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Central limit theorems for a class of irreducible multicolor urn models

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Abstract. We take a unified approach to central limit theorems for a class of irreducible multicolor urn models with constant replacement matrix. Depending on the eigenvalue, we consider appropriate linear combinations of the number of balls of different colors. Then under appropriate norming the multivariate distribution of the weak limits of these linear combinations is obtained and independence and dependence issues are investigated. Our approach consists of looking at the problem from the viewpoint of recursive equations.

Keywords. Central limit theorem; Markov chains; martingale; urn models.

1. Introduction

In this article we are going to study irreducible multicolor urn models. As an illustrative example we first start with an irreducible four color urn model, describe the evolution and state the results. This is done in the next three paragraphs. We will then proceed to generalize the results to the irreducible multicolor situation.

Consider a four-color urn model in which the replacement matrix is actually a stochastic matrix $\mathbf{R}$ in the manner of Gouet [9]. That is, we start with one ball of any color, which is the 0-th trial. Let $\mathbf{W}_n$ denote the column vector of the number of balls of the four colors up to the n-th trial, where the components of $\mathbf{W}_n$ are nonnegative real numbers. Then a color is observed by random sampling from a multinomial distribution with probabilities $(1/(n+1))\mathbf{W}_n$. Depending on the color that is observed, the corresponding row of $\mathbf{R}$ is added to $\mathbf{W}'_n$ and this gives $\mathbf{W}'_{n+1}$. A special case of the main theorem of Gouet [9] is that if the stochastic matrix $\mathbf{R}$ is irreducible, then $(1/(n+1))\mathbf{W}'_n$ converges a.s. to the stationary distribution π of the irreducible stochastic matrix $\mathbf{R}$ (it should be carefully noted that the multicolor urn model is vastly different from the Markov chain evolving according to the transition matrix equal to the stochastic matrix $\mathbf{R}$). Suppose the nonprincipal eigenvalues of $\mathbf{R}$ satisfy $\lambda_1 < 1/2,\ \lambda_2 = 1/2,\ \lambda_3 > 1/2$ respectively, which are assumed to be real (and hence lie in $(-1, 1)$), and $\xi_1,\ \xi_2,\ \xi_3$ be the corresponding eigenvectors. Using $\pi\xi_i = \pi\mathbf{R}\xi_i = \lambda_i\pi\xi_i$ it is seen that $(1/(n+1))\mathbf{W}'_n\xi_i \to 0$. Thus central limit theorems are the next interesting statistical results.

In this article we consider the joint limiting distribution of (X_n, Y_n, Z_n) where

$$X_n = \frac{\mathbf{W}'_n\xi_1}{\sqrt{n}},\quad Y_n = \frac{\mathbf{W}'_n\xi_2}{\sqrt{n\log n}},\quad Z_n = \frac{\mathbf{W}'_n\xi_3}{\Pi_0^{n-1}\left(1+\frac{\lambda_3}{j+1}\right)}. \tag{1}$$

Special cases of this result are known from Freedman [7], Gouet [8], Smythe [11] and Bai and Hu [5]. Freedman [7], as well as Gouet [8], consider two color urn, so that there is only one eigenvector and the corresponding nonprincipal eigenvalue can be one of the three types. Smythe [11] considers multicolor urn, but all the nonprincipal eigenvalues (or their real parts) are assumed to be less than $1/2$. Recently Bai and Hu [5] have considered the case when all the nonprincipal eigenvalues (or their real parts) are less than or equal to $1/2$. In this article we consider the joint limit when eigenvalues of all the three types occur. Analogous results for multitype branching processes are known, for example from Athreya [1, 2], and the recent paper by Janson [10] contains functional limit theorems as well as an extensive discussion of related results and applications. The limit theorems for urn models can be derived through an embedding of the urn model into a branching process (the Athreya–Karlin embedding) and applying the limit theorems of branching processes in the above-mentioned articles and the references therein. In particular, Athreya and Karlin [3], Athreya and Ney [4] and Janson [10] employ this embedding procedure and derive the results for urn models in various forms. We take a fresh look at this central limit problem for urn models directly through recursive equations with diagonal drift. The interesting feature is, which will be clear from the proof, the differences in the rates of the differences of the three components. Thus we get a direct Markov chain analysis of the problem without invoking the techniques from branching processes. Also the recursive equations with diagonal drift and multiple rates may be of independent interest since the rates $1/\sqrt{n}$ and $1/\sqrt{n \log n}$, particular to urn models, may be replaced with other rates. The main feature of these rates that we use is that an appropriate ratio, like $\sqrt{n_0}/\sqrt{n_0 \log n_0}$, goes to zero as n_0 goes to infinity (see for example (13) and (14)).

For the above four color set up the main result is as follows.

Theorem 1.1. *(X_n, Y_n, Z_n) converges in distribution to (X, Y, Z) where X, Y, Z are independent, X and Y are (independent) normals with zero means. The convergence of Z_n to Z is also in the almost sure sense.*

The variances of X and Y are identified in the proof. The proof indicates $EZ = 0$ and gives some idea about the variance, but does not say anything about the distribution of Z. Some features of this Z in a two-color case are discussed in Freedman [7]. For the above urn model, we also need to point out the connection of Theorem 1.1 with a class of results in the literature. These results consider norming the vector $(\mathbf{W}_n - E\mathbf{W}_n)$ and not the linear combinations from the eigenvectors. Now the eigenvectors ξ_1, ξ_2, ξ_3 and the principal eigenvector $\mathbf{u} = (1, 1, 1, 1)'$ span $\mathbb{R}^4$, so that any linear combination can be expressed in terms of them. But $\mathbf{W}_n'\mathbf{u} = n+1$, so its effect cancels out after the expectation is subtracted and we are left with the linear combinations corresponding to ξ_1, ξ_2, ξ_3. These results in the literature divide $(\mathbf{W}_n - E\mathbf{W}_n)$ by the largest rate, and in the case the real part of the nonprincipal eigenvalues is less than or equal to 1/2 (actually the rate in that case may be different from $\sqrt{n \log n}$ as will be clear in the later sections) which derive asymptotic normality (see for e.g. [5]).

We have stated the theorem for the four color model for the sake of notational simplicity in the proof. The theorem also extends to situations (with more than four colors) where there are more than one eigenvalue(s) of any one or more of the three types. These extensions involve the same technique, but require more calculations related to the Jordan form of the replacement matrix. So we have sketched some of these extensions in separate sections. These sections discuss the main theorem in increasing generality along with development of suitable notation, and we have indicated the generalizations inside these sections. First, all the eigenvalues are considered to be real, the Jordan form thus involves only real vectors. Next, the eigenvalues can be complex, so the Jordan form involves complex vectors and we deal with the real and imaginary parts of these vectors. Another interesting feature of these later sections dealing with the Jordan form is the role of nilpotent and rotation matrices. The final result is given as Theorem 5.1 along with subsequent discussion of asymptotic mixed normality for $\mathrm{Re}(\lambda) = 1/2$.

The proof of Theorem 1.1 for the above four color set up is given in the next section. It employs an iteration technique involving conditional characteristic functions (an example of these iterations occurs in Example 2, pp. 79–80 of [6]). We have written this proof in detail, however the proofs for the generalizations of the main theorem are only sketched in later sections as the ideas are the same.

2. Proof of Theorem 1.1

A quick guide through the proof is through equations (3), (10), (11), (13), (14), (17), (18) and (19) and the discussions following them.

We first collect a few computational details. The column vector of the indicator functions of balls of different colors obtained from the $n+1$-st trial is denoted by χ_{n+1}. It is clear that $E\{\chi_{n+1}|\mathcal{F}_n\} = (1/(n+1))\mathbf{W}_n$, where $\mathcal{F}_n$ denotes the σ-field of observations up to the n-th trial. This notation leads to

$$\mathbf{W}'_{n+1}\xi_i = \mathbf{W}'_n\xi_i + \chi'_{n+1}\mathbf{R}\xi_i = \mathbf{W}'_n\xi_i + \lambda_i\chi'_{n+1}\xi_i. \tag{2}$$

For the purpose of iteration we shall use a decomposition of the components of the Markov chain $(X_{n+1}, Y_{n+1}, Z_{n+1})$ illustrated with the first component as follows:

$$X_{n+1} = E\{X_{n+1}|\mathcal{F}_n\} + (X_{n+1} - E\{X_{n+1}|\mathcal{F}_n\}).$$

The first term will be expressed in terms of X_n and the second term is the martingale difference that will play an important role in our proof in analogy with the calculations for the central limit theorem for i.i.d. random variables.

To write the first term in terms of X_n (Y_n, Z_n respectively) we shall use the following approximations

$$(1+1/n)^{-1/2} = 1 - \frac{1}{2n} + O\left(\frac{1}{n^2}\right),$$

$$\frac{\log n}{\log(n+1)} = \frac{\log n}{\log n + 1/n + O(1/n^2)} = \frac{1}{1 + (1/n\log n) + O(1/n^2\log n)},$$

$$\sqrt{\frac{n\log n}{(n+1)\log(n+1)}} = \left\{1 - \frac{1}{2n} + O\left(\frac{1}{n^2}\right)\right\}\left\{1 - \frac{1}{2n\log n} + O\left(\frac{1}{n^2}\right)\right\} = 1 - \frac{1}{2n} - \frac{1}{2n\log n} + O\left(\frac{1}{n^2}\right),$$

$$\Pi_0^{n-1}(1 + \lambda_3/(j+1)) \sim \frac{n^{\lambda_3}}{\Gamma(\lambda_3+1)}.$$

Using these and the conditional expectation of (2) it follows that:

$$E\{X_{n+1}|\mathcal{F}_n\} = X_n\left(1 - \frac{1/2 - \lambda_1}{n}\right) + X_n O(1/n^2),$$

$$E\{Y_{n+1}|\mathcal{F}_n\} = Y_n\left(1 - \frac{1}{2n\log n}\right) + Y_n O(1/n^2),$$

$$E\{Z_{n+1}|\mathcal{F}_n\} = Z_n, \tag{3}$$

the second of which crucially uses $\lambda_2 = 1/2$. Now let us look at the martingale difference terms which are

$$M_{1,n+1} = X_{n+1} - E\{X_{n+1}|\mathcal{F}_n\} = \lambda_1 \frac{\chi'_{n+1}\xi_1}{\sqrt{n+1}} - \frac{\lambda_1}{n+1}\sqrt{\frac{n}{n+1}}X_n,$$

$$M_{2,n+1} = Y_{n+1} - E\{Y_{n+1}|\mathcal{F}_n\} = \lambda_2 \frac{\chi'_{n+1}\xi_2}{\sqrt{(n+1)\log(n+1)}} - \frac{\lambda_2}{n+1}Y_n\sqrt{\frac{n\log n}{(n+1)\log(n+1)}},$$

$$M_{3,n+1} = Z_{n+1} - E\{Z_{n+1}|\mathcal{F}_n\} = \lambda_3 \frac{\chi'_{n+1}\xi_3}{\Pi_0^n\left(1+\frac{\lambda_3}{j+1}\right)} - \frac{\frac{\lambda_3}{n+1}}{1+\frac{\lambda_3}{n+1}}Z_n. \tag{4}$$

It will be seen that the part involving $\chi'_{n+1}\xi_i$ plays a significant role in the second moment calculations.

2.1 *Main idea of the proof*

Now we are ready to start the proof of Theorem 1.1.

