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On the compactly locally uniformly rotund points of Orlicz spaces

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Abstract. In this paper, locally uniformly rotund points and compactly locally uniformly rotund points are introduced. Moreover, criteria for compactly locally uniformly rotund points in Orlicz spaces are given.

Keywords. CLUR points; LUR points; Orlicz spaces.

1. Introduction

Let X be a real Banach space, and let $B(X)$ and $S(X)$ be the closed unit ball and the unit sphere of X, respectively. By X^* we denote the dual space of X. In the sequel N and R will denote the set of natural numbers and the set of real numbers, respectively.

Before starting with our results, we need to recall some notions.

In 1936, J A Clarkson introduced the concept of uniform convexity (UR), and many started studying the kinds of properties related with it. A R Lovaglia first introduced the concept of locally uniform convexity, Panda and Kapoor [7] generalized the notion of local uniform convexity, and introduced the notion of compact local uniform convexity. These properties play an important role in some branches of mathematics. For example, for any Banach space X, both of them imply the Kadec-Klee property and this property, together with reflexivity, is equivalent to approximative compactness of X. Approximative compactness of X gives that any nonempty convex and closed set in X is proximinal in X and the projection $P_A(.)$ from X to A is a continuous operator (see [2]).

A Banach space X is said to be locally uniformly convex if and only if every point of $S(X)$ is a point of local uniform convexity (see [7]).

A point $x \in S(X)$ is said to be a LUR point of $B(X)$ if for any sequence $\{x_n\} \subset B(X)$, the condition $\|x_n + x_0\| \to 2$ implies $x_n \to x_0$.

A point $x \in S(X)$ is said to be a CLUR point of $B(X)$ if for any sequence $\{x_n\} \subset B(X)$, the condition $\|x_n + x_0\| \to 2$ implies that $\{x_n\}$ is relatively compact.

Obviously, a LUR point is a CLUR point. It has been proved that $x \in S(X)$ is a LUR point if and only if it is a strong U-point and CLUR point (see [2]).

A map Φ: $R \to [0, \infty]$ is said to be an Orlicz function if Φ is vanishing only at zero, even, convex and continuous on the whole of R^+, $\lim_{u\to 0} \frac{\Phi(u)}{u} = 0$, $\lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty$. Let $p(u)$ be the right-hand side derivative of $\Phi(u)$. For every Orlicz function Φ we define the complementary function $\Psi\colon R \to [0, \infty]$ by the formula

$$\Psi(v) = \sup\{u|v| - \Phi(u)\colon u \geq 0\}.$$

The complementary function Ψ is also an Orlicz function.

Let (G, Σ, μ) be a measure space with a finite, non-atomic and complete measure μ and $L^{\circ}(\mu)$ be the space of all μ-equivalence classes of real and Σ-measurable functions defined on G. For a given Orlicz function Φ we define on $L^{\circ}(\mu)$ a convex functional I_Φ by

$$I_\Phi(x) = \int_G \Phi(x(t))\mathrm{d}\mu.$$

We define the Orlicz space L_Φ generated by an Orlicz function Φ by the formula

$$L_\Phi = \{x \in L^{\circ}(\mu)\colon I_\Phi(cx) < \infty,\ \text{for some } c > 0 \text{ depending on } x\}$$

and its subspace is

$$E_\Phi = \{x \in L^{\circ}(\mu)\colon I_\Phi(cx) < \infty,\ \text{for any } c > 0\},$$

which will be considered with the norm induced from L_Φ. L_Φ is usually equipped with the Luxemburg norm

$$\|x\|_\Phi = \inf\left\{\varepsilon > 0\colon I_\Phi\left(\frac{x}{\varepsilon}\right) \leq 1\right\}$$

or with the equivalent one

$$\|x\|^{\circ}_\Phi = \sup\left\{\int_T |x(t)y(t)|\mathrm{d}\mu\colon y \in L_\Psi, I_\Psi(y) \leq 1\right\}$$

called the Orlicz norm.

