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Some nonlinear dynamic inequalities on time scales

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Abstract. The aim of this paper is to investigate some nonlinear dynamic inequalities on time scales, which provide explicit bounds on unknown functions. The inequalities given here unify and extend some inequalities in (B G Pachpatte, On some new inequalities related to a certain inequality arising in the theory of differential equation, *J. Math. Anal. Appl.* **251** (2000) 736–751).

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1. Introduction

In 1988, Stefan Hilger [10] introduced the calculus on time scales which unifies continuous and discrete analysis. Since then many authors have expounded on various aspects of the theory of dynamic equations on time scales. Recently, there has been much research activity concerning the new theory. For example, we refer the reader to the papers [1, 6–9, 14, 15], the monographes [4, 5] and the references cited therein. At the same time, a few papers [2, 3, 11] have studied the theory of dynamic inequalities on time scales.

In this paper, we investigate some nonlinear dynamic inequalities on time scales, which unify and extend some inequalities by Pachpatte in [13]. This paper is organized as follows: In §2 we give some preliminary results with respect to the calculus on time scales, which can also be found in [4, 5]. In §3 we deal with our nonlinear dynamic inequalities on time scales. In §4 we give an example to illustrate our main results.

2. Some preliminaries on time scales

In what follows, $\mathbf{R}$ denotes the set of real numbers, $\mathbf{Z}$ denotes the set of integers and $\mathbf{C}$ denotes the set of complex numbers.

A *time scale* $\mathbf{T}$ is an arbitrary nonempty closed subset of $\mathbf{R}$. The *forward jump operator* σ on $\mathbf{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbf{T}\colon s > t\} \in \mathbf{T} \quad \text{for all} \quad t \in \mathbf{T}.$$

In this definition we put $\inf \emptyset = \sup \mathbf{T}$, where $\emptyset$ is the empty set. If $\sigma(t) > t$, then we say that t is *right-scattered*. If $\sigma(t) = t$ and $t < \sup \mathbf{T}$, then we say that t is *right-dense*.

The *backward jump operator*, *left-scattered* and *left-dense* points are defined in a similar way. The *graininess* $\mu\colon \mathbf{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. The set $\mathbf{T}^\kappa$ is derived from $\mathbf{T}$ as follows: If $\mathbf{T}$ has a left-scattered maximum m, then $\mathbf{T}^\kappa = \mathbf{T} - \{m\}$; otherwise, $\mathbf{T}^\kappa = \mathbf{T}$. For $f\colon \mathbf{T} \to \mathbf{R}$ and $t \in \mathbf{T}^\kappa$, we define $f^\Delta(t)$ to be the number (provided it exists) such that given any $\varepsilon > 0$, there is a neighborhood U of t with

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \le \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

We call $f^\Delta(t)$ the *delta derivative* of f at t.

Remark 2.1. f^Δ is the usual derivative f' if $\mathbf{T} = \mathbf{R}$ and the usual forward difference Δf (defined by $\Delta f(t) = f(t+1) - f(t)$) if $\mathbf{T} = \mathbf{Z}$.

Theorem 2.1. *Assume $f, g\colon \mathbf{T} \to \mathbf{R}$ and let $t \in \mathbf{T}^\kappa$. Then we have the following conclusions*:

(i) *If f is differentiable at t, then f is continuous at t.*
(ii) *If f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) *If f is differentiable at t and t is right-dense, then*

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) *If f is differentiable at t, then*

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t).$$

(v) *If f and g are differentiable at t, then the product $fg\colon \mathbf{T} \to \mathbf{R}$ is differentiable at t with*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t).$$

We say that $f\colon \mathbf{T} \to \mathbf{R}$ is *rd-continuous* provided f is continuous at each right-dense point of $\mathbf{T}$ and has a finite left-sided limit at each left-dense point of $\mathbf{T}$. As usual, the set of rd-continuous functions is denoted by C_{rd}. A function $F\colon \mathbf{T} \to \mathbf{R}$ is called an *antiderivative* of $f\colon \mathbf{T} \to \mathbf{R}$ provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbf{T}^\kappa$. In this case we define the *Cauchy integral* of f by

$$\int_a^b f(t)\Delta t = F(b) - F(a) \quad \text{for } a, b \in \mathbf{T}.$$

Theorem 2.2. *If $a, b, c \in \mathbf{T}$, $\alpha \in \mathbf{R}$, and $f, g \in C_{\mathrm{rd}}$, then*

(i) $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$;
(ii) $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t$;
(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$;

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$;
(v) $\int_a^a f(t)\Delta t = 0$;
(vi) *if* $f(t) \geq 0$ *for all* $a \leq t \leq b$, *then* $\int_a^b f(t)\Delta t \geq 0$.