Step A. Using (3) and the inequality $|e^{ix} - 1| \le |x|$ for real number x, and remembering that $|\mathbf{W}'_n\xi_i| \le cn$, so that $X_n/\sqrt{n}$, $Y_n/\sqrt{n}$, $Z_n/n^{1-\lambda_3}$ are bounded, we can expand $e^{it_1 X_n O(1/n^2) + it_2 Y_n O(1/n^2)}$ to get

$$\left| E\{e^{i(t_1X_{n+1}+t_2Y_{n+1}+t_3Z_{n+1})}|\mathcal{F}_n\} - e^{i\{t_1\left(1-\frac{\frac{1}{2}-\lambda_1}{n}\right)X_n + t_2\left(1-\frac{1}{2n\log n}\right)Y_n + t_3Z_n\}} E\{e^{i(t_1M_{1,n+1}+t_2M_{2,n+1}+t_3M_{3,n+1})}|\mathcal{F}_n\}\right|$$

$$\le (|t_1||X_n| + |t_2||Y_n|)O(1/n^2)$$

$$\le \text{ const } \frac{1}{n^{3/2}}, \tag{5}$$

for n sufficiently large, say $n \ge n_0$.

Step B. Now we want to approximate $E\{e^{i(t_1M_{1,n+1}+t_2M_{2,n+1}+t_3M_{3,n+1})}|\mathcal{F}_n\}$ by

$$e^{-\frac{t_1^2}{2}\lambda_1^2\frac{\langle\pi,\xi_1^2\rangle}{n+1}-\frac{t_2^2}{2}\lambda_2^2\frac{\langle\pi,\xi_2^2\rangle}{(n+1)\log(n+1)}}. \tag{6}$$

We use the inequality $\left|e^{ix}-1-ix+\frac{1}{2}x^2\right| \le \text{const}|x|^3$ along with the observation that the martingale differences of (4) are bounded by $\text{const}/\sqrt{n}$, $\text{const}/\sqrt{n\log n}$ and $\text{const}/n^{\lambda_3}$ respectively (we approximate $\Pi_0^n(1+\lambda_3/(i+1)) \sim n^{\lambda_3}$). This gives

$$\begin{aligned}&\Big|E\{e^{i(t_1M_{1,n+1}+t_2M_{2,n+1}+t_3M_{3,n+1})}|\mathcal{F}_n\}\\ &\quad-\Big(1-\frac{1}{2}E\{(t_1^2M_{1,n+1}^2+t_2^2M_{2,n+1}^2+t_3^2M_{3,n+1}^2\\ &\quad+t_1t_2M_{1,n+1}M_{2,n+1}+t_1t_3M_{1,n+1}M_{3,n+1}+t_2t_3M_{2,n+1}M_{3,n+1})|\mathcal{F}_n\}\Big)\Big|\\ &\le \text{const}\,\frac{1}{n^{3/2}}\end{aligned} \tag{7}$$

for $n \ge n_0$.

To achieve (6) a detailed study of the terms of (7) is necessary. We have given the complete formulas, but to follow the proof one can start from the argument following (8) and come back to (8) as necessary. We denote by $\xi_i\xi_j$ the vector whose components are products of the corresponding components of ξ_i and ξ_j, and similarly ξ_i^2 denotes the vector whose components are products of the corresponding components of ξ_i and ξ_i. Remembering that χ_{n+1} consists of indicator functions of observations of balls of different colors, we get

$$\begin{aligned}E(M_{1,n+1}^2|\mathcal{F}_n) &= \lambda_1^2\frac{\langle\pi,\xi_1^2\rangle}{n+1}\\ &\quad+\left\{\lambda_1^2\frac{\left\langle\frac{\mathbf{W}_n'}{n+1}-\pi,\xi_1^2\right\rangle}{n+1}-\lambda_1^2\frac{n}{(n+1)^3}X_n^2\right\},\end{aligned}$$

$$\begin{aligned}E(M_{2,n+1}^2|\mathcal{F}_n) &= \lambda_2^2\frac{\langle\pi,\xi_2^2\rangle}{(n+1)\log(n+1)}\\ &\quad+\left\{\lambda_2^2\frac{\left\langle\frac{\mathbf{W}_n'}{n+1}-\pi,\xi_2^2\right\rangle}{(n+1)\log(n+1)}-\lambda_2^2\frac{n\log n}{(n+1)^3\log(n+1)}Y_n^2\right\},\end{aligned}$$

$$E(M_{3,n+1}^2|\mathcal{F}_n) = \lambda_3^2 \frac{\langle \pi, \xi_3^2 \rangle}{\left(\Pi_0^n \left(1 + \frac{\lambda_3}{j+1}\right)\right)^2}$$

$$+ \left\{ \lambda_3^2 \frac{\left\langle \frac{\mathbf{W}_n'}{n+1} - \pi, \xi_3^2 \right\rangle}{\left(\Pi_0^n \left(1 + \frac{\lambda_3}{j+1}\right)\right)^2} - \frac{\lambda_3^2}{(n+1)^2 \left(1 + \frac{\lambda_3}{n+1}\right)^2} Z_n^2 \right\},$$

$$E(M_{1,n+1} M_{2,n+1}|\mathcal{F}_n) = \lambda_1 \lambda_2 \frac{\langle \pi, \xi_1 \xi_2 \rangle}{\sqrt{n+1}\sqrt{(n+1)\log(n+1)}}$$

$$+ \left\{ \lambda_1 \lambda_2 \frac{\left\langle \frac{\mathbf{W}_n'}{n+1} - \pi, \xi_1 \xi_2 \right\rangle}{\sqrt{n+1}\sqrt{(n+1)\log(n+1)}} \right.$$

$$\left. - \lambda_1 \lambda_2 \frac{n\sqrt{\log n}}{(n+1)^3 \sqrt{\log(n+1)}} X_n Y_n \right\},$$

$$E(M_{1,n+1} M_{3,n+1}|\mathcal{F}_n) = \lambda_1 \lambda_3 \frac{\langle \pi, \xi_1 \xi_3 \rangle}{\sqrt{n+1} \left(\Pi_0^n \left(1 + \frac{\lambda_3}{j+1}\right)\right)}$$

$$+ \left\{ \lambda_1 \lambda_3 \frac{\left\langle \frac{\mathbf{W}_n'}{n+1} - \pi, \xi_1 \xi_3 \right\rangle}{\sqrt{n+1} \left(\Pi_0^n \left(1 + \frac{\lambda_3}{j+1}\right)\right)} \right.$$

$$\left. - \lambda_1 \lambda_3 \sqrt{\frac{n}{n+1}} \frac{\frac{1}{n+1}}{(n+1)\left(1 + \frac{\lambda_3}{n+1}\right)} X_n Z_n \right\},$$

$$E(M_{2,n+1} M_{3,n+1}|\mathcal{F}_n) = \lambda_2 \lambda_3 \frac{\langle \pi, \xi_2 \xi_3 \rangle}{\sqrt{(n+1)\log(n+1)} \left(\Pi_0^n \left(1 + \frac{\lambda_3}{j+1}\right)\right)}$$

$$+ \left\{ \lambda_2 \lambda_3 \frac{\left\langle \frac{\mathbf{W}_n'}{n+1} - \pi, \xi_2 \xi_3 \right\rangle}{\sqrt{(n+1)\log(n+1)} \left(\Pi_0^n \left(1 + \frac{\lambda_3}{j+1}\right)\right)} \right.$$

$$\left. - \lambda_2 \lambda_3 \sqrt{\frac{n \log n}{(n+1)\log(n+1)}} \frac{\frac{1}{n+1}}{(n+1)\left(1 + \frac{\lambda_3}{n+1}\right)} Y_n Z_n \right\}. \quad (8)$$

If $\sigma^2 \geq 0$, then we know that $\left|1 - \frac{\sigma^2}{2} - e^{-\sigma^2/2}\right| \leq \text{const}\,\sigma^4$. Using this on the constant terms of the first two equations of (8) we get

$$\left|1 - \frac{1}{2}t_1^2\lambda_1^2\frac{\langle\pi, \xi_1^2\rangle}{n+1} - \frac{1}{2}t_2^2\lambda_2^2\frac{\langle\pi, \xi_2^2\rangle}{(n+1)\log(n+1)} - e^{-\frac{t_1^2}{2}\lambda_1^2\frac{\langle\pi,\xi_1^2\rangle}{n+1} - \frac{t_2^2}{2}\lambda_2^2\frac{\langle\pi,\xi_2^2\rangle}{(n+1)\log(n+1)}}\right| \leq \text{const}\frac{1}{(n+1)^2}. \tag{9}$$

Step C. Combining (5), (7) and (9) we get the following basic inequality:

$$\left|E\{e^{i(t_1X_{n+1}+t_2Y_{n+1}+t_3Z_{n+1})}|\mathcal{F}_n\} - e^{i\left\{t_1\left(1-\frac{\frac{1}{2}-\lambda_1}{n}\right)X_n+t_2\left(1-\frac{1}{2n\log n}\right)Y_n+t_3Z_n\right\}} \times e^{-\frac{t_1^2}{2}\lambda_1^2\frac{\langle\pi,\xi_1^2\rangle}{n+1} - \frac{t_2^2}{2}\lambda_2^2\frac{\langle\pi,\xi_2^2\rangle}{(n+1)\log(n+1)}}\right| \leq \text{const}\frac{1}{n^{3/2}} + R_n, \tag{10}$$

where we use R_n to denote the sum of the other constant terms and random terms from the right of (8) which have not been used in (9) (this is also multiplied by exponentials of imaginary quantities, but those are bounded by 1 and will not make any difference). We also use the notation

$$C_n = -\frac{t_1^2}{2}\lambda_1^2\frac{\langle\pi, \xi_1^2\rangle}{n+1} - \frac{t_2^2}{2}\lambda_2^2\frac{\langle\pi, \xi_2^2\rangle}{(n+1)\log(n+1)}.$$

We then condition again on $\mathcal{F}_{n-1}$ and iterate backwards. While doing so, in the exponent the coefficients of t_i change as above, we get a sum of C_{n-j}'s in the exponent, and following iteration of (10) on the right we get a sum of conditional expectations of R_n's and const $\sum_{n_0}^{n} 1/(j+1)^{3/2}$. Note that the iteration from $n+1$ to n has changed the coefficient of X_n and Y_n, and these are assumed to be incorporated in C_{n-1} and R_{n-1}, and so on. R_{n-j} also involves terms like $e^{C_{n-j+1}+\cdots+C_n}$, but it will be seen from Steps 1 and 2 in the next section that these terms are bounded uniformly and will be absorbed in the 'const' term in (18). We should mention here that the constant term in (5), (7) and (9) and finally (10) can be taken independently of this iteration because during the iteration the coefficients of t_1 and t_2 decrease.

The main idea of the proof is to iterate the (conditional) characteristic function backwards up to a sufficiently large n_0, and first make $n \to \infty$. This will make the sum of C_n's independent of n_0, and the sum of the conditional expectations of the R_n's given $\mathcal{F}_{n_0}$ will be bounded by a random variable (which depends on the fixed n_0). Taking expectation of the conditional characteristic function we get the characteristic function. Then we let $n_0 \to \infty$, and a further argument gives us the characteristic function. Before we do this we provide a few ingredients of the proof in a separate subsection. However the reader may take a look at §2.3 at this point for an idea of the completion of the proof leading to the factorization of the characteristic function.

