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Positive linear operators generated by analytic functions

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Abstract. Let φ be a power series with positive Taylor coefficients $\{a_k\}_{k=0}^{\infty}$ and non-zero radius of convergence $r \le \infty$. Let ξ_x, $0 \le x < r$ be a random variable whose values α_k, $k = 0, 1, \ldots$, are independent of x and taken with probabilities $a_k x^k/\varphi(x)$, $k = 0, 1, \ldots$.

The positive linear operator $(A_\varphi f)(x) := \mathbf{E}[f(\xi_x)]$ is studied. It is proved that if $\mathbf{E}(\xi_x) = x$, $\mathbf{E}(\xi_x^2) = qx^2 + bx + c$, $q, b, c \in \mathbf{R}$, $q > 0$, then A_φ reduces to the Szász–Mirakyan operator in the case $q = 1$, to the limit q-Bernstein operator in the case $0 < q < 1$, and to a modification of the Lupaş operator in the case $q > 1$.

Keywords. Szász–Mirakyan operator; positive operator; limit q-Bernstein operator; q-integers; Poisson distribution; totally positive sequence.

1. Introduction

The Szász–Mirakyan operator is closely related to the Poisson distribution. If θ_x is a random variable having Poisson distribution with parameter nx, $x > 0$, $n \in \mathbf{N}$, then for a bounded continuous function $f\colon [0, \infty) \to \mathbf{R}$, the Szász–Mirakyan operator [7, 11] is defined by

$$S_n(f; x) = \mathbf{E}\left[f\left(\frac{\theta_x}{n}\right)\right] = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\frac{(nx)^k}{k!}\exp(-nx). \tag{1.1}$$

Since

$$\mathbf{E}\left[\frac{\theta_x}{n}\right] = x, \quad \mathbf{Var}\left[\frac{\theta_x}{n}\right] = \frac{x}{n}, \tag{1.2}$$

the uniform convergence of $S_n(f; x)$ to $f(x)$ on any compact subset of $[0, \infty)$ is justified by Feller's lemma (v. II, Ch. VII, §1, Lemma 1 of [3]). The approximation by the Szász–Mirakyan operator and its numerous generalizations have been studied by a vast number of authors (see, for example [1] and references therein).

In this paper, we consider a class of probability distributions generated by analytic functions with positive coefficients, and we show that under some conditions on the mathematical expectation and variance that are natural analogues of (1.2), these distributions lead to the construction of operators generalizing (1.1).

Let φ be a function with positive Taylor coefficients analytic in the disc $\{x\colon |x| < r\}$, $0 < r \le \infty$:

$$\varphi(x) = \sum_{k=0}^{\infty} a_k x^k, \; a_0 = 1, \; a_k > 0, \quad k \in \mathbf{N}. \tag{1.3}$$

Consider a random variable ξ_x $(0 \leq x < r)$ whose values $\{\alpha_k\}_{k=0}^{\infty}$ do not depend on x and which are taken with the following probabilities:

$$\mathbf{P}\{\xi_x = \alpha_k\} = \frac{a_k x^k}{\varphi(x)} =: p_k(x), \; k \in \mathbf{Z}_+ := \mathbf{N} \cup \{0\}. \tag{1.4}$$

Clearly, (1.4) defines a discrete probability distribution depending on parameter x. The sequence $\{\alpha_k\}$ of values will be chosen later on.

Let X be the linear space of functions defined on $\{\alpha_k\}$ so that the mathematical expectation $\mathbf{E}[f(\xi_x)]$ exists for each x. Then, we can define a linear operator on X as follows:

$$\mathbf{E}[f(\xi_x)] = \sum_{k=0}^{\infty} f(\alpha_k) p_k(x) =: (A_\varphi f)(x). \tag{1.5}$$

Clearly, $A_\varphi f$ is a function defined on $[0, r)$.

Suppose that the distribution (1.4) satisfies the following conditions:

(i) $$\mathbf{E}(\xi_x) = x, \tag{1.6}$$

that is A_φ leaves invariant linear functions;

(ii) $$\mathbf{E}(\xi_x^2) = qx^2 + bx + c, \; q, b, c \in \mathbf{R}, \quad q \neq 0, \tag{1.7}$$

that is A_φ takes a square polynomial to a square polynomial.