It is proved in [5] that for any Orlicz function Φ the Amemiya formula for the Orlicz norm

$$\|x\|^{\circ}_\Phi = \inf_{k>0} \frac{1}{k}(1 + I_\Phi(kx))$$

is true.

To simplify notations, we put $L_\Phi = (L_\Phi, \|\cdot\|_\Phi)$, $L^{\circ}_\Phi = (L_\Phi, \|\cdot\|^{\circ}_\Phi)$. For $x \in L^{\circ}_\Phi$ and $x \neq 0$, we denote $k^*_x = \inf\{k > 0\colon I_\Psi(p(k|x|)) \geq 1\}$, $k^{**}_x = \sup\{k > 0\colon I_\Psi(p(k|x|)) \leq 1\}$ and $K(x) = [k^*_x, k^{**}_x]$. It is well-known that $\|x\|^{\circ}_\Phi = \frac{1}{k}(1+I_\Phi(kx))$ if and only if $k \in K(x)$ (see [1]).

We say an Orlicz function Φ satisfies the Δ_2-condition ($\Phi \in \Delta_2$,for short) if there are positive constants k and $u_0 > 0$ such that $\Phi(2u) \leq k\Phi(u)$ for $|u| \geq u_0$.

We say an Orlicz function Φ satisfies the ∇_2-condition ($\Phi \in \nabla_2$, for short) if its complementary function Ψ satisfies Δ_2-condition.

We say an interval $[a, b]$ is an affine interval of Φ, if

$$\Phi\left(\frac{a+b}{2}\right) = \frac{1}{2}(\Phi(a) + \Phi(b)),\ \Phi\left(\frac{a-\varepsilon+b}{2}\right) < \frac{1}{2}(\Phi(a-\varepsilon) + \phi(b))$$

and $\Phi\left(\frac{a+b+\varepsilon}{2}\right) < \frac{1}{2}(\Phi(a) + \Phi(b+\varepsilon))$, for any $\varepsilon > 0$.

We say ω is a point of strict convexity of Φ (we write $\omega \in \mathrm{SC}(\Phi)$) if for every $u, v \in R$ such that $u \neq v$ there holds $\Phi\left(\frac{u+v}{2}\right) < \frac{1}{2}(\Phi(u) + \Phi(v))$.

For more details on Orlicz spaces, we refer to [1–10].

2. General results

We begin this section by giving the criteria for compactly locally uniformly rotund points in Orlicz spaces equipped with the Orlicz norm.

Theorem 1. *A point $x \in S(L^{\circ}_{\Phi})$ is a* CLUR *point of $B(L^{\circ}_{\Phi})$ if and only if*:

(1) $\Phi \in \Delta_2 \cap \nabla_2$,
(2) *If p is a constant on an interval I and $k \in K(x)$, then $\mu\{t \in G\colon k|x(t)| \in I\} = 0$,*
(3) *$I_{\Psi}(pk|x|) = 1$ or $\mu\{t \in G\colon k|x(t)| = b\} = 0$ for any affine interval $[a, b]$ of Φ, where $p \circ k|x|(t) = p(k|x(t)|)$ for any $t \in G$.*

Proof.

Sufficiency. Under these conditions, x is a LUR point (see [1]). Of course, x is then a CLUR point.

Necessity.

(i) Assume that condition (1) does not hold. Let us first suppose that $\Phi \notin \triangle_2$. Then $L^{\circ}_{\Phi} \neq E^{\circ}_{\Phi}$. We consider the following two cases.

(a) $x \notin E^{\circ}_{\Phi}$. We have $\lim_{n\to\infty} \|x\chi_{G\setminus G_n}\|^{\circ} = d(x, E^{\circ}_{\Phi}) = d > 0$, where $G_n = \{t \in G\colon |x(t)| \leq n\}$. Since $\|x\|^{\circ} > \frac{d}{2}$, there exists $G_{n_1} \subset G$ such that $\|x\chi_{G_{n_1}}\|^{\circ} > \frac{d}{2}$. Since $\|x\chi_{G\setminus G_{n_1}}\|^{\circ} > \frac{d}{2}$, there exists $G_{n_2} \supset G_{n_1}$ such that $\|x\chi_{G_{n_2}\setminus G_{n_1}}\|^{\circ} > \frac{d}{2}$; etc. In such a way, we get a sequence $\{G_{n_i}\}_{i=1}^{\infty}$ of subsets of G such that

$$G_{n_i} \subset G_{n_{i+1}}, \quad \|x\chi_{G_{n_{i+1}}\setminus G_{n_i}}\|^{\circ} > \frac{d}{2} \quad (i = 1, 2, \ldots).$$