2.2 *Important limits and estimates*

So *assume we have iterated backwards up to a sufficiently large n_0*. For ease of exposition we divide the calculations into a few steps. In Step 1 we concentrate on the nonrandom terms corresponding to t_1^2 and t_2^2, which gives the form of the characteristic function corresponding to X_n and Y_n. In Step 2 we consider the other nonrandom terms, and in Step 3 we handle the random (second bracketed) terms. Steps 2 and 3 contribute to the sum of R_n's.

Step 1. The calculations here will go into C_n. They come from the first (nonrandom) terms of the first two equations on the right of (8). Because of the presence of the term $\left(1-\frac{\frac{1}{2}-\lambda_1}{n}\right)$ in the characteristic function, it is seen that after iterating backwards up to n_0, the (nonrandom part of the) coefficient of $-(1/2)t_1^2$ is

$$\sum_{n_0}^{n} f_{n-j+1}\lambda_1^2 \frac{\langle \pi, \xi_1^2\rangle}{j+1},$$

where

$$f_{n-j+1} = \Pi_{i=j+1}^{n}\left(1-\frac{\frac{1}{2}-\lambda_1}{i}\right)^2.$$

As $n\to\infty$, the above sum goes to

$$\lambda_1^2\langle \pi, \xi_1^2\rangle \int_0^\infty \mathrm{e}^{-(1-2\lambda_1)x}\mathrm{d}x. \tag{11}$$

This can be seen from the following calculation. The above sum is bounded by $\sum_1^n f_{n-j+1}\lambda_1^2\frac{\langle \pi,\xi_1^2\rangle}{j+1}$, and we can write

$$\sum_{1}^{n} f_{n-j+1}\frac{1}{j+1} = \sum_{1}^{n_0-1} f_{n-j+1}\frac{1}{j+1} + \sum_{n_0}^{n} f_{n-j+1}\frac{1}{j+1}.$$

Fixing n_0 sufficiently large as we make $n\to\infty$, the first sum on the right goes to zero, but the terms of the second sum after approximating the product by an exponential give

$$\lim_{n\to\infty}\sum_{j=n_0}^{n} \mathrm{e}^{-(1-2\lambda_1)\sum_{j+1}^{n}\frac{1}{i}}\frac{1}{j+1} = \int_0^\infty \mathrm{e}^{-(1-2\lambda_1)x}\mathrm{d}x.$$

Similarly because of the presence of $\left(1-\frac{1}{2n\log n}\right)$ in the characteristic function, after iterating backwards up to n_0, the (nonrandom part of the) coefficient of $-\frac{1}{2}t_2^2$ is

$$\sum_{n_0}^{n} g_{n-j+1}\lambda_2^2\frac{\langle \pi, \xi_2^2\rangle}{(j+1)\log(j+1)},$$

where

$$g_{n-j+1} = \Pi_{i=j+1}^{n}\left(1-\frac{1}{2i\log i}\right)^2.$$

As $n \to \infty$, the above sum clearly goes to

$$\lambda_2^2 \langle \pi, \xi_2^2 \rangle \int_0^\infty \mathrm{e}^{-x} \mathrm{d}x. \tag{12}$$

Thus, irrespective of n_0, the (nonrandom part of the) coefficients of $-\frac{1}{2}t_1^2$ and $-\frac{1}{2}t_2^2$ go to constants as $n \to \infty$. At this point note that as we made $n \to \infty$ the coefficient of X_{n_0} in the characteristic function $t_1\sqrt{f_{n-n_0+1}}$ goes to zero and similarly for the coefficient of Y_{n_0}, which is $t_1\sqrt{g_{n-n_0+1}}$. Thus, fixing n_0, as we let $n \to \infty$, the characteristic function does not have X_{n_0}, Y_{n_0} and the nonrandom part of the coefficients of $-\frac{1}{2}t_1^2$ and $-\frac{1}{2}t_2^2$ go to constants independent of n_0. This takes care of the sum of C_{n-j}'s, $j = n_0, n_0 + 1, \ldots, n$, as we make $n \to \infty$.

Step 2. The calculations here will go into the upper bound for the sum of R_n's. The (nonrandom part of the) coefficient of $-(1/2)t_1t_2$ is

$$\sum_{n_0}^{n} h_{n-j+1}\lambda_1\lambda_2 \frac{\langle \pi, \xi_1\xi_2 \rangle}{\sqrt{j+1}\sqrt{(j+1)\log(j+1)}}, \tag{13}$$

where

$$h_{n-j+1} = \Pi_{i=j+1}^{n}\left(1 - \frac{1}{2i\log i}\right)\left(1 - \frac{\frac{1}{2}-\lambda_1}{i}\right).$$

Clearly

$$h_{n-j+1} \le \Pi_{i=j+1}^{n}\left(1 - \frac{\frac{1}{2}-\lambda_1}{i}\right),$$

and combining the $\sqrt{j+1}$ of $\sqrt{(j+1)\log(j+1)}$ with the other $\sqrt{j+1}$, it is seen that the term (13) is less than

$$\frac{1}{\sqrt{\log(n_0+1)}} \sum_{n_0}^{n} \Pi_{i=j+1}^{n}\left(1 - \frac{\frac{1}{2}-\lambda_1}{i}\right)\lambda_1\lambda_2\langle \pi, \xi_1\xi_2 \rangle . \frac{1}{j+1},$$

which goes to

$$\frac{1}{\sqrt{\log(n_0+1)}}\lambda_1\lambda_2\langle \pi, \xi_1\xi_2 \rangle \int_0^\infty \mathrm{e}^{-(\frac{1}{2}-\lambda_1)x}\mathrm{d}x \tag{14}$$

as $n \to \infty$. Actually here in the expansion of $(1 - 1/(2i\log i))(1 - ((1/2) - \lambda_1)/i)$ the important contribution comes from $1 - ((1/2) - \lambda_1)/i$, which can later be compared with the comments following Theorem 5.1.

The coefficient of $-(1/2)t_1t_3$ is (we approximate $\Pi_0^j(1 + \lambda_3/(l+1)) \sim j^{\lambda_3}$),

$$\sum_{n_0}^{n} f_{n-j+1}\lambda_1\lambda_3 \frac{\langle \pi, \xi_1\xi_3 \rangle}{\sqrt{j+1}\,j^{\lambda_3}},$$

where

$$f_{n-j+1} = \Pi_{i=j+1}^{n}\left(1 - \frac{\frac{1}{2} - \lambda_1}{i}\right).$$

Following the argument of the previous paragraph, as we let $n \to \infty$ this coefficient is less than

$$\frac{1}{n_o^{\lambda_3 - 1/2}} \lambda_1 \lambda_3 \langle \pi, \xi_1 \xi_3 \rangle \int_0^\infty \mathrm{e}^{-\left(\frac{1}{2} - \lambda_1\right)x} \mathrm{d}x. \tag{15}$$

Similarly as $n \to \infty$, the (nonrandom part of the) coefficient of $-(1/2)t_2 t_3$ is less than

$$\frac{\sqrt{(n_0+1)\log(n_0+1)}}{n_o^{\lambda_3}} \lambda_2 \lambda_3 \langle \pi, \xi_2 \xi_3 \rangle \int_0^\infty \mathrm{e}^{-x/2} \mathrm{d}x. \tag{16}$$

Also note that when we iterate backwards the coefficient of Z_{n_0} is still t_3 and keeping n_0 fixed as we let $n \to \infty$ the (nonrandom part of the) coefficient of $-\frac{1}{2}t_3^2$ goes to

$$\sum_{n_0}^{\infty} \lambda_3^2 \frac{\langle \pi, \xi_3^2 \rangle}{(j+1)^{2\lambda_3}}. \tag{17}$$

Thus, fixing n_0, the sum of $-t_1 t_2$, $-t_1 t_3$, $-t_2 t_3$, $-\frac{1}{2}t_3^2$, multiplied by their respective (constant part of the) coefficients, is bounded by a constant F_{n_0} as we let $n \to \infty$. The exact form of F_{n_0} is easily obtained from (14), (15), (16) and (17), however for us the important observation will be $F_{n_0} \to 0$ as we later make $n_0 \to \infty$.

Step 3. The calculations here will go into the upper bound for the sum of R_n's. We now concentrate on the random terms. First note that

$$\sup_{n_0 \le n < \infty} \left\| \frac{\mathbf{W}_n'}{n+1} - \pi \right\|,$$

where $\|.\|$ denotes the maximum, is a bounded random variable that converges to 0 a.s. Also $X_n/\sqrt{n} = \mathbf{W}_n' \xi_1 / n$ is bounded by a constant and converges to 0 a.s. as $n_0 \to \infty$, hence the same holds for

$$\sup_{n_0 \le n < \infty} X_n^2 / n.$$

These two observations show that when we iterate backwards the random terms in the coefficient of $-t_1^2/2$ contribute a random variable less in absolute value than

$$\text{const} \left\{ \sup_{n_0 \le n < \infty} \left\| \frac{\mathbf{W}_n'}{n+1} - \pi \right\| + \sup_{n_0 \le n < \infty} X_n^2 \Big/ n \right\} \sum_{n_0}^{n} f_{n-j+1} \frac{1}{j+1}. \tag{18}$$

The 'const' term here is an upper bound for $\mathrm{e}^{C_{n-n_0} + \cdots + C_n}$ and all the terms in Step 2 are also to be multiplied by this. Recall that fixing n_0 as we make $n \to \infty$, the sum $\sum_{n_0}^{n} f_{n-j+1} \frac{1}{j+1}$ converges to an integral (see (11), so that the above sum is bounded by a constant for all n), showing that as we make $n \to \infty$ keeping n_0 fixed, the contribution of the random terms to the coefficient of $-t_1^2/2$ is bounded by a bounded random variable. This random variable is constant times the conditional expectation of the random term in (18) given by $\mathcal{F}_{n_0}$, and its expectation converges to 0 using the dominated convergence theorem as we later make $n_0 \to \infty$ (see (19) and (20)).

Similarly, for the other terms involving Y_n and Z_n, we use that $\sqrt{\log n/n}Y_n$ and $Z_n/n^{1-\lambda_3}$ are bounded random variables. Then exactly as in the previous paragraph and following the calculations leading to (11), and the other coefficients (12), (14), (15) and (16) we see that fixing n_0 as we let $n \to \infty$, the contribution of the random terms is bounded by a bounded random variable, say the conditional expectation given by $\mathcal{F}_{n_0}$ of a certain G_{n_0} (whose expectation goes to 0 almost surely as we later make $n_0 \to \infty$).

