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Compact solutions to the equation $Tx = y$ in a weakly closed $\mathcal{T}(\mathcal{N})$-module

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Abstract. Given two vectors x, y in a Hilbert space and a weakly closed $\mathcal{T}(\mathcal{N})$-module $\mathcal{U}$, we provide a necessary and sufficient condition for the existence of a compact operator T in $\mathcal{U}$ satisfying $Tx = y$.

Keywords. Compact interpolation; weakly closed $\mathcal{T}(\mathcal{N})$-modules.

1. Introduction

Given a set $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ the one-vector interpolation problem for $\mathcal{A}$ is the following: for vectors x, y in $\mathcal{H}$, does an operator $T \in \mathcal{A}$ exist such that $Tx = y$? A variation, 'the multi-vector interpolation problem' seeks for an operator $T \in \mathcal{A}$ such that $Tx_\alpha = y_\alpha$ for fixed two families of vectors $\{x_\alpha\}$, $\{y_\alpha\}$ in a Hilbert space.

Problems of the above form have attracted some interest for several years and there are many related results in the literature. An early and particularly deep example of this type of result is Kadison's transitivity theorem. In the case of a nest algebra, the one-vector interpolation problem was solved by Lance [7]. Recently, in [2], the authors studied the problem of finding T so that $Tx = y$ and T is required to lie in certain ideals contained in the nest algebras. Hopenwasser [4,5] generalized some of the above results to arbitrary CSL algebras. On the other hand, Katsoulis, Moore and Trent [6] generalized Lance's theorem to multi-vectors and Anoussis [1] arrived at the same result independently. In this paper, we mainly consider the compact interpolation problem for a weakly closed $\mathcal{T}(\mathcal{N})$-module, and give a necessary and sufficient condition on vectors x, y in a Hilbert space for the existence of a compact operator T lying in a weakly closed $\mathcal{T}(\mathcal{N})$-module, such that $Tx = y$. In §2, we first consider the interpolation problem for a weakly closed $\mathcal{T}(\mathcal{N})$-module. After this paper is completed, we note that the results in §2 have already appeared in [8]. However for completeness and the need of §3, we state it here. In §3, we solve the compact interpolation question completely.

Let us introduce some notations and terminologies. $\mathcal{H}$ represents a complex Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$. A nest $\mathcal{N}$ is a chain of closed subspaces of Hilbert space $\mathcal{H}$ containing (0) and $\mathcal{H}$ which is closed under intersection and closed span. If $\mathcal{N}$ is a nest, then the nest algebra $\mathcal{T}(\mathcal{N})$ is the set of all operators T such that $TN \subseteq N$ for every element N in $\mathcal{N}$. Suppose that $N \to \tilde{N}$ is an order homomorphism of

$\mathcal{N}$ into itself (that is, $N \leq N'$ implies $\tilde{N} \leq \tilde{N'}$). Then the set

$$\mathcal{U} = \{X \in \mathcal{B}(\mathcal{H})\colon XN \subseteq \tilde{N},\ \forall N \in \mathcal{N}\}$$

is clearly a weakly closed (two-sided) $\mathcal{T}(\mathcal{N})$-module. In [3], Erdos and Power showed that every weakly closed $\mathcal{T}(\mathcal{N})$-module is of the above form, which the order homomorphism $N \to \tilde{N}$ is left continuous.

2. Interpolation in $\mathcal{U}$

Let $\mathcal{F} = \{(0) = N_0, N_1, \ldots, N_{n-1}, N_n = \mathcal{H}\}$ be a finite sub-nest of $\mathcal{N}$, and denote $\mathcal{U}(\mathcal{F}) = \{T \in \mathcal{B}(\mathcal{H})\colon TN \subseteq \tilde{N}, \forall N \in \mathcal{F}\}$. Denote the algebra of upper triangular matrices by $\mathcal{N}_n$.