Obviously, $\lim_{i\to\infty} \|x\chi_{G_{n_i}}\|^{\circ} = \|x\|^{\circ} = 1$, and

$$\varliminf_{i\to\infty} \|x + x\chi_{G_{n_i}}\|^{\circ} \geq 2 \lim_{i\to\infty} \|x\chi_{G_{n_i}}\|^{\circ} = 2.$$

Consequently, $\|x + x\chi_{G_{n_i}}\|^{\circ} \to 2$ as $i \to \infty$. We have for $x_i = x\chi_{G_{n_i}}$,

$$\|x_i - x_j\|^{\circ} = \|x\chi_{G_{n_i}} - x\chi_{G_{n_j}}\|^{\circ} \geq \|x\chi_{G_{n_j}\setminus G_{n_{j-1}}}\|^{\circ} > \frac{d}{2} \text{ for any } j > i.$$

We got a contradiction with the fact that x is a CLUR point.

(b) $x \in E^{\circ}_{\Phi}$. Take $y \notin E^{\circ}_{\Phi}$ such that $I_{\Phi}(y) < \infty$, and $d(y, E^{\circ}_{\Phi}) = d > 0$. Repeating the procedure as in the above method, we can get a sequence $\{G_{n_i}\}_{i=1}^{\infty}$ of subsets of G such that

$$G_{n_i} \subset G_{n_{i+1}}, \quad \|y\chi_{G_{n_{i+1}}\setminus G_{n_i}}\|^{\circ} > \frac{d}{2} \quad (i = 1, 2, \ldots).$$

Take $k > 1$ such that $1 = \|x\|^{\circ} = \frac{1}{k}(1 + I_{\Phi}(kx))$ and let

$$x_i(t) = \begin{cases} x(t), & t \in G\setminus(G_{n_{i+1}}\setminus G_{n_i}) \\ \frac{y(t)}{k}, & t \in G_{n_{i+1}}\setminus G_{n_i} \end{cases}, \tag{2.1}$$

for $i = 1, 2, \ldots$. Then $\|x_i\|^\circ \geq \|x\chi_{G_{n_i}}\|^\circ \to \|x\|^\circ = 1$. So we have $\underline{\lim}_{i\to\infty} \|x_i\|^\circ \geq 1$. Moreover,

$$\begin{aligned}\|x_i\|^\circ &\leq \frac{1}{k}(1 + I_\Phi(kx_i)) \leq \frac{1}{k}((1 + I_\Phi(kx)) + I_\Phi(y\chi_{G_{n_{i+1}}\setminus G_{n_i}}))\\ &= \|x\|^\circ + \frac{1}{k} I_\Phi(y\chi_{G_{n_{i+1}}\setminus G_{n_i}})\\ &\leq 1 + \frac{1}{k} I_\Phi(y\chi_{G\setminus G_{n_i}}) \to 1, \quad (i \to \infty).\end{aligned}$$

Hence, we get $\lim_{i\to\infty} \|x\|^\circ = 1$. Because of $x \in E_\Phi^\circ$, we can take $i_0 \in N$ such that $\|x\chi_{G\setminus G_{n_{i_0}}}\|^\circ < \frac{d}{4}$, when $i \geq i_0$. Moreover,

$$\begin{aligned}\|x_i - x_j\|^\circ \geq \|(x-y)\chi_{G_{n_{i+1}}\setminus G_{n_i}}\|^\circ &\geq \frac{1}{k}\|y\chi_{G_{n_{i+1}}\setminus G_{n_i}}\|^\circ - \|x\chi_{G_{n_{i+1}}\setminus G_{n_i}}\|^\circ\\ &> \frac{d}{4k} \quad (j > i \geq i_0).\end{aligned}$$

Since $\|x + x_i\|^\circ \geq 2\|x\chi_{G_{n_i}}\|^\circ \to 2 (i \to \infty)$, we have $\lim_{i\to\infty} \|x + x_i\|^\circ = 2$. But

$$\|x_i - x_j\|^\circ > \frac{d}{4k}, \quad j > i \geq i_0$$

which means that the sequence $\{x_i\}_{i=1}^\infty$ is not relatively compact, and so x is not a CLUR point.