2.3 *Completion of proof*

Let us now write $H_{n_0} = F_{n_0} + G_{n_0}$, that is the remainder term is bounded by the sum of a constant and a random term uniformly in n. Notice that H_{n_0} is actually $\mathcal{F}_\infty$ measurable and in the calculations what we really use is its conditional expectation given by $\mathcal{F}_{n_0}$. Combining Steps 1, 2 and 3, and fixing n_0 as we make $n \to \infty$, we get from (10) and the previous subsection

$$\limsup_{n\to\infty} |E\{e^{i(t_1X_n+t_2Y_n+t_3Z_n)}|\mathcal{F}_{n_0}\} - e^{it_3Z_{n_0}}e^{-\frac{\sigma_1^2}{2}t_1^2-\frac{\sigma_2^2}{2}t_2^2}|$$
$$\le E\{H_{n_0}|\mathcal{F}_{n_0}\} + \text{const}\sum_{n_0}^{\infty}\frac{1}{j^{3/2}}, \tag{19}$$

with σ_1^2 and σ_2^2 coming from (11) and (12) respectively. Taking expectation and using $|EV| = |EE\{V|\mathcal{F}_{n_0}\}| \le E|E\{V|\mathcal{F}_{n_0}\}|$, for any integrable random variable V, we get

$$\limsup_{n\to\infty}|Ee^{i(t_1X_n+t_2Y_n+t_3Z_n)} - Ee^{it_3Z_{n_0}}e^{-\frac{\sigma_1^2}{2}t_1^2-\frac{\sigma_2^2}{2}t_2^2}| \le EH_{n_0} + \text{const}\sum_{n_0}^{\infty}\frac{1}{j^{3/2}}. \tag{20}$$

Now Z_n is a martingale, and in the appendix we show that Z_n is L^2-bounded, so that Z_n converges to some Z a.s. In the calculation so far n_0 is arbitrary. We now let $n_0 \to \infty$, recalling that the nonrandom F_{n_0} converges to 0 and that the bounded random variable G_{n_0} also converges to 0 almost surely from Step 3, to get the limiting characteristic function

$$Ee^{it_3Z}e^{-\frac{\sigma_1^2}{2}t_1^2-\frac{\sigma_2^2}{2}t_2^2}.$$

This shows that Z is independent of X, Y, and that X and Y are independent normals. □

3. Case of real vectors

In the previous sections we have considered linear combinations corresponding to eigenvectors. To consider general vectors we need the Jordan form of the irreducible replacement matrix. For simplicity we assume that there are only three real eigenvalues. However now there exists a nonsingular matrix $\mathbf{T}$ such that

$$\mathbf{T}^{-1}\mathbf{R}\mathbf{T} = \begin{pmatrix} 1 & & & \\ & \Lambda_1 & & \\ & & \Lambda_2 & \\ & & & \Lambda_3 \end{pmatrix},$$

where

$$\Lambda_i = \begin{pmatrix} \lambda_i & 1 & 0 & \\ 0 & \lambda_i & 1 & \\ & & \ddots & \\ & & & \lambda_i \end{pmatrix}.$$

Let us consider the case of Λ_1. Let the dimension be d_1. Then the vectors $\xi_1 = (1, 0, 0, \ldots)'$, $\xi_2 = (0, 1, 0, \ldots)'$, $\ldots$, $\xi_{d_1} = (0, 0, \ldots, 1)'$ transform according to the equations $\Lambda_1\xi_1 = \lambda_1\xi_1$, $\Lambda_1\xi_2 = \xi_1 + \lambda_1\xi_2$, $\Lambda_1\xi_3 = \xi_2 + \lambda_1\xi_3, \ldots$, i.e. in matrix form $\Lambda_1(\xi_1, \xi_2, \ldots, \xi_{d_1}) = (\xi_1, \xi_2, \ldots, \xi_{d_1})\Lambda_1$. Denoting the matrix of ξ_i's for the three matrices Λ_1, Λ_2, Λ_3 by Ξ_1, Ξ_2, Ξ_3 respectively (and necessarily adding 0's for the other components) we have

$$\begin{pmatrix} 1 & & & \\ & \Lambda_1 & & \\ & & \Lambda_2 & \\ & & & \Lambda_3 \end{pmatrix} (\mathbf{u} : \Xi_1 : \Xi_2 : \Xi_3) = (\mathbf{u} : \Xi_1 : \Xi_2 : \Xi_3) \begin{pmatrix} 1 & & & \\ & \Lambda_1 & & \\ & & \Lambda_2 & \\ & & & \Lambda_3 \end{pmatrix},$$

where $\mathbf{u}$ denotes the vector $(1, 0, \cdots)$ of dimension $1 + d_1 + d_2 + d_3$. It may be noticed that $(\mathbf{u} : \Xi_1 : \Xi_2 : \Xi_3)$ is the identity matrix written in a suitable form.

In our case we have to work with not the above matrix of Λ_i's, but the stochastic matrix $\mathbf{R}$. In that case, using the above mentioned Jordan decomposition of $\mathbf{R}$, we have to use the vectors $\mathbf{T}(\mathbf{u} : \Xi_1 : \Xi_2 : \Xi_3)$, and the equation

$$\mathbf{R}\mathbf{T}(\mathbf{u} : \Xi_1 : \Xi_2 : \Xi_3) = \mathbf{T}(\mathbf{u} : \Xi_1 : \Xi_2 : \Xi_3) \begin{pmatrix} 1 & & & \\ & \Lambda_1 & & \\ & & \Lambda_2 & \\ & & & \Lambda_3 \end{pmatrix}.$$

As $\mathbf{R}$ has principal eigenvalue 1 corresponding to the eigenvector $\mathbf{1}$ consisting of 1's, we have $\mathbf{T}\mathbf{u} = \mathbf{1}$. This implies a trivial limit for $\mathbf{W}_n'\mathbf{T}\mathbf{u}/(n+1)$. However the limits for the other linear combinations corresponding to $\mathbf{W}_n'\mathbf{T}\Xi_i$, $i = 1, 2, 3$, are nontrivial and are discussed in the next three subsections. For simplicity with a slight abuse of notation we shall use the same notation Ξ_i to denote $\mathbf{T}\Xi_i$.

Notice that we can write $\Lambda_i = \lambda_i I_i + F_i$ where F_i is a nilpotent matrix. The presence of this nilpotent F_i changes our calculations in the previous section at certain places and we will discuss how. We first note that $\mathbf{W}'_{n+1}\Xi_i = \mathbf{W}'_n\Xi_i + \chi'_{n+1}\mathbf{R}\Xi_i = \mathbf{W}'_n\Xi_i + \chi'_{n+1}\Xi_i\Lambda_i$ (remember the abuse of notation mentioned before). We give the most important contributions, the higher order terms have been ignored for notational simplicity.

3.1 $\lambda_1 < 1/2$

For notational simplicity from now on we shall restrict ourselves to the highest order terms significant for the results to hold, and this will be denoted by the notation $\sim$. For $\lambda < 1/2$, the approximation $\sqrt{n/(n+1)} \sim (1 - 1/(2n))$ gives

$$E\left\{\frac{\mathbf{W}'_{n+1}\Xi_1}{\sqrt{n+1}}\middle|\mathcal{F}_n\right\} \sim \frac{\mathbf{W}'_n\Xi_1}{\sqrt{n}}\left(I_1 - \frac{\frac{1}{2}I_1 - \Lambda_1}{n}\right), \tag{21}$$

leading to the product terms when iterating backwards. On the other hand, the approximate form leading to the explicit computations for the conditional characteristic function comes from

$$\frac{\mathbf{W}'_{n+1}\Xi_1}{\sqrt{n+1}} - E\left\{\frac{\mathbf{W}'_{n+1}\Xi_1}{\sqrt{n+1}}\middle|\mathcal{F}_n\right\} \sim \frac{1}{\sqrt{n+1}}\left(\chi'_{n+1} - \frac{\mathbf{W}'_{n+1}}{n+1}\right)\Xi_1\Lambda_1. \tag{22}$$

As before the most important contribution in the conditional covariance comes from the first term of the right-hand side of (22) after removal of brackets. Notice that $E\{\chi_{n+1}\chi'_{n+1}|\mathcal{F}_n\}$ consists only of diagonal terms and is thus approximately (using the strong law and the dominated convergence theorem) D_π, meaning the diagonal matrix with components of π, namely $\pi_1, \pi_2, \ldots$, as diagonals. This gives for the conditional covariance of (22) the approximate expression

$$\frac{1}{n+1}\Lambda'_1\Xi'_1 D_\pi \Xi_1\Lambda_1.$$

This when iterated backwards with terms coming from (21), leads to the limiting covariance matrix of the asymptotically normal $\mathbf{W}'_n\Xi_1/\sqrt{n}$, given by

$$\begin{aligned}\lim_{n\to\infty}\sum_{n_0}^{n}\frac{1}{j+1}\Pi_{i=j+1}^{n}\left(I_1 - \frac{\frac{1}{2}I_1 - \Lambda_1}{i}\right)'\Lambda'_1\Xi'_1 D_\pi\Xi_1\Lambda_1\Pi_{i=j+1}^{n}\left(I_1 - \frac{\frac{1}{2}I_1 - \Lambda_1}{i}\right)\\ = \int_0^\infty e^{-\left(\frac{1}{2}I_1 - \Lambda_1\right)'s}\Lambda'_1\Xi'_1 D_\pi\Xi_1\Lambda_1 e^{-\left(\frac{1}{2}I_1 - \Lambda_1\right)s}\mathrm{d}s,\end{aligned} \tag{23}$$

which can be compared with (11) for the case of eigenvector ξ_1.

3.2 $\lambda_2 = 1/2$

In this case the norming for the central limit theorem is $\sqrt{n\log^{2d_2-1}n}$, where d_2 is the dimension of Λ_2. The reason for the $2d_2 - 1$ power will be clear towards the end. First note the approximation

$$\sqrt{\frac{n\log^{2d_2-1}n}{(n+1)\log^{2d_2-1}(n+1)}} \sim \left(1 - \frac{1}{2n}\right)\left(1 - \frac{2d_2-1}{2n\log n}\right).$$

With this we get

$$E\left\{\frac{\mathbf{W}'_{n+1}\,\Xi_2}{\sqrt{(n+1)\log^{2d_2-1}(n+1)}}\middle|\mathcal{F}_n\right\}$$
$$\sim \frac{\mathbf{W}'_n\,\Xi_2}{\sqrt{n\log^{2d_2-1}n}}\left(1-\frac{1}{2n}\right)\left(1-\frac{2d_2-1}{2n\log n}\right)+\frac{\mathbf{W}'_n}{n+1}\frac{\Xi_2\Lambda_2}{\sqrt{n\log^{2d_2-1}n}}$$
$$= \frac{\mathbf{W}'_n\,\Xi_2}{\sqrt{n\log^{2d_2-1}n}}\left(I_2\left(1-\frac{2d_2-1}{2n\log n}\right)+\frac{F_2}{n}\right), \tag{24}$$

where we have crucially used the form of Λ_2 to cancel the $1/(2n)$'s occurring with opposite signs. This F_2 plays an important role in the computations later explaining the $2d_2-1$ power. On the other hand, the martingale terms for the covariance computations come from

$$\frac{\mathbf{W}'_{n+1}\,\Xi_2}{\sqrt{(n+1)\log^{2d_2-1}(n+1)}}-E\left\{\frac{\mathbf{W}'_{n+1}\,\Xi_2}{\sqrt{(n+1)\log^{2d_2-1}(n+1)}}\middle|\mathcal{F}_n\right\}$$
$$\sim \frac{1}{\sqrt{(n+1)\log^{2d_2-1}(n+1)}}\left(\chi'_{n+1}-\frac{\mathbf{W}'_{n+1}}{n+1}\right)\Xi_2\Lambda_2. \tag{25}$$

This gives for the conditional covariance of (25) the approximate expression

$$\frac{1}{(n+1)\log^{2d_2-1}(n+1)}\Lambda'_2\,\Xi'_2 D_\pi\,\Xi_2\Lambda_2.$$

This when iterated backwards with terms coming from (24), leads to the limiting covariance matrix of the asymptotically normal $\mathbf{W}'_n\,\Xi_1/\sqrt{n\log^{2d_2-1}n}$, given by

$$\lim_{n\to\infty}\sum_{n_0}^{n}\frac{1}{(j+1)\log^{2d_2-1}(j+1)}\Pi_{i=j+1}^{n}\left(I_2\left(1-\frac{2d_2-1}{2i\log i}\right)+\frac{F_2}{i}\right)'$$
$$\times\,\Lambda'_2\,\Xi'_2 D_\pi\,\Xi_2\Lambda_2\Pi_{i=j+1}^{n}\left(I_2\left(1-\frac{2d_2-1}{2i\log i}\right)+\frac{F_2}{i}\right), \tag{26}$$