Lemma 2.1 [7]. *Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be vectors in C^n. Here C is the complex number field, and suppose that for $1 \leq r \leq n$,*

$$\sum_{i=r}^{n} |y_i|^2 \leq M^2 \sum_{i=r}^{n} |x_i|^2$$

for some $M \geq 0$. Then there is a matrix A in $\mathcal{N}_n$ such that $\|A\| \leq M$ and $Ax = y$. If $y_p = 0$ for some p $(1 \leq p \leq n)$, the matrix $A = (a_{ij})$ can be chosen so that $a_{pj} = 0$ $(1 \leq j \leq n)$. If $x_q = 0$ for some q $(1 \leq q \leq n)$, then A can be chosen so that $a_{iq} = 0$ $(1 \leq i \leq n)$.

PROPOSITION 2.1

Let $\mathcal{F}$ be a finite sub-nest of $\mathcal{N}$. Let x, y be vectors in $\mathcal{H}$. Let $M > 0$. The following statements are equivalent.

(1) *There exists a finite rank operator T in $\mathcal{U}(\mathcal{F})$ such that $Tx = y$ and $\|T\| \leq M$;*
(2) $\|P(\tilde{N})^{\perp} y\| \leq M \cdot \|P(N)^{\perp} x\|$ *for every element N in $\mathcal{F}$.*

Proof.

(1) $\Rightarrow$ (2). If there is an operator $T \in \mathcal{U}(\mathcal{F})$ such that $Tx = y$, then

$$\begin{aligned}\|P(\tilde{N})^{\perp} y\| &= \|P(\tilde{N})^{\perp} Tx\| \\ &= \|P(\tilde{N})^{\perp} TP(N)^{\perp} x\| \\ &\leq \|P(\tilde{N})^{\perp} T\|\|P(N)^{\perp} x\| \leq M\|P(N)^{\perp} x\|.\end{aligned}$$

(2) $\Rightarrow$ (1). Let $\mathcal{F} = \{N_i\colon 0 \leq i \leq n\}$, let $x_i = (P(N_i) - P(N_{i-1}))x$, $y_i = (P(\widetilde{N_i}) - P(\widetilde{N_{i-1}}))y$ $(1 \leq i \leq n)$. Let H_x, H_y be the subspaces of $\mathcal{H}$ spanned by $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ respectively. By the condition (2), the data of Lemma 2.1 is satisfied with $x = (\|x_1\|, \ldots, \|x_n\|)$, $y = (\|y_1\|, \ldots, \|y_n\|)$. Hence there is a matrix B in $\mathcal{N}_n$ such that $\|B\| \leq M$ and $\sum_{j=i}^{n} b_{ij}\|x_j\| = \|y_i\|$ $(1 \leq i \leq n)$. Also by Lemma 2.1, we may suppose that $b_{ij} = 0$ whenever x_j or y_i is zero. Define

$$a_{ij} = \begin{cases} \dfrac{b_{ij}}{\|x_j\|\|y_i\|}, & \text{when } b_{ij} \neq 0 \\ \\ 0, & \text{when } b_{ij} = 0 \end{cases}.$$

Thus $A = (a_{ij}) \in \mathcal{N}_n$. We define T_A in $\mathcal{B}(\mathcal{H})$ by

$$T_A = \sum\sum_{1 \le i \le j \le n} a_{ij} y_i \otimes x_j.$$

It is routine to prove that $T_A \in \mathcal{U}(\mathcal{F})$, and it is clear that $\|T_A\| = \sup_{z \in H_x} \|T_A z\| \|z\|^{-1}$.