It turns out that properties (1.6) and (1.7) specify a class of distributions of the form (1.4) depending on parameters $q > -1$ and $b > 0$. We show that the operator (1.5) reduces to the Szász–Mirakyan operator in the case $q = 1$, to the limit q-Bernstein operator in the case $0 < q < 1$, and to a modification of the Lupaş operator in the case $q > 1$. The limit q-Bernstein operator was introduced in [4] as a limit operator for a sequence of the q-Bernstein polynomials. Wang observed that the limit q-Bernstein operator arises also as a limit for a sequence of q-Meyer-König and Zeller operators considered by Trif in [12]. It has been shown that it is a shape-preserving positive linear operator which reproduces linear functions and approximates continuous functions on [0, 1] as $q \to 1^-$. Its various properties has been studied in [8], [9], [13], [14] and [16]. It is a basic example for Wang's Korovkin type theorem (see [15]). The Lupaş operator emerges as a limit for a sequence of the Lupaş q-analogue of the Bernstein operator. Its approximation and shape-preserving properties have been studied in [6] and [10].

Therefore, in this paper, we have obtained a common approach to different operators studied previously without any connections among them. We also consider the approximation with operator (1.5).

2. Statement of results

Our first result describes admissible values of the random variable ξ_x subject to (1.6) and (1.7).

Lemma 2.1. *Let φ be a function given by* (1.3), *and ξ_x be a random variable with distribution* (1.4). *If* (1.6) *holds*, *then*

$$\alpha_0 = 0; \quad \alpha_k = \frac{a_{k-1}}{a_k}, \quad k \in \mathbf{N}. \tag{2.1}$$

The next lemma gives the values of parameters q, b, c via the coefficients of φ under conditions (1.6) and (1.7).

Lemma 2.2. *Let φ be a function given by* (1.3), *and ξ_x be a random variable with distribution* (1.4). *If both* (1.6) *and* (1.7) *hold, then*

$$q = \frac{a_1^2}{a_2} - 1, \quad b = \frac{1}{a_1}, \ c = 0. \tag{2.2}$$

It can be readily seen from (2.2) that $q > -1, \ b > 0$.

In the sequel we use the following standard notation (see for e.g. Ch. 10, §10.2 of [2]):

$$(z;q)_0 := 1; \quad (z;q)_n := \prod_{k=0}^{n-1}(1 - zq^k); \quad (z;q)_\infty = \prod_{k=0}^{\infty}(1 - zq^k).$$

For any $n \in \mathbf{Z}_+$, the *q-integer* $[n]_q$ is defined by

$$[n]_q := 1 + q + \cdots + q^{n-1} \ (n \in \mathbf{N}), \ [0]_q := 0;$$

and the *q-factorial* $[n]_q!$ by

$$[n]_q! := [1]_q[2]_q \ldots [n]_q \ (n \in \mathbf{N}), \ [0]_q! := 1.$$

Clearly, for $q = 1$,

$$[n]_1 = n, \quad [n]_1! = n!.$$

We will use the following two basic identities of the q-calculus attributed to Euler (see for e.g. Ch. 10, §10.2, Corollary 10.2.2 of [2]):

$$(x;q)_\infty = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)/2} x^k}{(q;q)_k}, \quad |q| < 1, \quad x \in \mathbf{C} \tag{2.3}$$

and

$$\frac{1}{(x;q)_\infty} = \sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k}, \ |q| < 1, \ |x| < 1. \tag{2.4}$$

The function defined by

$$e_q(x) := \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} \tag{2.5}$$

can be considered as a q-analogue of the exponential function. Obviously, $e_1(x) = \exp(x)$. We note that if $|q| < 1$, then the series (2.5) converges for $|x| < 1$, whereas for $|q| \geq 1$, the series converges for all $x \in \mathbf{C}$.

Now, we can describe the class of probability distributions discussed previously.

Theorem 2.3. *Let ξ_x be random variables depending on x, whose distribution* (1.4) *satisfy conditions* (1.6) *and* (1.7). *Then*

$$\alpha_k = b[k]_q \ \ \textit{and} \ \ \varphi(x) = e_q\left(\frac{x}{b}\right). \tag{2.6}$$

Therefore, the distribution of ξ_x *has the form*:

$$\mathbf{P}\{\xi_x = b[k]_q\} = \frac{1}{e_q(x/b)} \cdot \frac{x^k}{b^k [k]_q!}, \ k \in \mathbf{Z}_+. \tag{2.7}$$

Remark 2.1. If $q = b = 1$, then ξ_x has Poisson distribution with parameter x.