F_2 being nilpotent. In the above products only a few terms will be nonzero. The consideration of the limits of the nonzero terms will explain the $\log^{2d_2-1}n$ term in the norming. We shall now use exponentiation to simplify the calculations. Observe that

$$\Pi_{i=j+1}^{n}\left(I_2\left(1-\frac{2d_2-1}{2i\log i}\right)+\frac{F_2}{i}\right)\sim \Pi_{i=j+1}^{n}\mathrm{e}^{-\frac{2d_2-1}{2i\log i}I_2+\frac{F_2}{i}}$$
$$=\mathrm{e}^{-\sum_{i=j+1}^{n}\frac{2d_2-1}{2i\log i}I_2+\sum_{i=j+1}^{n}\frac{F_2}{i}}\sim \mathrm{e}^{-\frac{2d_2-1}{2}\log\frac{\log n}{\log(j+1)}I_2+F_2\log\left(\frac{n}{j+1}\right)}$$

$$= \mathrm{e}^{-\frac{2d_2-1}{2}\log\frac{\log n}{\log(j+1)}} I_2\, \mathrm{e}^{F_2 \log\left(\frac{n}{j+1}\right)}$$

$$= \mathrm{e}^{-\frac{2d_2-1}{2}\log\frac{\log n}{\log(j+1)}} I_2 \left[\sum_{k=0}^{d_2-1} \left(F_2 \log\left(\frac{n}{j+1}\right)\right)^k \Big/ k!\right]. \tag{27}$$

A summand of (26) is thus approximated using (27) as

$$\frac{1}{(j+1)\log^{2d_2-1}(j+1)} \Pi_{i=j+1}^{n} \left(I_2\left(1 - \frac{2d_2-1}{2i\log i}\right) + \frac{F_2}{i}\right)'$$

$$\times\, \Lambda_2' \Xi_2' D_\pi\, \Xi_2 \Lambda_2 \Pi_{i=j+1}^{n} \left(I_2\left(1 - \frac{2d_2-1}{2i\log i}\right) + \frac{F_2}{i}\right)$$

$$\sim \frac{1}{(j+1)\log^{2d_2-1} n} \left[\sum_{k=0}^{d_2-1} \left(F_2' \log\left(\frac{n}{j+1}\right)\right)^k \Big/ k!\right]$$

$$\times\, \Lambda_2' \Xi_2' D_\pi\, \Xi_2 \Lambda_2 \left[\sum_{k=0}^{d_2-1} \left(F_2 \log\left(\frac{n}{j+1}\right)\right)^k \Big/ k!\right]. \tag{28}$$

When we sum (28) from n_0 to n and make $n \to \infty$, only the highest powers of F_2 survive, as can be seen from the following calculation (by considering $(\log n - \log(j+1))^m$ for $m < 2d_2 - 2$) which is done for the highest power only

$$\frac{1}{\log^{2d_2-1} n} \sum_{j=n_0}^{n} \frac{1}{j+1} \frac{(\log n - \log(j+1))^{2d_2-2}}{((d_2-1)!)^2}$$

$$\sim \frac{1}{\log^{2d_2-1} n} \int_0^{\log n - \log n_0} \frac{u^{2d_2-2}}{((d_2-1)!)^2}$$

$$= \frac{1}{(2d_2-1)\,((d_2-1)!)^2} \left(\frac{(\log n - \log n_0)^{2d_2-1}}{\log^{2d_2-1} n}\right)$$

$$\to \frac{1}{(2d_2-1)\,((d_2-1)!)^2}. \tag{29}$$

Thus the limiting covariance matrix obtained from (26) becomes

$$\frac{1}{(2d_2-1)\,((d_2-1)!)^2} (F_2')^{d_2-1} \Lambda_2' \Xi_2' D_\pi\, \Xi_2 \Lambda_2 F_2^{d_2-1}. \tag{30}$$

3.3 $\lambda_3 > 1/2$

We expect to get an L^2-bounded martingale sequence. Notice first that $E\{\mathbf{W}_{n+1}' \Xi_3 | \mathcal{F}_n\} = \mathbf{W}_n' \Xi_3\big(I_3 + \frac{1}{n+1}\Lambda_3\big)$. Hence the martingale sequence we work with is

$$\mathbf{Z}_n = \mathbf{W}_n' \Xi_3 \left\{\Pi_0^{n-1}\left(I_3 + \frac{1}{j+1}\Lambda_3\right)\right\}^{-1} = \mathbf{W}_n' \Xi_3 \mathbf{A}_n^{-1}. \tag{31}$$

The following calculation is similar to the calculation in the Appendix and we have used some approximations for notational convenience. $\mathbf{Z}_n$ satisfies the following equation:

$$\begin{aligned}\mathbf{Z}_{n+1} - \mathbf{Z}_n &= \mathbf{W}_n' \Xi_3 \left(\left(I_3 + \frac{1}{n+1}\Lambda_3 \right)^{-1} - I_3 \right) \mathbf{A}_n^{-1} + \chi_{n+1}' \Xi_3 \Lambda_3 \mathbf{A}_{n+1}^{-1} \\ &\sim -\frac{1}{n+1}\mathbf{Z}_n \mathbf{A}_n \Lambda_3 \mathbf{A}_n^{-1} + \chi_{n+1}' \Xi_3 \mathbf{A}_n^{-1} \mathbf{A}_n \Lambda_3 \mathbf{A}_n^{-1} \\ &\sim -\frac{1}{n+1}\mathbf{Z}_n \Lambda_3 + \chi_{n+1}' \Xi_3 \mathbf{A}_n^{-1} \Lambda_3, \end{aligned} \tag{32}$$

noting that $\mathbf{A}_n, \mathbf{A}_n^{-1}$ and Λ_3 commute. To prove L^2-boundedness consider $EE\{\mathbf{Z}_{n+1}\mathbf{Z}_{n+1}' | \mathcal{F}_n\}$. Using the martingale property and the above decomposition it follows that

$$\begin{aligned} E\{\mathbf{Z}_{n+1}\mathbf{Z}_{n+1}' | \mathcal{F}_n\} &\sim \mathbf{Z}_n\mathbf{Z}_n' - \frac{1}{(n+1)^2}\mathbf{Z}_n \Lambda_3 \Lambda_3' \mathbf{Z}_n' \\ &\quad + E\{\chi_{n+1}' \Xi_3 \mathbf{A}_n^{-1} \Lambda_3 \Lambda_3' (\mathbf{A}_n^{-1})' \Xi_3' \chi_{n+1} | \mathcal{F}_n\} \\ &\leq \mathbf{Z}_n\mathbf{Z}_n' \left(1 - \frac{\beta}{(n+1)^2} \right) \\ &\quad + \mathrm{Tr}\{\Xi_3 \mathbf{A}_n^{-1} \Lambda_3 \Lambda_3' (\mathbf{A}_n^{-1})' \Xi_3' E\{\chi_{n+1}\chi_{n+1}' | \mathcal{F}_n\}\}, \end{aligned} \tag{33}$$

where β denotes the minimum eigenvalue of $\Lambda_3\Lambda_3'$ and we have used properties of the trace of a matrix. Approximating $E\{E\{\chi_{n+1}\chi_{n+1}' | \mathcal{F}_n\}\}$ by D_π, further expectation of the above inequality gives

$$\begin{aligned} E\mathbf{Z}_{n+1}\mathbf{Z}_{n+1}' &\leq E\mathbf{Z}_n\mathbf{Z}_n' \left(1 - \frac{\beta}{(n+1)^2} \right) \\ &\quad + \text{const Tr}\, \{\Xi_3 \mathbf{A}_n^{-1} \Lambda_3 \Lambda_3' (\mathbf{A}_n^{-1})' \Xi_3' D_\pi\}. \end{aligned} \tag{34}$$

We need to find the order of the last matrix so that the above equation can be iterated as in the one dimensional case of the Appendix, giving L^2-boundedness of $\mathbf{Z}_n$. We show this by showing that the terms of $\mathbf{A}_n^{-1}$ are $O(n^{-\lambda_3} \log^{d_3 - 1} n)$. First note that

$$\begin{aligned} \mathbf{A}_n &= \Pi_1^n \left(I_3 + \frac{1}{j+1}\Lambda_3 \right) \\ &= \Pi_1^n \left(I_3 \left(1 + \frac{\lambda_3}{j+1} \right) + \frac{1}{j+1} F_3 \right). \end{aligned} \tag{35}$$

Using the commutativity of I_3 and F_3 and the fact that $F_3^{d_3} = \mathbf{0}$, $\mathbf{A}_n$ can be approximated as

$$\mathbf{A}_n \sim \mathrm{e}^{\lambda_3 \log n I_3 + F_3 \log n}$$

Hence

$$\mathbf{A}_n^{-1} \sim \mathrm{e}^{-\lambda_3 \log n} I_3 \times \left[\sum_{k=0}^{d_3-1} F_3^k (-\log n)^k \Big/ k! \right]$$

$$\sim n^{-\lambda_3} \log^{d_3-1} n \frac{F_3^{d_3-1}}{(d_3-1)!}. \tag{36}$$

Thus, $\mathbf{A}_n^{-1} = O(n^{-\lambda_3} \log^{d_3-1} n)$, the terms of $\mathbf{A}_n^{-1}(\mathbf{A}_n^{-1})'$ are $O(n^{-2\lambda_3} \log^{2d_3-2} n)$. With this we now go back to (34) to prove L^2-boundedness using $2\lambda_3 > 1$.

Then the analysis of §2 proceeds to show independence of the weak limits (strong limit for $\mathbf{Z}_n$). We may state the analogue of Theorem 1.1 as follows:

Theorem 3.1. *In case all eigenvalues are real, we consider the linear combinations corresponding to the columns of* $\mathbf{T}$ *as identified at the beginning of* §3. *The weak limits of the normalized linear combinations corresponding to eigenvalues* $\lambda < 1/2, \lambda = 1/2$ *and* $\lambda > 1/2$ *are independent.*

For the different eigenvalues all of which are less than 1/2, there may be dependence among the weak limits coming from the Jordan blocks for different eigenvalues (see Theorem 5.1 later). For real $\lambda = 1/2$ there is only one Jordan block (the situation for complex λ with real part 1/2 is somewhat different). For $\lambda > 1/2$ the weak limits coming from the Jordan blocks corresponding to different λ's may be correlated. One instance of this limiting covariance has been computed in the Appendix although we cannot say definitely that the limit is nonzero.