Let $z = \sum_{i=1}^n \xi_i x_i \in H_x$ ($\xi_i \in C$). Then we have

$$\|z\|^2 = \sum_{i=1}^n |\xi_i|^2 \|x_i\|^2.$$

Now

$$\begin{aligned} \|T_A\|^2 &= \sup_{z \in H_x, \|z\|=1} \left\| \sum\sum_{1 \le i \le j \le n} a_{ij} \langle z, x_j \rangle y_i \right\|^2 \\ &= \sup_{z \in H_x, \|z\|=1} \left\| \sum\sum_{1 \le i \le j \le n} a_{ij} \xi_j \|x_j\|^2 y_i \right\|^2 \\ &= \sup_{z \in H_x, \|z\|=1} \sum_{i=1}^n \left| \sum_{j=i}^n a_{ij} \xi_j \|x_j\|^2 \right|^2 \cdot \|y_i\|^2 \\ &= \sup_{z \in H_x, \|x\|=1} \sum_{i=1}^n \left| \sum_{j=i}^n (a_{ij} \|x_j\| \cdot \|y_i\|) \xi_j \|x_j\| \right|^2 \\ &\le \sup_{w \in C^n, \|w\|=1} \sum_{i=1}^n \left| \sum_{j=i}^n (a_{ij} \|x_j\| \cdot \|y_i\|) w_j \right|^2. \end{aligned}$$

Hence $\|T_A\| \le \|B\| \le M$, since $b_{ij} = a_{ij} \|x_j\| \cdot \|y_i\|$ by the definition of a_{ij}. Since $\sum_{j=i}^n b_{ij} \|x_j\| = \|y_i\|$, we have

$$\sum_{j=i}^n a_{ij} \|x_j\|^2 \|y_i\| = \|y_i\|,$$

and so

$$\sum_{j=i}^n a_{ij} \|x_j\|^2 = 1, \quad \text{whenever } \; y_i \neq 0.$$

It follows that

$$\begin{aligned} T_A x &= \sum\sum_{1\le i\le j\le n} a_{ij}\langle x, x_j\rangle y_i \\ &= \sum\sum_{1\le i\le j\le n} a_{ij}\|x_j\|^2 y_i \\ &= \sum_{i=1}^{n} y_i = P(\tilde{H})y. \end{aligned}$$

It follows from the hypothesis of (2) that $P(\tilde{H})^{\perp}y = 0$ and $P(\tilde{H})y = y$. Hence $T_A x = y$. Thus with T_A for T, the lemma is proved. □

After this present paper is completed, we note that the following result has been proved in [8]. For completeness, we state it here.

Theorem 2.1 [8]. *Let $\mathcal{U}$ be a weakly closed $\mathcal{T}(\mathcal{N})$- module, and let $x, y \in \mathcal{H}$. The following statements are equivalent.*

(1) $Tx = y$ *for some* T *in* $\mathcal{U}$.

(2) $M = \sup_{N\in\mathcal{N}} \dfrac{\|P(\tilde{N})^{\perp}y\|}{\|P(N)^{\perp}x\|} < +\infty.$

If these conditions are satisfied, then T can be chosen so that $\|T\| = M$.

3. Compact interpolation in $\mathcal{U}$

Given two vectors x and y in $\mathcal{H}$, is there a compact operator T in $\mathcal{U}$ which maps x to y? In order to solve this problem we define

$$N_x = \wedge\{N \in \mathcal{N}: x \in N\}$$

and we examine two cases.

Case 1. $(N_x)_- \neq N_x$. In this case, the interpolation problem is trivial; the condition that $y \in \widetilde{N_x}$ is necessary and sufficient for the existence of an interpolating operator for x and y.

Indeed, if $y \in \widetilde{N_x}$ the rank one operator

$$T = y \otimes \frac{(P(N_x) - P((N_x)_-))x}{\|(P(N_x) - P((N_x)_-))x\|^2}$$

belongs to $\mathcal{U}$ and maps x to y.

On the other hand, if there exists an operator T in $\mathcal{U}$ such that $Tx = y$ then

$$P(\widetilde{N_x})^{\perp}y = P(\widetilde{N_x})^{\perp}Tx = P(\widetilde{N_x})^{\perp}TP(N_x)^{\perp}x = 0$$

and hence $y \in \widetilde{N_x}$.

Case 2. $(N_x)_- = N_x$. In this case our result reads as follows.