Remark 2.2. If $|q| < 1$, then distribution (2.7) is well-defined for $0 \leq x < b/(1-q)$. If $q \geq 1$, then (2.7) is well-defined for all $x \geq 0$.

The following theorems give interesting properties of distribution (1.4).

Theorem 2.4. *The sequence* $\{a_k\}_{k=0}^{\infty}$ *generated by the function* $\varphi(x) = e_q(x/b)$ *is totally positive.*

Theorem 2.5. *For any* $m \in \mathbf{N}$, *the mathematical expectation* $\mathbf{E}(\xi_x^m)$ *is a polynomial of degree* m *in variable* x.

3. Special cases

In this section, we consider some important particular cases of the distribution (1.4), and show that, in this way, we may obtain some known operators as particular cases of (1.5).

Example 3.1. Let $q = 1, \ b = 1/n$. In this case we recover the Szász–Mirakyan operator (1.1).

Indeed, for these values of q and b, we have by Theorem 2.3:

$$\alpha_k = \frac{k}{n} \quad \text{and} \quad e_q(x/b) = \exp(nx).$$

Hence,

$$(A_\varphi f)(x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!} \exp(-nx) = S_n(f; x).$$

Clearly, the distribution of ξ_x coincides with the distribution of θ_x/n, where θ_x has the Poisson distribution with parameter nx (see (1.1)). It has already been mentioned that if $f \in C[0, \infty)$ is bounded, then $S_n(f; x) \to f(x)$ as $n \to \infty$ uniformly on any compact subset of $[0, \infty)$.

Example 3.2. We may slightly generalize Example 3.1 in the following way. Let $q = 1$. The distribution of ξ_x in this case is given by

$$\mathbf{P}\{\xi_x = bk\} = \frac{x^k}{b^k k!} \exp(-x/b), \quad k \in \mathbf{Z}_+$$

with $\mathbf{E}[\xi_x] = x$ and $\mathbf{Var}[\xi_x] = bx$. We construct $A_\varphi = A_b$ according to (1.5):

$$(A_b f)(x) = \sum_{k=0}^{\infty} f(bk) \frac{x^k}{b^k k!} \exp(-x/b).$$

Feller's lemma implies that for any bounded $f \in C[0,\infty)$, $(A_b f)(x)$ converges uniformly to f on any compact subset of $[0,\infty)$ whenever $b \to 0$.

In particular, taking a sequence $b_n \to 0$, we obtain an approximating sequence of positive linear operators:

$$(A_n f)(x) = \sum_{k=0}^{\infty} f(b_n k) \frac{x^k}{b_n^k k!} \exp(-x/b_n).$$

Example 3.3. Let $0 < q < 1$, $b = 1 - q$. Then with

$$\varphi(x) = \frac{1}{(x;q)_\infty}, \quad |x| < 1,$$

we have by Euler's identity (2.4):

$$\mathbf{P}\{\xi_x = 1 - q^k\} = \frac{x^k}{(q;q)_k}(x;q)_\infty, \ k \in \mathbf{Z}_+.$$

In this case, A_φ represents the limit q-Bernstein operator $B_\infty(f,q;x)$. This operator is given by

$$B_\infty(f,q;x) = \sum_{k=0}^{\infty} f(1-q^k)\frac{x^k}{(q;q)_k}(x;q)_\infty, \quad |x| < 1.$$

The operator arises naturally while considering the limit of a sequence of q-Bernstein polynomials, $0 < q < 1$. It was introduced in [4] and studied in a number of papers afterwards. It has been proved that for $f \in C[0,1]$, $B_\infty(f,q;x) \to f(x)$ as $q \to 1^-$ uniformly on $[0,1]$. Estimates for the rate of approximation along with Voronovskaya type theorems are given in [13] and [14]. Other properties of B_∞ can be found, e.g., in [8] and [9].