We say that $p\colon \mathbf{T} \to \mathbf{R}$ is *regressive* provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbf{T}$. We denote by $\mathcal{R}$ the set of all regressive and rd-continuous functions. We define the set of all positively regressive functions by $\mathcal{R}^+ = \{p \in \mathcal{R}\colon 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbf{T}\}$.

For $h > 0$, we define the cylinder transformation $\xi_h\colon \mathbf{C}_h \to \mathbf{Z}_h$ by

$$\xi_h(z) = \frac{1}{h}\log(1 + zh),$$

where log is the principal logarithm function, and

$$\mathbf{C}_h = \left\{z \in \mathbf{C}\colon z \neq -\frac{1}{h}\right\}, \quad \mathbf{Z}_h = \left\{z \in \mathbf{C}\colon -\frac{\pi}{h} < \mathrm{Im}(z) \leq \frac{\pi}{h}\right\}.$$

For $h = 0$, we define $\xi_0(z) = z$ for all $z \in \mathbf{C}$.

If $p \in \mathcal{R}$, then we define the *exponential function* by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \quad \text{for } s, t \in \mathbf{T}.$$

Theorem 2.3. *If* $p \in \mathcal{R}$ *and* $t_1 \in \mathbf{T}$ *fixed, then the exponential function* $e_p(\cdot, t_1)$ *is the unique solution of the initial value problem*

$$x^\Delta = p(t)x, \quad x(t_1) = 1 \ \textit{on} \ \mathbf{T}.$$

Theorem 2.4. *If* $p \in \mathcal{R}$, *then*

(i) $e_p(t, t) \equiv 1$ *and* $e_0(t, s) \equiv 1$;
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
(iii) *if* $p \in \mathcal{R}^+$, *then* $e_p(t, t_0) > 0$ *for all* $t \in \mathbf{T}$.

Remark 2.2. It is easy to see that, if $\mathbf{T} = \mathbf{R}$, the exponential function is given by

$$e_p(t, s) = \mathrm{e}^{\int_s^t p(\tau)\mathrm{d}\tau}, \quad e_\alpha(t, s) = \mathrm{e}^{\alpha(t-s)}, \quad e_\alpha(t, 0) = \mathrm{e}^{\alpha t}$$

for $s, t \in \mathbf{R}$, where $\alpha \in \mathbf{R}$ is a constant and $p\colon \mathbf{R} \to \mathbf{R}$ is a continuous function; if $\mathbf{T} = \mathbf{Z}$, the exponential function is given by

$$e_p(t, s) = \prod_{\tau=s}^{t-1}[1 + p(\tau)], \quad e_\alpha(t, s) = (1 + \alpha)^{t-s}, \quad e_\alpha(t, 0) = (1 + \alpha)^t$$

for $s, t \in \mathbf{Z}$ with $s < t$, where $\alpha \neq -1$ is a constant and $p\colon \mathbf{Z} \to \mathbf{R}$ is a sequence satisfying $p(t) \neq -1$ for all $t \in \mathbf{Z}$.

Theorem 2.5. *If* $p \in \mathcal{R}$ *and* $a, b, c \in \mathbf{T}$, *then*

$$\int_a^b p(t)e_p(c, \sigma(t))\Delta t = e_p(c, a) - e_p(c, b).$$

Theorem 2.6. *Let $t_1 \in \mathbf{T}^\kappa$ and $k\colon \mathbf{T} \times \mathbf{T}^\kappa \to \mathbf{R}$ be continuous at (t, t), where $t \in \mathbf{T}^\kappa$ with $t > t_1$. Assume that $k^\Delta(t, \cdot)$ is rd-continuous on $[t_1, \sigma(t)]$. If for any $\varepsilon > 0$, there exists a neighborhood U of t, independent of $\tau \in [t_1, \sigma(t)]$, such that*