Theorem 3.1. *Let $\mathcal{U}$ be a weakly closed $\mathcal{T}(\mathcal{N})$-module and x, y be vectors in $\mathcal{H}$. If $(N_x)_- = N_x$ then the following are equivalent.*

(1) *There exists a compact operator T in $\mathcal{U}$ such that $Tx = y$.*
(2) *$y \in \widetilde{N_x}$ and moreover, the net*

$$\left\{ \frac{\|P(\tilde{N})^\perp y\|}{\|P(N)^\perp x\|} \right\}_{N \in \mathcal{N},\ N < N_x}$$

converges to zero as N increases to N_x.

In addition, if statement (2) *is true then, given $\varepsilon > 0$, the interpolating operator T can be chosen such that $\|T\| \le \sqrt{2}M + \varepsilon$, where*

$$M = \sup \left\{ \frac{\|P(\tilde{N})^\perp y\|}{\|P(N)^\perp x\|}, N \in \mathcal{N}\ N < N_x \right\}.$$

It is easy to see that (1) implies (2). Indeed, if there is a compact operator T in $\mathcal{U}$ such that $Tx = y$, then

$$\frac{\|P(\tilde{N})^\perp y\|}{\|P(N)^\perp x\|} = \frac{\|P(\tilde{N})^\perp Tx\|}{\|P(N)^\perp x\|} = \frac{\|P(\tilde{N})^\perp TP(N)^\perp P(N_x)x\|}{\|P(N)^\perp x\|}$$

$$= \frac{\|P(\tilde{N})^\perp TP(N_x)P(N)^\perp x\|}{\|P(N)^\perp x\|} \le \|TP(N_x)P(N)^\perp\|$$

and the conclusion that the quotient converges to zero follows from the compactness of T.

In order to prove (2) $\Rightarrow$ (1), we need several lemmas.

Lemma 3.1. *Let f and g be non-increasing functions mapping* [0,1] *into the non-negative real numbers, and let g be continuous. Suppose that*

$$L = \sup \left\{ \frac{f(t)}{g(t)} : t \in [0, 1] \right\} < +\infty,$$

and assume that $g(t) = 0$ if and only if $t = 1$. Then for a given $\delta > 0$, there exists a sequence $\{t_n\}_{n=0}^{+\infty}$, increasing to 1, *such that for each $n \in \mathbb{N}$,*

$$\frac{f(t_{n-1}) - f(t_n)}{g(t_n) - g(t_{n+1})} \le L + \delta.$$

Proof. By examining the proof of Lemma 1.2 of [2], we know that the hypothesis of the continuity of f is not needed. □

The condition (2) in Theorem 3.1 implies that the quantity M is finite, since, if N were bounded away from N_x, the relevant quotients would certainly be bounded. Thus, if the condition (2) holds, then so does the condition (2) in Theorem 2.1 (where we understand that $0/0 = 0$ for $N \ge N_x$ in $\mathcal{N}$).

Lemma 3.2. *If* $\mathcal{N}$ *is a continuous nest, the statement* (2) *of Theorem* 3.1 *implies statement* (1). *Moreover, given* $\varepsilon > 0$, *the interpolating operator* T *can be chosen to have norm no greater than* $M + \varepsilon$.

Proof. We assume without loss of generality that $\mathcal{N}$ is ordered by the interval [0,1], and we denote the elements in $\mathcal{N}$ by $\{N_t\}_{t\in[0,1]}$. Let x and y be vectors which satisfy condition (2) of Theorem 3.1. If $\varepsilon > 0$, let $\varepsilon_0 = M + \frac{\varepsilon}{2}$ and $\varepsilon_n = \frac{\varepsilon}{2^{n-1}}$, $n \geq 1$. For each $n \geq 0$, we shall find an increasing sequence $\{N_n\}_{n=0}^{+\infty}$ in $\mathcal{N}$ and finite-rank operators T_n in $\mathcal{U}$ such that

(a) $\sup\left\{\frac{\|P(\tilde{N})^{\perp}y\|}{\|P(N)^{\perp}x\|} \colon N \in \mathcal{N} \text{ and } N_n \leq N < N_x\right\} < \varepsilon_n$, and
(b) $T_n x = (P(\widetilde{N_n}) - P(\widetilde{N_{n-1}}))y$ and $\|T_n\| \leq \varepsilon_{n-1}$ for $n > 0$.