Suppose that $\Phi \notin \nabla_2$, i.e. $\Psi \notin \Delta_2$. Then there exists $y_0 \in S(L_\Psi)$ such that $\|y_0\chi_{G\setminus G_n}\|_\Psi = 1$ for any $n \in N$, where $G_n = \{t \in G\colon |y_0(t)| \leq n\}$.

Since $\|y_0\chi_{G\setminus G_n}\|_\Psi = 1$, let G_{n_1} be such that $\|y_0\chi_{G\setminus G_{n_1}}\|_\Psi = 1$, and there exists $n_2 > n_1$ such that $\|y_0\chi_{G_{n_2}\setminus G_{n_1}}\|_\Psi > \frac{1}{2}$.

Since $\|y_0\chi_{G\setminus G_{n_2}}\|_\Psi = 1$, there exists $n_3 > n_2$ such that $\|y_0\chi_{G_{n_3}\setminus G_{n_2}}\|_\Psi > \frac{2}{3}$; etc.

Thus, we can get a sequence $\{G_{n_i}\}_{i=1}^\infty$ of subsets of G such that

$$G_{n_i} \subset G_{n_{i+1}} \text{ and } 1 \geq \|y_0\chi_{G_{n_{i+1}}\setminus G_{n_i}}\| > \frac{i}{i+1} \quad (i = 1, 2, \ldots).$$

We have $\lim_{i\longrightarrow\infty} \mu(G_{n_{i+1}}\setminus G_{n_i}) = 0$. Otherwise, we can assume without loss of generality that $\mu(G_{n_{i+1}}\setminus G_{n_i}) \geq \delta > 0$ for all $i \in N$. In consequence, $I_\Phi(y_0) \geq I_\Phi(n_i\chi_{G_{n_{i+1}}\setminus G_{n_i}}) = \Phi(n_i)\mu(G_{n_{i+1}}\setminus G_{n_i}) \geq \delta\mu(G_{n_{i+1}}\setminus G_{n_i}) \to \infty$ as $i \to \infty$, a contradiction.

Let $y_i = y_0\chi_{G_{n_{i+1}}\setminus G_{n_i}}$. It is easy to see that $y_i \in E_\Psi$, then there exists $x_i \in S(L_\Phi^\circ)$ such that $\langle x_i, y_i\rangle = \|y_i\|_\Psi$ and $\mathrm{supp}(x_i) \subseteq G_{n_{i+1}}\setminus G_{n_i}$. Obviously, $\|x_i - x_j\|^\circ \geq \|x_i\|^\circ = 1$ for $i \neq j$.

Since $\Phi \in \Delta_2$, there exists $y \in S(L_\Psi)$ such that y is a supporting functional for x and $I_\Psi(y) = 1$.

Let

$$z_i(t) = \begin{cases} y(t), & t \in G\setminus(G_{n_{i+1}}\setminus G_{n_i}),\\ y_0(t), & t \in G_{n_{i+1}}\setminus G_{n_i}.\end{cases} \tag{2.2}$$

Then $I_\Psi(z_i) \leq I_\Psi(y) + I_\Psi(y_0\chi_{G_{n_{i+1}}\setminus G_{n_i}}) \to I_\Psi(y) = 1$. Consequently, $I_\Psi(z_i) \leq 1$.