4. Complex eigenvalues

For complex eigenvalues we consider another canonical form which is similar to the Jordan canonical form. This form comes from considering the real vectors coming from the real and imaginary parts of the complex vectors corresponding to the complex Jordan form. Special cases of this decomposition has been studied in Smythe [11]. We first consider three types of eigenvalues, one of each type as before (i.e. with real part less than 1/2, equal to 1/2 and greater than 1/2). There exists a nonsingular matrix $\mathbf{S}$ such that

$$\mathbf{S}^{-1}\mathbf{R}\mathbf{S} = \begin{pmatrix} 1 & & & \\ & \Lambda_{c1} & & \\ & & \Lambda_{c2} & \\ & & & \Lambda_{c3} \end{pmatrix},$$

where

$$\Lambda_{ci} = \begin{pmatrix} B_i & I & & \\ & B_i & \ddots & \\ & & \ddots & I \\ & & & B_i \end{pmatrix}$$

and

$$B_i = \begin{pmatrix} \lambda_{ir} & \lambda_{ic} \\ -\lambda_{ic} & \lambda_{ir} \end{pmatrix}.$$

I is a 2-dimensional identity matrix and rest of the elements are 0. Let the dimension of Λ_{ci} be $2d_{ci}$. As before we partition the matrix **SI** (this **I** has dimension $1 + 2(d_{c1} + d_{c2} + d_{c3})$) into a vector of ones and S_i, $i = 1, 2, 3$ with number of columns in S_i equal to $2d_{ci}$. These vectors give us the linear combinations.

Notice that, here we can write

$$\Lambda_{ci} = \lambda_{ir} I_{ci} + \lambda_{ic} C_{ci} + F_{ci},$$

where I_{ci} is an identity matrix of dimension $2d_{ci}$, C_{ci} is a block diagonal matrix of the same dimension as Λ_{ci}. Each block, say D_i, is of dimension 2, where

$$D_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

F_{ci} is a nilpotent matrix of order d_{ci}, i.e., $F_{ci}^{d_{ci}} = \mathbf{0}$ and d_{ci} is the least such integer.

First observe that the rotation matrix D_i satisfies $D_i^2 = -I$, $D_i^3 = -D_i$, $D_i^4 = I, \ldots,$ where I is the identity matrix of the same dimension as D_i. Also, it is to be noted that the matrices I_{ci}, C_{ci} and F_{ci} commute with each other. Thus,

$$\begin{aligned} \mathrm{e}^{k_1 I_{ci} + k_2 C_{ci} + k_3 F_{ci}} &= \mathrm{e}^{k_1 I_{ci}} \mathrm{e}^{k_2 C_{ci}} \mathrm{e}^{k_3 F_{ci}} \\ &= \mathrm{e}^{k_1} I_{ci} [\cos(k_2) I_{ci} + \sin(k_2) C_{ci}] \left[\sum_{j=1}^{d_{ci}-1} (k_3^j F_{ci}^j) \Big/ j! \right]. \end{aligned} \tag{37}$$

We will mention briefly how the proof of Theorem 1.1 go for the complex roots with the presence of the nilpotent matrix and the rotation matrix. We note that $\mathbf{W}'_{n+1} S_i = \mathbf{W}'_n S_i + \chi'_{n+1} \mathbf{R} S_i = \mathbf{W}'_n S_i + \chi'_{n+1} S_i \Lambda_{ci}$. We give the most important contributions, the higher order terms have been ignored for notational simplicity.

4.1 $\lambda_{1r} < 1/2$

In this case, since $\sqrt{n/(n+1)} \sim (1 - 1/(2n))$, it is to be noted that

$$E\left\{ \frac{\mathbf{W}'_{n+1} S_1}{\sqrt{n+1}} \middle| \mathcal{F}_n \right\} \sim \frac{\mathbf{W}'_n S_1}{\sqrt{n}} \left(I_{c1} - \frac{\frac{1}{2} I_{c1} - \Lambda_{c1}}{n} \right). \tag{38}$$

Now iterating backwards we get the product terms as before. Thus,

$$\frac{\mathbf{W}'_{n+1} S_1}{\sqrt{n+1}} - E\left\{ \frac{\mathbf{W}'_{n+1} S_1}{\sqrt{n+1}} \middle| \mathcal{F}_n \right\} \sim \frac{1}{\sqrt{n+1}} \left(\chi'_{n+1} - \frac{\mathbf{W}'_n}{n+1} \right) S_1 \Lambda_{c1}. \tag{39}$$

As before the most important contribution in the conditional covariance comes from the first term of the above. Notice that $E\{\chi_{n+1} \chi'_{n+1} | \mathcal{F}_n\}$ consists only of diagonal terms and is thus approximately (using the strong law and the dominated convergence theorem) D_π. This gives for the conditional covariance of (39) the approximate expression

$$\frac{1}{n+1}\Lambda'_{c1}S'_1 D_\pi S_1 \Lambda_{c1}.$$

This when iterated backwards with terms coming from (38), leads to the limiting covariance matrix of the asymptotically normal $\mathbf{W}'_n S_1/\sqrt{n}$, given by

$$\lim_{n\to\infty}\sum_{n_0}^{n}\frac{1}{j+1}\left\{\Pi_{i=j+1}^{n}\left(I_{c1}-\frac{\frac{1}{2}I_{c1}-\Lambda_{c1}}{i}\right)'\right.$$
$$\left.\times\Lambda'_{c1}S'_1 D_\pi S_1\Lambda_{c1}\Pi_{i=j+1}^{n}\left(I_{c1}-\frac{\frac{1}{2}I_{c1}-\Lambda_{c1}}{i}\right)\right\}$$
$$=\int_0^\infty \mathrm{e}^{-\left(\frac{1}{2}I_{c1}-\Lambda_{c1}\right)'s}\Lambda'_{c1}S'_1 D_\pi S_1\Lambda_{c1}\mathrm{e}^{-\left(\frac{1}{2}I_{c1}-\Lambda_{c1}\right)s}\mathrm{d}s, \tag{40}$$

which can be compared with (11) for the case of eigenvector ξ_1. From the calculation in (37), it can be seen that

$$\mathrm{e}^{-\left(\frac{1}{2}I_{c1}-\Lambda_{c1}\right)s}$$
$$=\mathrm{e}^{-\left(\frac{1}{2}-\lambda_{1r}\right)s}I_{c1}[\cos(s\lambda_{1c})I_{c1}+\sin(s\lambda_{1c})C_{c1}]\left[\sum_{j=1}^{d_{c1}-1}(sF_{c1})^j\Big/ j!\right],$$

which is an integrable function, and hence (40) is finite.

4.2 $\lambda_{2r}=1/2$

In this case the norming for the central limit theorem is $\sqrt{n\log^{2d_{c2}-1}n}$, where $2d_{c2}$ is the dimension of Λ_{c2}. From the calculation of the covariance matrix the reason for the $2d_{c2}-1$ power of the the logarithm will be clear. The approximation

$$\sqrt{\frac{n\log^{2d_{c2}-1}n}{(n+1)\log^{2d_{c2}-1}(n+1)}}\sim\left(1-\frac{1}{2n}\right)\left(1-\frac{2d_{c2}-1}{2n\log n}\right)$$

leads to

$$E\left\{\frac{\mathbf{W}'_{n+1}S_2}{\sqrt{(n+1)\log^{2d_{c2}-1}(n+1)}}\Bigg|\mathcal{F}_n\right\}$$
$$\sim\frac{\mathbf{W}'_n S_2}{\sqrt{n\log^{2d_{c2}-1}n}}\left(1-\frac{1}{2n}\right)\left(1-\frac{2d_{c2}-1}{2n\log n}\right)+\frac{\mathbf{W}'_n}{n+1}\frac{S_2\Lambda_2}{\sqrt{n\log^{2d_{c2}-1}n}}$$
$$=\frac{\mathbf{W}'_n S_2}{\sqrt{n\log^{2d_{c2}-1}n}}\left(I_{c2}\left(1-\frac{2d_{c2}-1}{2n\log n}\right)+\frac{\lambda_{2c}}{n}C_{c2}+\frac{F_{c2}}{n}\right), \tag{41}$$

where the form of Λ_{c2} is used to cancel the $1/(2n)$'s occurring with opposite signs. We later discuss the role of C_{c2} and F_{c2} in the computations that explains the power of the logarithm. Notice that the computation of the covariance matrix depends on the martingale terms

$$\frac{\mathbf{W}'_{n+1}S_2}{\sqrt{(n+1)\log^{2d_{c2}-1}(n+1)}} - E\left\{\frac{\mathbf{W}'_{n+1}S_2}{\sqrt{(n+1)\log^{2d_{c2}-1}(n+1)}}\bigg|\mathcal{F}_n\right\}$$
$$\sim \frac{1}{\sqrt{(n+1)\log^{2d_{c2}-1}(n+1)}}\left(\chi'_{n+1} - \frac{\mathbf{W}'_{n+1}}{n+1}\right)S_2\Lambda_{c2}. \tag{42}$$

Thus, the approximate expression for the conditional covariance of (42) is found as

$$\frac{1}{(n+1)\log^{2d_{c2}-1}(n+1)}\Lambda'_{c2}S'_2 D_\pi S_2\Lambda_{c2}.$$

Iterating backwards with terms coming from (41) leads to the limiting covariance matrix of the asymptotically normal $\mathbf{W}'_n S_2/\sqrt{n\log^{2d_{c2}-2} n}$, given by

$$\lim_{n\to\infty}\sum_{n_0}^{n}\frac{1}{(j+1)\log^{2d_{c2}-1}(j+1)}$$
$$\left\{\Pi_{i=j+1}^{n}\left(I_{c2}\left(1-\frac{2d_{c2}-1}{2i\log i}\right)+\frac{\lambda_{2c}}{i}C'_{c2}+\frac{F_{c2}}{i}\right)'\Lambda'_{c2}S'_2 D_\pi S_2\Lambda_{c2}\right.$$
$$\left.\times\Pi_{i=j+1}^{n}\left(I_{c2}\left(1-\frac{2d_{c2}-1}{2i\log i}\right)+\frac{\lambda_{2c}}{i}C_{c2}+\frac{F_{c2}}{i}\right)\right\}. \tag{43}$$

We shall now use exponentiation to simplify the calculations. Observe that,

$$\Pi_{i=j+1}^{n}\left(I_{c2}\left(1-\frac{2d_{c2}-1}{2i\log i}\right)+\frac{\lambda_{2c}}{i}C_{c2}+\frac{F_{c2}}{i}\right)$$
$$\sim \Pi_{i=j+1}^{n}\mathrm{e}^{-\frac{2d_{c2}-1}{2i\log i}I_{c2}+\frac{\lambda_{2c}}{i}C_{c2}+\frac{F_{c2}}{i}}$$
$$= \mathrm{e}^{-\sum_{i=j+1}^{n}\frac{2d_{c2}-1}{2i\log i}I_{c2}+\sum_{i=j+1}^{n}\frac{\lambda_{2c}}{i}C_{c2}+\sum_{i=j+1}^{n}\frac{F_{c2}}{i}}$$
$$\sim \mathrm{e}^{-\frac{2d_{c2}-1}{2}\log\frac{\log n}{\log(j+1)}I_{c2}+\lambda_{2c}C_{c2}\log\left(\frac{n}{j+1}\right)+F_{c2}\log\left(\frac{n}{j+1}\right)}$$
$$= \mathrm{e}^{-\frac{2d_{c2}-1}{2}\log\frac{\log n}{\log(j+1)}}I_{c2}\ \mathrm{e}^{C_{c2}\lambda_{2c}\log\left(\frac{n}{j+1}\right)}\ \mathrm{e}^{F_{c2}\log\left(\frac{n}{j+1}\right)}$$
$$= \mathrm{e}^{-\frac{2d_{c2}-1}{2}\log\frac{\log n}{\log(j+1)}}I_{c2}\left[I_{c2}\cos\left(\lambda_{2c}\log\left(\frac{n}{j+1}\right)\right)\right.$$
$$\left.+C_{c2}\sin\left(\lambda_{2c}\log\left(\frac{n}{j+1}\right)\right)\right]\left[\sum_{k=0}^{d_{c2}-1}\left(F_{c2}\log\left(\frac{n}{j+1}\right)\right)^k\Big/k!\right]. \tag{44}$$