We proceed by induction. First, set $N_0 = 0$ and $T_0 = 0$. Suppose that we have found elements $N_1, N_2, \ldots, N_{n-1}$ and operators $T_1, T_2, \ldots, T_{n-1}$ satisfying the required conditions. Let $(P(N_x) - P(N_{n-1}))\mathcal{N}$ be the compression of the nest $\mathcal{N}$ to the subspace $N_x \ominus N_{n-1}$, the compression is still a continuous nest, and we again order its elements by the unit interval, that is, $(P(N_x) - P(N_{n-1}))\mathcal{N} = \{N_{n,t} \colon t \in [0, 1]\}$. Let $f(t) = \|P(\widetilde{N_{n-1} \oplus N_{n,t}})^{\perp}y\|^2$ and $g(t) = \|P(N_{n-1} \oplus N_{n,t})^{\perp}x\|^2$. Since

$$\sup\left\{\frac{f(t)}{g(t)} \colon t \in [0, 1]\right\}$$

$$= \sup\left\{\frac{\|P(\widetilde{N})^{\perp}y\|^2}{\|P(N)^{\perp}x\|^2} \colon N \in \mathcal{N} \text{ and } N_{n-1} \leq N < N_x\right\} < \varepsilon_{n-1}^2$$

and $g(t) = 0$ if and only if $t = 1$, we observe that functions f, g satisfy the hypothesis of Lemma 3.1. Therefore there exists a sequence $\{t_k\}_{k=0}^{+\infty}$ such that

$$\frac{\|(P(\widetilde{N_{n-1} \oplus N_{n,t_k}}) - P(\widetilde{N_{n-1} \oplus N_{n,t_{k-1}}}))y\|^2}{\|(P(N_{n,t_{k+1}}) - P(N_{n,t_k}))x\|^2} = \frac{f(t_{k-1}) - f(t_k)}{g(t_k) - g(t_{k+1})} < (\varepsilon_{n-1})^2,$$

for every $k > 0$.

Next, choose an integer K such that

$$\sup\left\{\frac{\|P(\widetilde{N_t})^{\perp}y\|}{\|P(N_t)^{\perp}y\|} \colon N_t \in \mathcal{N} \text{ and } N_x > N_t \geq N_{n-1} \oplus N_{n,t_K}\right\} \leq \varepsilon_n.$$

We now define $N_n = N_{n-1} \oplus N_{n,t_K}$ and

$$T_n = \sum_{k=1}^{K} (P(\widetilde{N_{n-1} \oplus N_{n,t_k}}) - P(\widetilde{N_{n-1} \oplus N_{n,t_{k-1}}}))y$$

$$\otimes \left(\frac{(P(N_{n,t_{k+1}}) - P(N_{n,t_k}))x}{\|(P(N_{n,t_{k+1}}) - P(N_{n,t_k}))x\|^2}\right).$$

By construction, T_n is a finite rank operator in $\mathcal{U}$, $T_n x = (P(\widetilde{N_n}) - P(\widetilde{N_{n-1}}))y$ and $\|T_n\| \leq \varepsilon_{n-1}$. Now let $T = \sum_{n=1}^{+\infty} T_n$. It is easy to see that T is compact and $\|T\| \leq M + \varepsilon$.

Since the order homomorphism $N \to \tilde{N}$ is left-order continuous and the sequence $\{N_n\}_{n=0}^{+\infty}$ converges to $N_x = (N_x)_-$ increasingly, we have that

$$Tx = \sum_{n=1}^{+\infty}(P(\widetilde{N_n}) - P(\widetilde{N_{n-1}}))y = \lim_{N_n \uparrow N_x} P(\widetilde{N_n})y$$
$$= P(\widetilde{N_x})y = y. \qquad \square$$

Lemma 3.3. *Let* $\mathcal{N}$ *be a nest that contains no finite-dimensional atoms. Then the statement* (2) *of Theorem* 3.1 *implies statement* (1). *Moreover, given* $\varepsilon > 0$, *the interpolating operator* T *can be chosen to have norm no greater than* $\sqrt{2}M + \varepsilon$.