Then

$$\begin{aligned}\|x+x_i\|^\circ &\geq \langle x+x_i, z_i\rangle \\ &= \langle x_i, z_i\rangle + \langle x, z_i\rangle \\ &= \int_{G_{n_{i+1}}\setminus G_{n_i}} x_i(t)y_0(t)\mathrm{d}t + \int_{(G\setminus G_{n_{i+1}})\cup G_{n_i}} x_i(t)y(t)\mathrm{d}t \\ &\quad + \int_{G_{n_{i+1}}\setminus G_{n_i}} x(t)y_0(t)\mathrm{d}t + \int_{(G\setminus G_{n_{i+1}})\cup G_{n_i}} x(t)y(t)\mathrm{d}t \\ &> \frac{i}{i+1} + 1 - \int_{G_{n_{i+1}}\setminus G_{n_i}} x(t)y(t)\mathrm{d}t + \int_{G_{n_{i+1}}\setminus G_{n_i}} x(t)y_0(t)\mathrm{d}t.\end{aligned}$$

Since $\int_G x(t)y(t)\mathrm{d}t = 1$, for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$\left|\int_e x(t)y(t)\mathrm{d}t\right| < \varepsilon$$

when $\mu(e) < \delta_1$.

Using the same argument, there exists $\delta_2 > 0$ such that

$$\left|\int_e x(t)y_0(t)\mathrm{d}t\right| < \varepsilon$$

when $\mu(e) < \delta_2$.

Take $\delta = \min\{\delta_1, \delta_2\}$, then

$$\|x+x_i\|^\circ > \frac{i}{i+1} + 1 - \varepsilon - \varepsilon \to 2 - \varepsilon \text{ as } i \to \infty.$$

Letting $\varepsilon \to 0$, we get $\underline{\lim}_{i\to\infty} \|x_i + x\|^\circ \geq 2$. So we have $\lim_{i\to\infty} \|x_i + x\|^\circ = 2$. But

$$\|x_i - x_j\|^\circ \geq 1 \text{ for any } i \neq j.$$

This contradiction shows that $\Phi \in \nabla_2$.

(ii) If condition (2) does not hold, there exists an affine interval $[a, b]$ of Φ and $k \in K(x)$ such that $\mu\{t \in G\colon k|x(t)| \in (a, b)\} > 0$. Take $\varepsilon > 0$ small enough such that $\mu\{t \in G\colon k|x(t)| \in [a+\varepsilon, b-\varepsilon]\} > 0$ and let $E = \{t \in G\colon k|x(t)| \in [a+\varepsilon, b-\varepsilon]\}$, and $\Phi(u) = Au + B$ for $u \in E$. Divide E into two subsets E_1^1 and E_2^1 such that $\mu E_1^1 = \mu E_2^1$, $E_1^1 \cap E_2^1 = \emptyset$, $E_1^1 \cup E_2^1 = E$. Divide E_1^1 into two subsets E_1^2 and E_2^2 such that $\mu E_1^2 = \mu E_2^2$, $E_1^2 \cap E_2^2 = \emptyset$, $E_1^2 \cup E_2^2 = E_1^1$. Divide E_2^1 into two subsets E_3^2 and E_4^2 such that $\mu E_3^2 = \mu E_4^2$, $E_3^2 \cap E_4^2 = \emptyset$, $E_3^2 \cup E_4^2 = E_2^1$; etc. Thus, we get two sequences $\{E_{2k-1}^n\}$, $\{E_{2k}^n\}$ of subsets of E with $\mu E_{2k-1}^n = \mu E_{2k}^n$, $E_{2k-1}^n \cup E_{2k}^n = E_k^{n-1} (n = 1, 2, \ldots; k = 1, 2, \ldots, 2^{n-1})$.

Let

$$x_n(t) = \begin{cases} x(t), & t \in G\setminus E, \\ x(t) - \frac{\varepsilon}{k}, & t \in \cup_{k=1}^{2^{n-1}} E_{2k-1}^n, \\ x(t) + \frac{\varepsilon}{k}, & t \in \cup_{k=1}^{2^{n-1}} E_{2k}^n \end{cases} \tag{2.3}$$

for $n = 1, 2, \ldots$. Then $I_\Phi(x_n - x_m) = \Phi\left(\frac{2\varepsilon}{k}\right)\frac{\mu E}{2}$ for any $m \neq n$.