Combining the contribution of the term in (44) to the two sides of (43) we get

$$\frac{1}{(j+1)\log^{2d_{c2}-1}(j+1)}\Pi_{i=j+1}^{n}\left(I_{c2}\left(1-\frac{2d_{c2}-1}{2i\log i}\right)+\frac{\lambda_{2c}}{i}C_{c2}+\frac{F_{c2}}{i}\right)'$$

$$\times\Lambda_{c2}'S_2'D_\pi S_2\Lambda_{c2}\Pi_{i=j+1}^{n}\left(I_{c2}\left(1-\frac{2d_{c2}-1}{2i\log i}\right)+\frac{\lambda_{2c}}{i}C_{c2}+\frac{F_{c2}}{i}\right)$$

$$\sim\frac{1}{(j+1)\log^{2d_{c2}-1}(j+1)}\mathrm{e}^{-(2d_{c2}-1)\log\frac{\log n}{\log(j+1)}}$$

$$\times\left[\sum_{k=0}^{d_{c2}-1}\left(F_{c2}'\log\left(\frac{n}{j+1}\right)\right)^k\Big/k!\right]$$

$$\times\left[I_{c2}\cos\left(\lambda_{2c}\log\left(\frac{n}{j+1}\right)\right)+C_{c2}'\sin\left(\lambda_{2c}\log\left(\frac{n}{j+1}\right)\right)\right]$$

$$\times\Lambda_{c2}'S_2'D_\pi S_2\Lambda_{c2}\left[I_{c2}\cos\left(\lambda_{2c}\log\left(\frac{n}{j+1}\right)\right)\right.$$

$$\left.+C_{c2}\sin\left(\lambda_{2c}\log\left(\frac{n}{j+1}\right)\right)\right]\left[\sum_{k=0}^{d_{c2}-1}\left(F_{c2}\log\left(\frac{n}{j+1}\right)\right)^k\Big/k!\right]$$

$$=\frac{1}{(j+1)\log^{2d_{c2}-1}n}\left[\sum_{k=0}^{d_{c2}-1}\left(F_{c2}'\log\left(\frac{n}{j+1}\right)\right)^k\Big/k!\right]$$

$$\times\left[I_{c2}\cos\left(\lambda_{2c}\log\left(\frac{n}{j+1}\right)\right)+C_{c2}'\sin\left(\lambda_{2c}\log\left(\frac{n}{j+1}\right)\right)\right]$$

$$\times\Lambda_{c2}'S_2'D_\pi S_2\Lambda_{c2}\left[I_{c2}\cos\left(\lambda_{2c}\log\left(\frac{n}{j+1}\right)\right)\right.$$

$$\left.+C_{c2}\sin\left(\lambda_{2c}\log\left(\frac{n}{j+1}\right)\right)\right]\left[\sum_{k=0}^{d_{c2}-1}\left(F_{c2}\log\left(\frac{n}{j+1}\right)\right)^k\Big/k!\right] \tag{45}$$

Now observing that the terms involving sine and cosine are all bounded, one finds that except the coefficient of the highest power term of F_{c2} i.e. $F_{c2}^{d_{c2}-1}$, the coefficients of other terms go to zero when $n\to\infty$. Observe that the highest power terms of F_{c2} would be multiplied by $\cos^2\left(\lambda_{2c}\log\left(\frac{n}{j+1}\right)\right)$ (i.e. $(1/2)\left[1+\cos\left(2\lambda_{2c}\log\left(\frac{n}{j+1}\right)\right)\right]$), $\sin^2\left(\lambda_{2c}\log\left(\frac{n}{j+1}\right)\right)$ (i.e. $(1/2)\left[1-\cos\left(2\lambda_{2c}\log\left(\frac{n}{j+1}\right)\right)\right]$), or, terms such as $\sin\left(\lambda_{2c}\log\left(\frac{n}{j+1}\right)\right)\cos\left(\lambda_{2c}\log\left(\frac{n}{j+1}\right)\right)$ (i.e. $(1/2)\left[\sin\left(2\lambda_{2c}\log\left(\frac{n}{j+1}\right)\right)\right]$), separately. Thus, the highest power terms of

F_{c2} with sine function give the coefficient

$$\frac{1}{\log^{2d_{c2}-1} n} \sum_{j=n_0}^{n} \frac{1}{j+1} \left\{ \frac{(\log n - \log(j+1))^{2d_{c2}-2}}{((d_{c2}-1)!)^2} \times \frac{\sin(2\lambda_{2c}(\log n - \log(j+1)))}{2} \right\}$$

$$\sim \frac{1}{\log^{2d_{c2}-1} n} \int_0^{\log n - \log n_0} \frac{u^{2d_{c2}-2}}{((d_{c2}-1)!)^2} \frac{\sin(2\lambda_{2c} u)}{2}$$

$$= O\left(\frac{(\log n - \log n_0)^{2d_{c2}-2}}{\log^{2d_{c2}-1} n} \right) \to 0 \tag{46}$$

(seen by integration by parts) as $n \to \infty$. Similarly, with cosine function, it gives

$$\frac{1}{\log^{2d_{c2}-1} n} \sum_{j=n_0}^{n} \frac{1}{j+1} \left\{ \frac{(\log n - \log(j+1))^{2d_{c2}-2}}{((d_{c2}-1)!)^2} \times \frac{\cos(2\lambda_{2c}(\log n - \log(j+1)))}{2} \right\}$$

$$\sim \frac{1}{\log^{2d_{c2}-1} n} \int_0^{\log n - \log n_0} \frac{u^{2d_{c2}-2}}{((d_{c2}-1)!)^2} \frac{\cos(2\lambda_{2c} u)}{2}$$

$$= O\left(\frac{(\log n - \log n_0)^{2d_{c2}-2}}{\log^{2d_{c2}-1} n} \right) \to 0 \quad \text{as } n \to \infty. \tag{47}$$

Now, the terms that involve multiplying by $1/2$ only, give

$$\frac{1}{\log^{2d_{c2}-1} n} \sum_{j=n_0}^{n} \frac{1}{j+1} \frac{(\log n - \log(j+1))^{2d_{c2}-2}}{((d_{c2}-1)!)^2}$$

$$\sim \frac{1}{\log^{2d_{c2}-1} n} \int_0^{\log n - \log n_0} \frac{u^{2d_{c2}-2}}{((d_{c2}-1)!)^2}$$

$$= \frac{1}{(2d_{c2}-1)\,((d_{c2}-1)!)^2} \left(\frac{(\log n - \log n_0)^{2d_{c2}-1}}{\log^{2d_{c2}-1} n} \right)$$

$$\to \frac{1}{(2d_{c2}-1)\,((d_{c2}-1)!)^2} \tag{48}$$

as $n \to \infty$. Thus, adding two of these terms one obtains the limiting covariance matrix from (43) as

$$\frac{1}{(2d_{c2}-1)\,((d_{c2}-1)!)^2} \times (F'_{c2})^{d_{c2}-1} \left(\frac{1}{2} \Lambda'_{c2} S'_2 D_\pi S_2 \Lambda_{c2} + \frac{1}{2} C'_{c2} \Lambda'_{c2} S'_2 D_\pi S_2 \Lambda_{c2} C_{c2} \right) F_{c2}^{d_{c2}-1}. \tag{49}$$

Notice that it does not involve λ_{2c}.

4.3 $\lambda_{3r} > 1/2$

Here also, we show that $\mathbf{W}'_n S_3 A_n^{-1}$ is an L^2-bounded martingale sequence, where $A_n = \Pi_0^{n-1}\big(I_{c3} + \frac{1}{j+1}\Lambda_{c3}\big)$. Notice first that $E\{\mathbf{W}'_{n+1}S_3|\mathcal{F}_n\} = \mathbf{W}'_n S_3\big(I_{c3} + \frac{1}{n+1}\Lambda_{c3}\big)$. Hence the martingale $\mathbf{Z}_n$ satisfies the following equation

$$\begin{aligned}\mathbf{Z}_{n+1} - \mathbf{Z}_n &= \mathbf{W}'_n S_3 \left(\left(I_{c3} + \frac{1}{n+1}\Lambda_{c3}\right)^{-1} - I_{c3}\right)\mathbf{A}_n^{-1} + \chi'_{n+1}S_3\Lambda_{c3}\mathbf{A}_{n+1}^{-1} \\ &\sim -\frac{1}{n+1}\mathbf{Z}_n\mathbf{A}_n\Lambda_{c3}\mathbf{A}_n^{-1} + \chi'_{n+1}S_3\mathbf{A}_n^{-1}\mathbf{A}_n\Lambda_{c3}\mathbf{A}_n^{-1} \\ &\sim -\frac{1}{n+1}\mathbf{Z}_n\Lambda_{c3} + \chi'_{n+1}S_3\mathbf{A}_n^{-1}\Lambda_{c3}, \end{aligned} \tag{50}$$

since $\mathbf{A}_n, \mathbf{A}_n^{-1}$ and Λ_{c3} commute. To prove L^2-boundedness, first observe

$$\begin{aligned}E\{\mathbf{Z}_{n+1}\mathbf{Z}'_{n+1}|\mathcal{F}_n\} &\sim \mathbf{Z}_n\mathbf{Z}'_n - \frac{1}{(n+1)^2}\mathbf{Z}_n\Lambda_{c3}\Lambda'_{c3}\mathbf{Z}'_n \\ &\quad + E\{\chi'_{n+1}S_3\mathbf{A}_n^{-1}\Lambda_{c3}\Lambda'_{c3}(\mathbf{A}_n^{-1})'S'_3\chi_{n+1}|\mathcal{F}_n\} \\ &\le \mathbf{Z}_n\mathbf{Z}'_n\left(1 - \frac{\beta_{c3}}{(n+1)^2}\right) \\ &\quad + \mathrm{Tr}\{S_3\mathbf{A}_n^{-1}\Lambda_{c3}\Lambda'_{c3}(\mathbf{A}_n^{-1})'S'_3E\{\chi_{n+1}\chi'_{n+1}|\mathcal{F}_n\}\}, \end{aligned} \tag{51}$$

where β_{c3} denotes the minimum eigenvalue of $\Lambda_{c3}\Lambda'_{c3}$. Approximating $E\{E\{\chi_{n+1}\chi'_{n+1}|\mathcal{F}_n\}\}$ by D_π, further expectation of the above inequality gives

$$\begin{aligned}E\mathbf{Z}_{n+1}\mathbf{Z}'_{n+1} &\le E\mathbf{Z}_n\mathbf{Z}'_n\left(1 - \frac{\beta_{c3}}{(n+1)^2}\right) \\ &\quad + \text{const. } \mathrm{Tr}\{S_3\mathbf{A}_n^{-1}\Lambda_{c3}\Lambda'_{c3}(\mathbf{A}_n^{-1})'S'_3D_\pi\}. \end{aligned} \tag{52}$$

We now find the order of the last matrix so that the above equation can be iterated as in the one-dimensional case of the Appendix, giving L^2-boundedness of $\mathbf{Z}_n$. We show this by showing that the terms of $\mathbf{A}_n^{-1}$ are $O(n^{-\lambda_3}\log^{d_3-1} n)$.

$$\begin{aligned}\mathbf{A}_n &= \Pi_1^n\left(I_{c3} + \frac{1}{j}\Lambda_{c3}\right) \\ &= \Pi_1^n\left(I_{c3}\left(1 + \frac{\lambda_{3r}}{j}\right) + \frac{\lambda_{3r}}{j}C_{c3} + \frac{1}{j}F_{c3}\right). \end{aligned} \tag{53}$$

Using commutativity of I_{c3}, C_{c3} and F_{c3} and the fact that $F_{c3}^{d_{c3}} = \mathbf{0}$, $\mathbf{A}_n$ can be approximated as

$$\mathbf{A}_n \sim \mathrm{e}^{\lambda_{3r}\log n I_{c3} + C_{c3}\log n + F_{c3}\log n}$$

Hence

$$\mathbf{A}_n^{-1} \sim \mathrm{e}^{-\lambda_{3r}\log n} I_{c3}[\cos(-\lambda_{3r}\log n) I_{c3} + \sin(-\lambda_{3r}\log n) C_{c3}] \times \left[\sum_{k=0}^{d_{c3}-1} F_{c3}^k (-\log n)^k \Big/ k!\right]$$
$$\sim n^{\lambda_{3r}} \log^{d_{c3}-1} n \frac{F_{c3}^{d_{c3}-1}}{(d_{c3}-1)!}. \qquad (54)$$

Thus, $\mathbf{A}_n^{-1} = O(n^{-\lambda_{3r}} \log^{d_{c3}-1} n)$ and from (52) one gets L^2-boundedness of $\mathbf{Z}_n$ since $2\lambda_{3r} > 1$.