Proof. Let x, y be vectors satisfying condition (2) and let $\varepsilon > 0$. Since the quotients $\left\{\frac{\|P(\tilde{N})^{\perp}y\|}{\|P(N)^{\perp}x\|}: N \in \mathcal{N},\ N < N_x\right\}$ converge to zero, we can choose an increasing sequence of elements $0 = N_0 < N_1 < N_2 < \cdots$ (with limit N_x) and for each $n \geq 1$,

$$\sup\left\{\frac{\|P(\tilde{N})^{\perp}y\|}{\|P(N)^{\perp}x\|}: N \in \mathcal{N}, N_n \leq N < N_x\right\} \leq \frac{\varepsilon}{2^{n+2}\sqrt{2}}.$$

We will be working separately on the intervals $N_{n+1} \ominus N_n$. Such an interval may contain some atoms from the nest, but there are at most countably many atoms inside each interval. Enumerate the atoms that are contained in $N_{n+1} \ominus N_n$ and let $\mathcal{A}_n = \{Q_k^{(n)}: Q_k^{(n)} = M_k^{(n)} \ominus N_k^{(n)}$ is an atom contained in $N_{n+1} \ominus N_n$, with $M_k^{(n)}$ and $N_k^{(n)}$ elements in $\mathcal{N}$, $k \geq 0\}$.

Let $y_d = \sum_{n=0}^{+\infty}\sum_{k=0}^{+\infty} \widetilde{Q_k^{(n)}}y$ and let $y_c = y - y_d$. We proceed in two steps.

Step 1. There exists a compact $T_c \in \mathcal{U}$ such that $T_c x = y_c$ and $\|T_c\| \leq M + \frac{\varepsilon}{\sqrt{2}}$. In order to prove this fact, we observe that, because the atoms in $\mathcal{N}$ are all infinite-dimensional, there exists a continuous nest $\mathcal{M}$ containing $\mathcal{N}$. If $N \ominus N_-$ is an infinite atom of $\mathcal{N}$ and $N_- < M \leq N$, we define $\tilde{M} = \tilde{N}$. It is easy to see that the map $M \to \tilde{M}$ of $\mathcal{M}$ into $\mathcal{M}$ is a left-continuous order homomorphism and is an extension of the map $N \to \tilde{N}$ of $\mathcal{N}$ into $\mathcal{N}$. Set $\mathcal{U}(\mathcal{M})$ the weakly closed $\mathcal{T}(\mathcal{M})$-module determined by the map $M \to \tilde{M}$. We claim that the net $\left\{\frac{\|P(\tilde{M})^{\perp}y_c\|}{\|P(M)^{\perp}x\|}: M < N_x, M \in \mathcal{M}\right\}$ converges to zero as $M \uparrow N_x$. To see this, let $\delta > 0$ and choose $N' \in \mathcal{N}$ such that

$$\sup\left\{\frac{\|P(\tilde{N})^{\perp}y\|}{\|P(N)^{\perp}x\|}: N \in \mathcal{N},\ N' \leq N < N_x\right\} \leq \delta.$$

It will be enough to prove that for any $M \in \mathcal{M}$ such that $N_x > M \geq N'$, we have $\frac{\|P(\tilde{M})^{\perp}y_c\|}{\|P(M)^{\perp}x\|} \leq \delta$. If the element M actually lies in $\mathcal{N}$, then

$$\frac{\|P(\tilde{M})^{\perp}y_c\|}{\|P(M)^{\perp}x\|} \leq \frac{\|P(\tilde{M})^{\perp}y\|}{\|P(M)^{\perp}x\|} \leq \delta.$$

On the other hand, suppose that M does not lie in $\mathcal{N}$. Then M is one of the new element which fill in the atoms of $\mathcal{N}$, and so there exist integer $n, k \geq 0$ such that $N_k^{(n)} < M < M_k^{(n)}$.