Example 3.4. Let $q > 1$, $b = 1 - 1/q$. We set $q_1 := 1/q$. In this case, Euler's identity (2.3) implies that

$$\varphi(x) = (-x;q_1)_\infty.$$

Hence, the distribution of ξ_x is given by

$$\mathbf{P}\left\{\xi_x = \frac{1-q_1^k}{q_1^{k-1}}\right\} = \frac{x^k q_1^{k(k-1)/2}}{(q_1;q_1)_k} \cdot \frac{1}{(-x;q_1)_\infty}, \ k \in \mathbf{Z}_+. \tag{3.1}$$

The mathematical expectation and variance of ξ_x in this case are

$$\mathbf{E}[\xi_x] = x, \quad \mathbf{Var}[\xi_x] = (1-q_1) \cdot \frac{x^2 + q_1 x}{q_1}.$$

Using again Feller's lemma, we prove the following statement:

If f is bounded and continuous on $[0,\infty)$, then $(A_\varphi f)(x) \to f(x)$ as $q_1 \to 1^-$ uniformly on any compact subset of $[0,\infty)$.

Remark 3.1. It is worth noting that the probabilities in (3.1) coincide with probabilities appearing in the Lupaş operator (see [6, 10]). The Lupaş operator is given by

$$L(f, q_1; x) := \frac{1}{(-x; q_1)_\infty} \sum_{k=0}^{\infty} \frac{f(1-q_1^k)q_1^{k(k-1)/2}x^k}{(q_1; q_1)_k}.$$

Operators A_φ are not exactly the same as the Lupaş operators. This is because the Lupaş operators do not satisfy (1.7).

4. Proofs of the results

Proofs of Lemmas 2.1 *and* 2.2. Since $\mathbf{E}[\xi_x] = \sum_{k=0}^{\infty} \alpha_k p_k(x)$, we get $\sum_{k=0}^{\infty} \alpha_k a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+1}$. Identifying the coefficients of x^k, we obtain (2.1).

Condition (1.7) implies that

$$\sum_{k=1}^{\infty} \alpha_k^2 a_k x^k = ca_0 + (ca_1 + ba_0)x + \sum_{k=2}^{\infty} (qa_{k-2} + ba_{k-1} + ca_k)\, x^k.$$

Comparing the coefficients of $1, x, x^2$ on both sides, we get (2.2). □

Proof of Theorem 2.3. It follows from Lemma 2.2 that $\mathbf{E}[\xi_x^2] = qx^2 + bx$. Since $\mathbf{E}[\xi_x^2] = \sum_{k=0}^{\infty} \alpha_k^2 p_k(x)$, we obtain

$$\sum_{k=1}^{\infty} \alpha_k^2 a_k x^k = \sum_{k=0}^{\infty} (qa_k x^{k+2} + ba_k x^{k+1}).$$

Equating the coefficients of x^k for $k \geq 2$, we get

$$\alpha_k^2 a_k = qa_{k-2} + ba_{k-1} \quad \text{for all} \quad k = 2, 3, \ldots.$$

Dividing by a_{k-1} and taking into account that by Lemma 2.1, $\alpha_k = a_{k-1}/a_k$, we obtain the following difference equation for α_k:

$$\alpha_{k+1} = q\alpha_k + b, \quad \alpha_1 = b, \quad k \in \mathbf{N}. \tag{4.1}$$

The solution of (4.1) is

$$\alpha_k = b\frac{1-q^k}{1-q} = b[k]_q, \quad k \in \mathbf{Z}_+.$$

For $k = 0$, the formula is true due to (2.1). Since by Lemma 2.1, $\alpha_k = a_{k-1}/a_k$ for $k \in \mathbf{N}$, it follows that

$$a_k = \frac{a_{k-1}}{\alpha_k} = \frac{a_{k-2}}{\alpha_k \alpha_{k-1}} = \cdots = \frac{a_0}{\alpha_k \alpha_{k-1} \ldots \alpha_1} = \frac{1}{b^k [k]_q!}, \quad k \in \mathbf{N}.$$

Obviously, the formula is true for $k = 0$ as well.

Thus,

$$\varphi(x) = \sum_{k=0}^{\infty} \frac{x^k}{b^k [k]_q!} = e_q\left(\frac{x}{b}\right).$$

□

Proof of Theorem 2.4. By the AESW theorem (see Ch. 8, §5, Theorem 5.3 of [5]) a sequence $\{a_k\}$ is totally positive if and only if the following representation is true:

$$\sum_{k=0}^{\infty} a_k x^k = \mathrm{e}^{\gamma x} \frac{\prod_{k=0}^{\infty}(1+\mu_k x)}{\prod_{k=0}^{\infty}(1-\nu_k x)}, \tag{4.2}$$

where $\gamma \geq 0$, $\mu_k \geq 0$, $\nu_k \geq 0$ and $\sum(\mu_k+\nu_k) < \infty$.