$$|k(\sigma(t), \tau) - k(s, \tau) - k^\Delta(t, \tau)(\sigma(t) - s)| \le \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U,$$

where k^Δ denotes the derivative of k with respect to the first variable. Then

$$g(t) := \int_{t_1}^{t} k(t, \tau)\Delta\tau$$

implies

$$g^\Delta(t) = \int_{t_1}^{t} k^\Delta(t, \tau)\Delta\tau + k(\sigma(t), t).$$

The following theorem is a foundational result in the theory of dynamic inequalities. For convenience of notation, throughout this paper, we always assume that $t_0 \in \mathbf{T}$, $\mathbf{T}_0 = [t_0, \infty) \bigcap \mathbf{T}$.

Theorem 2.7. *Suppose $u, b \in C_{\mathrm{rd}}$, $a \in \mathcal{R}^+$. Then*

$$u^\Delta(t) \le a(t)u(t) + b(t) \quad \text{for all } t \in \mathbf{T}_0$$

implies

$$u(t) \le u(t_0)e_a(t, t_0) + \int_{t_0}^{t} e_a(t, \sigma(\tau))b(\tau)\Delta\tau \quad \text{for all } t \in \mathbf{T}_0.$$

3. Main results

In this section, we deal with dynamic inequalities on time scales. Throughout this section, let $p > 1$ be a real constant.

The following lemma, which is proved in [12], is useful in our main results.

Lemma 3.1. *Assume that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$x^{\frac{1}{p}} y^{\frac{1}{q}} \le \frac{x}{p} + \frac{y}{q} \quad \text{for } x, y \in \mathbf{R}_+ = [0, \infty).$$

Theorem 3.1. *Assume that $u, a, b, g, h \in C_{\mathrm{rd}}$, u, a, b, g, h are nonnegative. Then*

$$(u(t))^p \le a(t) + b(t) \int_{t_0}^{t} [g(\tau)(u(\tau))^p + h(\tau)u(\tau)]\Delta\tau \quad \text{for all } t \in \mathbf{T}_0 \tag{3.1}$$

which implies

$$u(t) \le \left\{a(t) + b(t) \int_{t_0}^{t} \left[a(\tau)g(\tau) + h(\tau)\left(\frac{p - 1 + a(\tau)}{p}\right)\right] e_{bm}(t, \sigma(\tau))\Delta\tau\right\}^{\frac{1}{p}} \quad \text{for all } t \in \mathbf{T}_0, \tag{3.2}$$

where

$$m(t) = g(t) + \frac{h(t)}{p}. \tag{3.3}$$

Proof. Define a function $z(t)$ by

$$z(t) = \int_{t_0}^{t} [g(\tau)(u(\tau))^p + h(\tau)u(\tau)]\Delta\tau \quad \text{for all } t \in \mathbf{T}_0. \tag{3.4}$$

Then $z(t_0) = 0$ and (3.1) can be restated as

$$(u(t))^p \leq a(t) + b(t)z(t) \quad \text{for all } t \in \mathbf{T}_0. \tag{3.5}$$

Using Lemma 3.1, from (3.5), we easily obtain

$$\begin{aligned} u(t) &\leq (a(t) + b(t)z(t))^{\frac{1}{p}}(1)^{\frac{p-1}{p}} \\ &\leq \frac{a(t)}{p} + \frac{b(t)}{p}z(t) + \frac{p-1}{p} \quad \text{for all } t \in \mathbf{T}_0. \end{aligned} \tag{3.6}$$

Combining (3.4)–(3.6), we get

$$\begin{aligned} z^{\Delta}(t) &= g(t)(u(t))^p + h(t)u(t) \\ &\leq g(t)[a(t) + b(t)z(t)] + h(t)\left(\frac{p-1+a(t)}{p} + \frac{b(t)}{p}z(t)\right) \\ &= \left[a(t)g(t) + \frac{a(t)+p-1}{p}h(t)\right] + b(t)m(t)z(t) \quad \text{for all } t \in \mathbf{T}_0, \end{aligned} \tag{3.7}$$

where $m(t)$ is defined as in (3.3).