Then the analysis of §2 proceeds to show independence of the weak limits (with strong limit for $\mathbf{Z}_n$). We may state the analogue of Theorem 1.1 as follows:

Theorem 4.1. *In case eigenvalues are complex, we consider the linear combinations corresponding to the columns of* $\mathbf{S}$ *as identified at the beginning of* §4. *The weak limits of the normalized linear combinations corresponding to eigenvalues* $\mathrm{Re}(\lambda) < 1/2$, $\mathrm{Re}(\lambda) = 1/2$ *and* $\mathrm{Re}(\lambda) > 1/2$ *are independent.*

For the different eigenvalues all of which have real parts less than 1/2, there may be dependence among the weak limits coming from the (modified) Jordan blocks for different eigenvalues (see Theorem 5.1 later). For $\mathrm{Re}(\lambda) = 1/2$, there may be different (modified) Jordan blocks corresponding to different $\mathrm{Im}(\lambda)$. However, inside $\mathrm{Re}(\lambda) = 1/2$, the weak limits coming from (modified) Jordan blocks of different dimensions are not independent, in general. For $\lambda > 1/2$, the weak limits coming from the Jordan blocks corresponding to different λ's may be correlated (similar to the real eigenvalue case computed in the Appendix), although we cannot say definitely that they are.

5. General case

In the general case we decompose the replacement matrix into a (modified) Jordan form as in the previous two sections. That is, corresponding to real eigenvalues we take the form as in §3, and corresponding to complex eigenvalues by considering the real and imaginary parts of vectors we take the form as in §4. Without loss of generality, we can now consider only the real parts of the eigenvalues, and the linear combinations will come from the (modified) Jordan form.

There are now three types of blocks: for $\mathrm{Re}(\lambda) < 1/2$, for $\mathrm{Re}(\lambda) = 1/2$ and the last type is for $\mathrm{Re}(\lambda) > 1/2$. According to our previous notation, there exists a nonsingular matrix $\mathbf{M}$ such that

$$\mathbf{M}^{-1}\mathbf{R}\mathbf{M} = \begin{pmatrix} 1 & & & \\ & G_1 & & \\ & & G_2 & \\ & & & G_3 \end{pmatrix},$$

where

$$G_i = \begin{pmatrix} \Lambda_{i,1} & & & \\ & \Lambda_{i,2} & & \\ & & \ddots & \\ & & & \Lambda_{i,n_i} \end{pmatrix}$$

and $\Lambda_{i,j}$s are either of the form of Λ_i as in §3 or Λ_{ci} as in §4. Also notice that, for each $i = 1, 2, 3$, there is a positive integer $0 \le k_i \le n_i$ such that $\Lambda_{i,1}, \ldots, \Lambda_{i,k_i}$ blocks correspond to real eigenvalues and the rest of the $n_i - k_i$ blocks correspond to complex eigenvalues. It can be observed that $k_2 \le 1$, and it is also assumed that the blocks inside G_2 which have the same dimension (i.e. same d_2 or d_{c2}) are arranged next to one another and put into the same subblock.

Let us recall that the linear combinations come from the columns of **MI** which we write with an abuse of notation as $(\mathbf{1} : M_1 : M_2 : M_3)$. With appropriate normalizations they decompose into the following three classes, independent in the limit.

Theorem 5.1.

(1) $\mathrm{Re}(\lambda) < 1/2$: *For the linear combinations corresponding to columns of* M_1, *the normalization is* $\sqrt{n}$ *and the limit is normal. The covariance is given by* (40) *with* G_1 *replacing* Λ_{c1} (*and* M_1 *replacing* S_1) *and we have to use the decomposition of* G_1 *combining the features of the real and the complex cases.*

(2) $\mathrm{Re}(\lambda) = 1/2$: *Recalling the arrangement inside* G_2, *in this case the linear combinations correspond to columns of* M_2. *For the subblock of* G_2 *having dimension* d_2 *or* d_{c2} *for the original* $\Lambda_{2,k}$*'s* (*of the same dimension*), *the normalization for the corresponding columns of* M_2 *is* $\sqrt{n \log^{2d_{c2}-1} n}$ (*or* $\sqrt{n \log^{2d_2-1} n}$ *as appropriate*) *and the limit is normal. The limits for different subblocks are not independent*, *in general*, *and for each subblock the covariance can be found from* (30) *and* (49) *by decomposing the subblock of* G_2 *combining the features of the real and the complex cases* (*and replacing* S_2 *by the column submatrix of* M_2 *corresponding to the subblock of* G_2).

(3) $\mathrm{Re}(\lambda) > 1/2$: *For the linear combinations corresponding to columns of* M_3, $\mathbf{W}'_n M_3 A_n^{-1}$ *is an* L^2*-bounded martingale sequence*, *where* $A_n = \Pi_0^{n-1}\left(I_3 + \frac{1}{j+1} G_3\right)$, *and* I_3 *is an identity matrix of the same dimension as* G_3. *The covariance between some of the components of the* (*almost sure*) *limit may be nonzero* (*although we cannot say definitely*), *even though rates are different.*

To summarize parts one and two of the above theorem, observe that $(1/\sqrt{n})$ is the only normalization for part one, i.e., for $W'_n M_1$ and (not necessarily zero) covariances are obtained between different Jordan blocks in this part. Whereas, for part two, let us take $M_2 = [M_{2,1} : \ldots : M_{2,n_2}]$ where $M_{2,j}$'s correspond to different Jordan subblocks. Then

$$W'_n M_2 \mathbf{P}_{n2} = \left(W'_n M_{2,1}, W'_n M_{2,2}, \ldots, W'_n M_{2,n_2}\right) \begin{pmatrix} P_{n2,1} & & & \\ & P_{n2,2} & & \\ & & \ddots & \\ & & & P_{n2,n_i} \end{pmatrix}$$

is asymptotically normal with covariance matrix given below, where $P_{n2,j}$ is a diagonal matrix of dimension p_{mj} with each entry as $(1/\sqrt{n\log^{2d_{mj}-1} n})$. Here d_{mj} equals to d_{2j} if it corresponds to a real eigenvalue (as in §3), and it is d_{c2j} if it corresponds to a complex case (as in §4), whereas p_{mj} equals to d_{2j} if it corresponds to a real eigenvalue, and it is $2d_{c2j}$ if it corresponds to a complex case as in §4). This is a case for asymptotic mixed normality. In this case, typical entries of the limiting covariance matrix of $W_n' M_2 \mathbf{P}_{n2}$, say V_2, can be seen in (30) and (49) as follows:

$$V_2(j,l) = \frac{1}{(d_{mj}+d_{ml}-1)\,((d_{mj}-1)!(d_{ml}-1)!)} \times (F'_{mj})^{d_{mj}-1}\left(\frac{1}{2}\Lambda'_{mj}M'_{2,j}D_\pi M_{2,l}\Lambda_{ml} + \frac{1}{2}C'_{mj}\Lambda'_{mj}M'_{2,j}D_\pi M_{2,l}\Lambda_{ml}C_{ml}\right)F_{ml}^{d_{ml}-1},$$

where Λ_{mj} is the subblock of G_2 corresponding to $M_{2,j}$.

6. Appendix

Suppose U_n and V_n are normalized linear combinations corresponding to eigenvectors ξ_3, ξ_4, with eigenvalues λ_3, λ_4, respectively both of which are real and greater than 1/2. We want to show that the limit of EU_nV_n exists. This technique has been used in the proof of Lemma 3.1 of Freedman [7]. U_n and V_n satisfy the following equations:

$$U_{n+1} - U_n = \lambda_3 \frac{\chi'_{n+1}\xi_3}{\Pi_0^n\left(1+\frac{\lambda_3}{j+1}\right)} - \frac{\frac{\lambda_3}{n+1}}{1+\frac{\lambda_3}{n+1}}U_n,$$

$$V_{n+1} - V_n = \lambda_4 \frac{\chi'_{n+1}\xi_4}{\Pi_0^n\left(1+\frac{\lambda_4}{j+1}\right)} - \frac{\frac{\lambda_4}{n+1}}{1+\frac{\lambda_4}{n+1}}V_n. \tag{55}$$

Using the martingale property it follows that

$$E\{U_{n+1}V_{n+1}|\mathcal{F}_n\} = U_nV_n\left(1 - \frac{\frac{\lambda_3}{n+1}}{1+\frac{\lambda_3}{n+1}}\frac{\frac{\lambda_4}{n+1}}{1+\frac{\lambda_4}{n+1}}\right) + \frac{\lambda_3\lambda_4}{\Pi_0^n\left(1+\frac{\lambda_3}{j+1}\right)\Pi_0^n\left(1+\frac{\lambda_4}{j+1}\right)}\left\langle\frac{\mathbf{W}_n}{n+1}, \xi_3\xi_4\right\rangle,$$

$$EU_{n+1}V_{n+1} = EU_nV_n\left(1 - \frac{\frac{\lambda_3}{n+1}}{1+\frac{\lambda_3}{n+1}}\frac{\frac{\lambda_4}{n+1}}{1+\frac{\lambda_4}{n+1}}\right) + \frac{\lambda_3\lambda_4}{\Pi_0^n\left(1+\frac{\lambda_3}{j+1}\right)\Pi_0^n\left(1+\frac{\lambda_4}{j+1}\right)}\left\langle E\frac{\mathbf{W}_n}{n+1}, \xi_3\xi_4\right\rangle. \tag{56}$$

Notice that by the dominated convergence theorem and the strong law, $E\frac{\mathbf{W}_n}{n+1}$ converges to $\langle \pi, \xi_3\xi_4 \rangle$. Iterating the above equation and using $\Pi_0^n\big(1+\frac{\lambda_3}{j+1}\big) \sim \frac{1}{\Gamma(\lambda_3+1)} n^{\lambda_3}$, we get (remembering $\lambda_3, \lambda_4 > 1/2$) that EU_nV_n converges, although we cannot definitely say that the limit is nonzero. In particular, the same technique yields the L^2-boundedness of Z_n of §1.

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