We have to prove that the function φ satisfies (4.2). By Theorem 2.3,

$$\varphi(x) = e_q\left(\frac{x}{b}\right).$$

For $q = 1$, $e_q(x/b) = \exp(x/b)$ and the statement is obvious (with $\mu_k = \nu_k = 0$).

For $|q| < 1$, by Euler's identity (2.4), we get

$$e_q\left(\frac{x}{b}\right) = \frac{1}{\left(\frac{1-q}{b}x; q\right)_\infty} = \prod_{k=0}^{\infty} \frac{1}{\left(1 - \frac{q^k(1-q)}{b}x\right)}.$$

In this case, (4.2) is true with $\gamma = 0$, $\mu_k = 0$, and $\nu_k = q^k(1-q)/b > 0$.

For $q > 1$, we write

$$e_q\left(\frac{x}{b}\right) = \sum_{k=0}^{\infty} \frac{(1-q)^k x^k}{b^k (q;q)_k} = \sum_{k=0}^{\infty} \frac{q_1^{k(k-1)/2}(1-q_1)^k x^k}{b^k (q_1;q_1)_k},$$

where $q_1 = 1/q$. By Euler's identity (2.3), we get

$$e_q\left(\frac{x}{b}\right) = \left(-\frac{(1-q_1)x}{b}; q_1\right)_\infty = \prod_{k=0}^{\infty}\left(1 + q_1^k \frac{(1-q_1)x}{b}\right).$$

Juxtaposing with (4.2), we observe that $\gamma = 0$, $\mu_k = q_1^k(1-q_1)/b > 0$, $\nu_k = 0$. Since in all of the above cases the conditions of AESW theorem hold, it follows that $\{a_k\}$ is a totally positive sequence. □

Proof of Theorem 2.5. We prove the statement by induction on m. For $j = 1, 2$ the statement is true by (1.6) and (1.7). Suppose that $\mathbf{E}[\xi_x^j] := P_j(x)$ is a polynomial of degree j for $j = 1, 2, \ldots, m$. Consider

$$\mathbf{E}[\xi_x^{m+1}] = \sum_{k=0}^{\infty} \alpha_k^{m+1} p_k(x) = \frac{1}{\varphi(x)} \sum_{k=1}^{\infty} \alpha_k^{m+1} a_k x^k.$$

By virtue of (2.6) and (2.7), $\alpha_k a_k = a_{k-1}$, $k \in \mathbf{N}$, and we obtain, therefore:

$$\mathbf{E}[\xi_x^{m+1}] = \frac{1}{\varphi(x)} \sum_{k=1}^{\infty} \alpha_k^m a_{k-1} x^k = \frac{x}{\varphi(x)} \sum_{k=0}^{\infty} \alpha_{k+1}^m a_k x^k.$$

We notice that

$$\alpha_{k+1} = b[k+1]_q = b + bq[k]_q = b + q\alpha_k.$$

Hence

$$\mathbf{E}[\xi_x^{m+1}] = \frac{x}{\varphi(x)} \sum_{k=0}^{\infty} (b + q\alpha_k)^m a_k x^k = x \sum_{k=0}^{\infty} (b + q\alpha_k)^m p_k(x)$$

$$= x \sum_{j=0}^{m} \binom{m}{k} b^{m-j} q^j \sum_{k=0}^{\infty} \alpha_k^j p_k(x) = x \sum_{j=0}^{m} \binom{m}{j} q^j b^{m-j} \mathbf{E}[\xi_x^j].$$

By the induction assumption, $\mathbf{E}[\xi_x^j] = P_j(x)$ for $j = 0, 1, \ldots, m$. Thus,

$$\mathbf{E}[\xi_x^{m+1}] = x \sum_{j=0}^{m} \binom{m}{j} q^j b^{m-j} P_j(x) =: P_{m+1}(x),$$

where $P_{m+1}(x)$ is a polynomial of degree $m + 1$. □

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