We have

$$\|x_n - x_m\|^\circ > \|x_n - x_m\| \geq \min(1, I_\Phi(x_n - x_m))$$
$$\geq \min\left(1, \Phi\left(\frac{2\varepsilon}{k}\right)\frac{\mu E}{2}\right) > 0, \text{ for any } m \neq n.$$

For any $\eta > 0$,

$$I_\Psi(p((1+\eta)kx_n)) \geq I_\Psi(p((1+\eta)kx)) \geq 1,$$
$$I_\Psi(p((1-\eta)kx_n)) \leq I_\Psi(p((1-\eta)kx)) \leq 1.$$

So we have $k \in K(x)$. Then

$$\|x_n\|^\circ = \frac{1}{k}(1 + I_\Phi(kx_n))$$
$$= \frac{1}{k}\Bigg(1 + I_\Phi(kx\chi_{G\setminus E}) + \sum_{i=1}^{2^{n-1}} \int_{E^n_{2i-1}} \Phi(kx(t) - \varepsilon)\mathrm{d}t$$
$$+ \sum_{i=1}^{2^{n-1}} \int_{E^n_{2i}} \Phi(kx(t) + \varepsilon)\mathrm{d}t\Bigg)$$
$$= \frac{1}{k}\left(1 + I_\Phi(kx\chi_{G\setminus E}) + \sum_{i=1}^{2^{n-1}}\left(\int_{E^n_{2i-1}} \Phi(kx(t))\mathrm{d}t - A\varepsilon\mu E^n_{2i-1}\right)\right)$$
$$= \frac{1}{k}\Bigg(1 + I_\Phi(kx\chi_{G\setminus E}) + \sum_{i=1}^{2^{n-1}}\left(\int_{E^n_{2i-1}} \Phi(kx(t))\mathrm{d}t - A\varepsilon\mu E^n_{2i-1}\right)$$
$$+ \sum_{i=1}^{2^{n-1}}\left(\int_{E^n_{2i}} \Phi(kx(t))\mathrm{d}t + A\varepsilon\mu E^n_{2i}\right)\Bigg)$$
$$= \frac{1}{k}(1 + I_\Phi(kx\chi_{G\setminus E}) + I_\Phi(kx\chi_E)) = \|x\|^\circ = 1$$

for $n = 1, 2, \ldots$.

Using the same argument, we get $k \in K\left(\frac{x_n+x}{2}\right)$ for $n = 1, 2, \ldots$. Then

$$\left\|\frac{x_n + x}{2}\right\|^\circ = \frac{1}{k}\left(1 + I_\Phi\left(k\frac{x_n + x}{2}\right)\right)$$
$$= \frac{1}{k}\left(1 + I_\Phi(kx\chi_{G\setminus E}) + \int_E \Phi\left(\frac{kx_n(t) + kx(t)}{2}\right)\mathrm{d}t\right)$$

$$= \frac{1}{k}\left(1 + I_\Phi(kx\chi_{G\backslash E}) + \frac{\int_E \Phi(kx_n(t))\mathrm{d}t + \int_E \Phi(kx(t))\mathrm{d}t}{2}\right)$$

$$= \left(\frac{\|x\|^\circ + \|x_n\|^\circ}{2}\right) = 1,$$

for $n = 1, 2, \ldots$. This shows that condition (2) holds.

(iii) If condition (3) does not hold, there exists a right endpoint of affine interval $[a, b]$ of Φ with $\mu\{t \in G\colon k|x(t)| = b\} > 0$ and $I_\Psi(p(k|x|)) < 1$. Then $1 > I_\Psi(p(k|x|)) = I_\Psi(p(k|x|\chi_{G\backslash E})) + \Psi(p(b))\mu E$. Let $\{E_i^n\}_{i=1}^{2^n}$ be a partition of E as above and put

$$x_n(t) = \begin{cases} x(t), & t \in G\backslash E, \\ \frac{b-\varepsilon}{k}, & t \in \cup_{i=1}^{2^{n-1}} E_{2i-1}^n, \\ \frac{b+\varepsilon}{k}, & t \in \cup_{i=1}^{2^{n-1}} E_{2i}^n \end{cases} \tag{2.4}$$

