldots, b_{n-1}$ are finite complex numbers. On the other hand, since (3.33) can be rewritten by

$$f^{(n)} - cf = (1 - c)a, \tag{3.41}$$

by substituting (3.39) and (3.40) into (3.41) we can get

$$\sum_{j=1}^{k} \left(1 - \frac{c}{\lambda_j^n}\right) \gamma_j \mathrm{e}^{\lambda_j z} = \frac{cbz^n}{n!} + \sum_{j=1}^{n-1} cb_j z^j + cb_0 + (1 - c)a - b. \tag{3.42}$$

Since $\lambda_1, \lambda_2, \ldots, \lambda_k$ are k distinct finite nonzero complex numbers satisfying (3.38), and $c \neq 0$, from (3.38), (3.42) and Lemma 2.5 we can deduce (1.12),

$$\lambda_j{}^n = c \, (j = 1, 2, \ldots, k) \tag{3.43}$$

and

$$b_j = b = 0 \quad (1 \leq j \leq n - 1). \tag{3.44}$$

From (3.40), (3.43) and (3.44) we can get (1.13). On the other hand, from (3.38) and (3.39) we can get

$$f^{(n+k)} = \sum_{j=1}^{k} \gamma_j d \mathrm{e}^{\lambda_j z}. \tag{3.45}$$

Substituting (3.39) and (3.45) into (3.34) and combining (3.44) we can get

$$\frac{\sum_{j=1}^{k} \gamma_j d \mathrm{e}^{\lambda_j z} - a}{\sum_{j=1}^{k} \gamma_j \mathrm{e}^{\lambda_j z} - a} \equiv d,$$

which implies that $d = 1$. Combining (3.38) and (3.43) we get (1.11). From (1.11), (1.12) and (1.13) we get the conclusion of Theorem 1.3.

Theorem 1.3 is thus completely proved.

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