Using Theorem 2.7 and noting $z(t_0) = 0$, from (3.7) we obtain

$$z(t) \leq \int_{t_0}^{t} \left[a(\tau)g(\tau) + h(\tau)\left(\frac{p-1+a(\tau)}{p}\right)\right] e_{bm}(t, \sigma(\tau))\Delta\tau \quad \text{for all } t \in \mathbf{T}_0. \tag{3.8}$$

Clearly, the desired inequality (3.2) follows from (3.5) and (3.8). The proof is complete.

Remark 3.1. If $\mathbf{T} = \mathbf{R}$, then the inequality established in Theorem 3.1 reduces to the inequality established by Pachpatte in Theorem 1(a_1) of [13]. Letting $\mathbf{T} = \mathbf{Z}$, from Theorem 3.1, we easily obtain Theorem 3(c_1) in [13].

COROLLARY 3.1

Assume that $u, h \in C_{\text{rd}}$, $u, h \geq 0$. *If* $\beta \geq 0$ *is a real constant*, *then*

$$(u(t))^p \leq \beta + \int_{t_0}^{t} h(\tau)u(\tau)\Delta\tau \quad \textit{for all } t \in \mathbf{T}_0 \tag{3.9}$$

implies

$$u(t) \leq \{(p-1+\beta)e_{\bar{h}}(t, t_0) - (p-1)\}^{1/p} \quad \textit{for all } t \in \mathbf{T}_0, \tag{3.10}$$

where

$$\bar{h}(t) = \frac{h(t)}{p}. \tag{3.11}$$

Proof. Using Theorem 3.1, it follows from (3.9) that

$$\begin{aligned} u(t) &\leq \left\{\beta + \int_{t_0}^{t} h(\tau)\frac{p-1+\beta}{p} e_{h/p}(t,\sigma(\tau))\Delta\tau\right\}^{1/p} \\ &= \left\{\beta + (p-1+\beta)\int_{t_0}^{t} \frac{h(\tau)}{p} e_{h/p}(t,\sigma(\tau))\Delta\tau\right\}^{1/p} \\ &= \{\beta + (p-1+\beta)[e_{\bar{h}}(t,t_0) - e_{\bar{h}}(t,t)]\}^{1/p} \\ &= \{\beta + (p-1+\beta)e_{\bar{h}}(t,t_0) - p + 1 - \beta\}^{1/p} \\ &= \{(p-1+\beta)e_{\bar{h}}(t,t_0) - (p-1)\}^{1/p} \quad \text{for all } t \in \mathbf{T}_0, \end{aligned}$$

where the second equation holds because of Theorem 2.5, and the third equation holds because of Theorem 2.4(i). This completes the proof.

Using Corollary 3.1, it is not difficult to obtain the following result.

COROLLARY 3.2

Let $k > 0$ and $\mathbf{T} = k\mathbf{Z} \bigcap [0, \infty)$. If u is a nonnegative function defined on $\mathbf{T}$, $\beta \geq 0$ and $\gamma > 0$ are real constants. Then

$$(u(t))^p \leq \beta + k\gamma \sum_{\tau=0}^{\frac{t}{k}-1} u(k\tau) \quad \textit{for all } t \in \mathbf{T} \tag{3.12}$$

implies

$$u(t) \leq \left\{(p-1+\beta)\left(1 + \frac{k\gamma}{p}\right)^{t/k} - (p-1)\right\}^{1/p} \quad \textit{for all } t \in \mathbf{T}. \tag{3.13}$$

Theorem 3.2. *Assume that $u, a, b, g, h \in C_{\mathrm{rd}}$, u, a, b, g, h are nonnegative. If $k(t,s)$ is defined as in Theorem 2.6 such that $k(\sigma(t), t) \geq 0$ and $k^{\Delta}(t,s) \geq 0$ for $t, s \in \mathbf{T}$ with $s \leq t$, then*

$$(u(t))^p \leq a(t) + b(t)\int_{t_0}^{t} k(t,\tau)[g(\tau)(u(\tau))^p + h(\tau)u(\tau)]\Delta\tau \quad \textit{for all } t \in \mathbf{T}_0 \tag{3.14}$$

implies

$$u(t) \leq \left\{a(t) + b(t)\int_{t_0}^{t} e_A(t,\sigma(\tau))B(\tau)\Delta\tau\right\}^{1/p} \quad \textit{for all } t \in \mathbf{T}_0, \tag{3.15}$$