Consequently, because $(P(\widetilde{M_k^{(n)}}) - P(\widetilde{N_k^{(n)}}))y_c = 0$,

$$\frac{\|P(\tilde{M})^\perp y_c\|}{\|P(M)^\perp x\|} = \frac{\|P(\widetilde{M_k^{(n)}})^\perp y_c\|}{\|P(M)^\perp x\|} < \frac{\|P(\widetilde{M_k^{(n)}})^\perp y\|}{\|P(M_k^{(n)})^\perp x\|} \leq \delta.$$

Thus we have established our claim.

Combining the claim with Lemma 3.2, we obtain the existence of a compact operator $T_c \in \mathcal{U}(\mathcal{M}) \subseteq \mathcal{U}$ such that $T_c x = y_c$ and $\|T_c\| \leq M + \frac{\varepsilon}{\sqrt{2}}$. This finishes Step 1.

Step 2. There exists a compact operator $T_d \in \mathcal{U}$ such that $T_d x = y_d$ and $\|T_d\| \leq M + \frac{\varepsilon}{\sqrt{2}}$.

To prove Step 2, we hope to find, for each non-negative integer n, a compact operator $T_n \in \mathcal{U}$ such that $T_n x = \sum_{k=0}^{+\infty} \widetilde{Q_k^{(n)}} y$ and $\|T_n\| \leq \varepsilon_n$, where $\varepsilon_0 = M + \frac{\varepsilon}{2\sqrt{2}}$ and $\varepsilon_n = \frac{\varepsilon}{2^{n+1}\sqrt{2}}$ for $n \geq 1$. The sum of these operators will be the desired compact operator T_d.

First, choose an integer $n \geq 0$. Since the sum $\sum_{k=0}^{+\infty} Q_k^{(n)} y$ converges, we can choose a positive k_n so that

$$\left\| \sum_{k=k_n+1}^{+\infty} Q_k^{(n)} y \right\| \leq \left(\frac{\varepsilon}{2^{n+2}\sqrt{2}} \right) \|P(N_{n+1})^\perp x\|.$$

Set

$$u_n = \sum_{k=0}^{k_n} \widetilde{Q_k^{(n)}} y \quad \text{and} \quad v_n = \sum_{k=k_n+1}^{+\infty} \widetilde{Q_k^{(n)}} y.$$

Now consider the finite sub-nest $\mathcal{F}_n = \{0, I\} \cup \{M_k^{(n)}, N_k^{(n)}: 0 \leq k \leq k_n\}$. Clearly,

$$\sup \left\{ \frac{\|P(\tilde{N})^\perp u_n\|}{\|P(N)^\perp x\|} : N \in \mathcal{F}_n \right\} \leq \begin{cases} M + \dfrac{\varepsilon}{4\sqrt{2}}, & \text{if } n = 0 \\ \dfrac{\varepsilon}{2^{n+2}\sqrt{2}}, & \text{if } n > 0 \end{cases}.$$

Thus Proposition 2.1 implies that there exists a finite rank operator $S_n \in \mathcal{U}(\mathcal{F}_n)$ (see the beginning of §2) such that $S_n x = u_n$ and $\|S_n\| \leq \frac{\varepsilon}{2^{n+2}\sqrt{2}}$ if $n > 0$ $\left(\|S_0\| \leq M + \frac{\varepsilon}{4\sqrt{2}}\right)$. Multiplying on the left (if necessary) by $\sum_{k=1}^{k_n} (\widetilde{Q_k^{(n)}})$, we may assume that S_n is a finite rank operator in $\mathcal{U}$. On the other hand, the operator

$$W_n = v_n \otimes \left(\frac{P(N_{n+1})^\perp x}{\|P(N_{n+1})^\perp x\|^2} \right)$$

is a compact operator in $\mathcal{U}$ with norm less than $\frac{\varepsilon}{2^{n+2}\sqrt{2}}$ which maps x to v_n. Therefore, the operator $T_n = S_n + W_n$ is a compact operator in $\mathcal{U}$ which maps x to $\sum_{k=0}^{+\infty} \widetilde{Q_k^{(n)}} y$ and $\|T_n\| \leq \varepsilon_n$. Step 2 is now complete.