where

$$A(t) = k(\sigma(t),t)b(t)\left(g(t) + \frac{h(t)}{p}\right) + \int_{t_0}^{t} k^{\Delta}(t,\tau)b(\tau)\left(g(\tau) + \frac{h(\tau)}{p}\right)\Delta\tau \tag{3.16}$$

and

$$B(t) = k(\sigma(t), t)\left[a(t)g(t) + h(t)\left(\frac{p-1+a(t)}{p}\right)\right] + \int_{t_0}^{t} k^{\Delta}(t,\tau)\left[a(\tau)g(\tau) + h(\tau)\left(\frac{p-1+a(\tau)}{p}\right)\right]\Delta\tau \tag{3.17}$$

for all $t \in \mathbf{T}_0$.

Proof. Define a function $z(t)$ by

$$z(t) = \int_{t_0}^{t} k(t,\tau)[g(\tau)(u(\tau))^p + h(\tau)u(\tau)]\Delta\tau \quad \text{for all } t \in \mathbf{T}_0. \tag{3.18}$$

Then $z(t_0) = 0$. As in the proof of Theorem 3.1, we easily obtain (3.5) and (3.6). Using Theorem 2.5 and combining (3.18), (3.5) and (3.6), we have

$$\begin{aligned}
z^{\Delta}(t) &= k(\sigma(t), t)[g(t)(u(t))^p + h(t)u(t)] + \int_{t_0}^{t} k^{\Delta}(t,\tau)[g(\tau)(u(\tau))^p + h(\tau)u(\tau)]\Delta\tau \\
&\le k(\sigma(t), t)\left[a(t)g(t) + h(t)\left(\frac{p-1}{p} + \frac{a(t)}{p}\right) + b(t)\left(g(t) + \frac{h(t)}{p}\right)z(t)\right] \\
&\quad + \int_{t_0}^{t} k^{\Delta}(t,\tau)\left[a(\tau)g(\tau) + h(\tau)\left(\frac{p-1}{p} + \frac{a(\tau)}{p}\right)\right. \\
&\quad \left. + b(\tau)\left(g(\tau) + \frac{h(\tau)}{p}\right)z(\tau)\right]\Delta\tau \\
&\le \left[k(\sigma(t), t)b(t)\left(g(t) + \frac{h(t)}{p}\right) + \int_{t_0}^{t} k^{\Delta}(t,\tau)b(\tau)\left(g(\tau) + \frac{h(\tau)}{p}\right)\Delta\tau\right]z(t) \\
&\quad + k(\sigma(t), t)\left[a(t)g(t) + h(t)\left(\frac{p-1}{p} + \frac{a(t)}{p}\right)\right] \\
&\quad + \int_{t_0}^{t} k^{\Delta}(t,\tau)\left[a(\tau)g(\tau) + h(\tau)\left(\frac{p-1}{p} + \frac{a(\tau)}{p}\right)\right]\Delta\tau \\
&= A(t)z(t) + B(t) \quad \text{for all } t \in \mathbf{T}_0.
\end{aligned}$$

Therefore, using Theorem 2.7 and noting $z(t_0) = 0$, we get

$$z(t) \le \int_{t_0}^{t} e_A(t, \sigma(\tau))B(\tau)\Delta\tau \quad \text{for all } t \in \mathbf{T}_0. \tag{3.19}$$

It is easy to see that the desired inequality (3.15) follows from (3.5) and (3.19). The proof of Theorem 3.2 is complete.

Remark 3.2. Clearly, if $\mathbf{T} = \mathbf{R}$, then the inequality established in Theorem 3.2 reduces to the inequality established by Pachpatte in Theorem 1(a_3) of [13]. Letting $\mathbf{T} = \mathbf{Z}$ in Theorem 3.2, we easily obtain Theorem 3(c_3) in [13].