The proof of the lemma follows easily, when we consider the operator

$$T = \left(\sum_{k,n} \widetilde{Q_k^{(n)}} \right) T_d + \left(\sum_{k,n} \widetilde{Q_k^{(n)}} \right)^\perp T_c.$$

□

Now we are in a position to conclude the proof of Theorem 3.1. The remaining task is to prove sufficiency. Suppose, then, that x and y are vectors in the Hilbert space for which condition (2) satisfies. The theorem has been proved in case all the atoms in $\mathcal{N}$ are infinite-dimensional. Our plan is to inflate the nest so that all the atoms become infinite-dimensional. Let $\hat{\mathcal{N}} = \mathcal{N}\hat{\otimes} I$ be the infinite inflation of $\mathcal{N}$, acting on the Hilbert space $\mathcal{H}\hat{\otimes}\mathcal{H}$. Set $\hat{\mathcal{U}}$ as the weakly closed $\mathcal{T}(\hat{\mathcal{N}})$-module determined by the map $N\hat{\otimes} I \to \tilde{N}\hat{\otimes} I$. Choose any unit vector u in $\mathcal{H}$ and consider vectors $\hat{x} = x\hat{\otimes} u$ and $\hat{y} = y\hat{\otimes} u$. Since

$$\frac{\|P(\tilde{N}\hat{\otimes} I)^{\perp}(y\hat{\otimes} u)\|}{\|P(N\hat{\otimes} I)^{\perp}(x\hat{\otimes} u)\|} = \frac{\|(P(\tilde{N})^{\perp}\hat{\otimes} I)(y\hat{\otimes} u)\|}{\|(P(N)^{\perp}\hat{\otimes} I)(x\hat{\otimes} u)\|} = \frac{\|P(\tilde{N})^{\perp} y\|}{\|P(N)^{\perp} x\|},$$

the net

$$\left\{ \frac{\|P(\tilde{N}\hat{\otimes} I)^{\perp}(y\hat{\otimes} u)\|}{\|P(N\hat{\otimes} I)^{\perp}(x\hat{\otimes} u)\|} : N \in \mathcal{N}, N < N_x \right\}$$

converges to zero as N increases to N_x. Since $\hat{\mathcal{N}}$ has no finite-dimensional atoms, Lemma 3.3 implies the existence of a compact operator $X \in \hat{\mathcal{U}}$ such that $X(x\hat{\otimes} u) = y\hat{\otimes} u$ and $\|X\| \leq \sqrt{2}M + \varepsilon$, where M is the quantity defined in the statement of the theorem.

Let P_u be the orthogonal projection of $\mathcal{H}$ onto $\overline{\text{span}}\{u\}$. Then the operator $I\hat{\otimes} P_u$ belongs to $\mathcal{T}(\hat{\mathcal{N}})$, and therefore $(I\hat{\otimes} P_u)X(I\hat{\otimes} P_u)$ is a compact operator in $\hat{\mathcal{U}}$, and which maps $x\hat{\otimes} u$ to $y\hat{\otimes} u$. But the fact that P_u is of rank one implies that there exists an operator $T \in \mathcal{B}(\mathcal{H})$ such that $(I\hat{\otimes} P_u)X(I\hat{\otimes} P_u) = T\hat{\otimes} P_u$, and, moreover,

$$\|T\| = \|T\hat{\otimes} P_u\| \leq \|X\| \leq \sqrt{2}M + \varepsilon.$$

It is easy to see that T is the desired operator. □

Given two families $\{x_\alpha\}$, $\{y_\alpha\}$ of vectors, which condition will ensure the existence of a compact interpolation operator T in $\mathcal{U}$ satisfying $Tx_\alpha = y_\alpha$? We will examine this question in a subsequent paper.

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