COROLLARY 3.3

Suppose that $u(t)$, $a(t)$ and $k(t,s)$ are defined as in Theorem 3.2. If $a(t)$ is nondecreasing for all $t \in \mathbf{T}_0$, then

$$(u(t))^p \le a(t) + \int_{t_0}^{t} k(t,\tau)u(\tau)\Delta\tau \quad \textit{for all} \;\; t \in \mathbf{T}_0 \tag{3.20}$$

implies

$$u(t) \le \{[p-1+a(t)]e_{\bar{A}}(t,t_0) - (p-1)\}^{1/p} \quad \textit{for all} \;\; t \in \mathbf{T}_0, \tag{3.21}$$

where

$$\bar{A}(t) = \frac{1}{p}\left(k(\sigma(t),t) + \int_{t_0}^{t} k^{\Delta}(t,\tau)\Delta\tau\right) \quad \textit{for all} \;\; t \in \mathbf{T}_0. \tag{3.22}$$

Proof. Letting $b(t) = 1$, $g(t) = 0$ and $h(t) = 1$ in Theorem 3.2, we obtain

$$A(t) = \frac{1}{p}\left(k(\sigma(t),t) + \int_{t_0}^{t} k^{\Delta}(t,\tau)\Delta\tau\right) = \bar{A}(t) \quad \text{for all} \;\; t \in \mathbf{T}_0 \tag{3.23}$$

and

$$\begin{aligned} B(t) &= \frac{1}{p}\left\{k(\sigma(t),t)[p-1+a(t)] + \int_{t_0}^{t} k^{\Delta}(t,\tau)[p-1+a(\tau)]\Delta\tau\right\} \\ &\le \frac{p-1+a(t)}{p}\left\{k(\sigma(t),t) + \int_{t_0}^{t} k^{\Delta}(t,\tau)\Delta\tau\right\} \\ &= [p-1+a(t)]\bar{A}(t) \quad \text{for all} \;\; t \in \mathbf{T}_0, \end{aligned} \tag{3.24}$$

where the inequality holds because $a(t)$ is nondecreasing for all $t \in \mathbf{T}_0$. Therefore, by Theorem 3.2, using (3.23) and (3.24), we easily get

$$\begin{aligned} u(t) &\le \left\{a(t) + \int_{t_0}^{t} e_A(t,\sigma(\tau))B(\tau)\Delta\tau\right\}^{1/p} \\ &\le \left\{a(t) + \int_{t_0}^{t} e_{\bar{A}}(t,\sigma(\tau))[p-1+a(\tau)]\bar{A}(\tau)\Delta\tau\right\}^{1/p} \\ &\le \left\{a(t) + [p-1+a(t)]\int_{t_0}^{t} e_{\bar{A}}(t,\sigma(\tau))\bar{A}(\tau)\Delta\tau\right\}^{1/p} \\ &= \{a(t) + [p-1+a(t)][e_{\bar{A}}(t,t_0) - e_{\bar{A}}(t,t)]\}^{1/p} \\ &= \{[p-1+a(t)]e_{\bar{A}}(t,t_0) - (p-1)\}^{1/p} \quad \text{for all} \;\; t \in \mathbf{T}_0. \end{aligned}$$

The proof of Corollary 3.3 is complete.

4. An application

In this section, we present an example to illustrate our main results.

Example 4.1. Consider the dynamic equation

$$(u^p(t))^\Delta = M(t, u(t)),\ t \in \mathbf{T}_0, \tag{4.1}$$

where $p > 1$ is a constant, $M\colon \mathbf{T}_0 \times \mathbf{R} \to \mathbf{R}$ is a continuous function.

Assume that

$$|M(t, u(t))| \le h(t)|u(t)|, \tag{4.2}$$

where $h(t)$ is as defined in Corollary 3.1. If $u(t)$ is a solution of eq. (4.1), then

$$|u(t)| \le \{(p - 1 + |C|)e_{\bar{h}}(t, t_0) - (p - 1)\}^{1/p} \quad \text{for all } \ t \in \mathbf{T}_0, \tag{4.3}$$

where

$$C = u^p(t_0),\ \bar{h}(t) = \frac{h(t)}{p}. \tag{4.4}$$

In fact, the solution $u(t)$ of eq. (4.1) satisfies the following equivalent equation:

$$u^p(t) = C + \int_{t_0}^{t} M(\tau, u(\tau))\Delta\tau, \quad t \in \mathbf{T}_0. \tag{4.5}$$

Using assumption (4.2), we have

$$|u^p(t)| \le |C| + \int_{t_0}^{t} h(\tau)|u(\tau)|\Delta\tau, \quad t \in \mathbf{T}_0. \tag{4.6}$$

Now a suitable application of Corollary 3.1 to (4.6) yields (4.3).

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