for $n = 1, 2, \ldots$. For any $\eta > 0$,

$$I_\Psi(p((1+\eta)kx_n)) \geq I_\Psi(p((1+\eta)kx\chi_{G\backslash E})) + \Psi(p((1+\eta)b))\mu(E)$$
$$= I_\Psi(p((1+\eta)kx)) \geq 1$$
$$I_\Psi(p((1-\eta)kx_n)) \leq I_\Psi(p((1-\eta)kx)) < 1.$$

So we have $k \in K(x_n)$. Then

$$\begin{aligned}\|x_n\|^\circ &= \frac{1}{k}(1 + I_\Phi(kx_n)) \\ &= \frac{1}{k}\left(1 + I_\Phi(kx\chi_{G\backslash E}) + \sum_{i=1}^{2^{n-1}} \int_{E_{2i-1}^n} \Phi(kx(t) - \varepsilon)\mathrm{d}t \right.\\ &\quad \left. + \sum_{i=1}^{2^{n-1}} \int_{E_{2i}^n} \Phi(kx(t) + \varepsilon)\mathrm{d}t\right) \\ &= \frac{1}{k}\left(1 + I_\Phi(kx\chi_{G\backslash E}) + \sum_{i=1}^{2^{n-1}} \left(\int_{E_{2i-1}^n} \Phi(kx(t))\mathrm{d}t - A\varepsilon\mu E_{2i-1}^n\right)\right.\\ &\quad \left. + \sum_{i=1}^{2^{n-1}} \left(\int_{E_{2i}^n} \Phi(kx(t))\mathrm{d}t + A\varepsilon\mu E_{2i}^n\right)\right) \\ &= \frac{1}{k}(1 + I_\Phi(kx\chi_{G\backslash E}) + I_\Phi(kx\chi_E)) = \|x\|^\circ = 1\end{aligned}$$

for $n = 1, 2, \ldots$. Using the same argument, we get $k \in K\left(\frac{x_n+x}{2}\right)$ and $\|x_n + x\|^\circ = 2$, $n = 1, 2, \ldots$.

Since $I_\Phi(x_n - x_m) = \Phi\left(\frac{2\varepsilon}{k}\right)\frac{\mu E}{2}$, there exists $\delta > 0$ such that $\|x_n - x_m\|^\circ > \|x_n - x_m\| \geq \delta (n \neq m)$, which contradicts the fact that x is a CLUR point. Thus, the proof is finished.

Now, criteria for compactly locally uniformly rotund points in Orlicz spaces equipped with the Luxemburg norm are shown as in the following.

Theorem 2. *A point $x \in S(L_\Phi)$ is a CLUR point of $B(L_\Phi)$ if and only if:*

(1) $\Phi \in \Delta_2$,
(2) $x(t) \in SC(\Phi)\mu - a.e.\ t \in G$,
(3) *If* $\mu\{t \in G: |x(t)| = b\} > 0$, *then* $\mu\{t \in G: |x(t)| = a\} = 0$ *for every affine interval* $[a, b]$ *of* Φ *and* $\Phi \in \nabla_2$.

Proof.

Sufficiency. Under these conditions, x is a LUR point (see [1]), x is a CLUR point as well.

Necessity.

(i) Assume that condition (1) does not hold. Then $L_\Phi \neq E_\Phi$. We consider two cases separately.
(a) $x \notin E_\Phi$. We have $\lim_{n\to\infty} \|x\chi_{G\setminus G_n}\| = d(x, E_\Phi) = d > 0$, where $G_n = \{t \in G: |x(t)| \leq n\}$.

Since $\|x\| > \frac{d}{2}$, there exists $G_{n_1} \subset G$ such that $\|x\chi_{G_{n_1}}\| > \frac{d}{2}$. Since $\|x\chi_{G\setminus G_{n_1}}\| > \frac{d}{2}$, there exists $G_{n_2} \supset G_{n_1}$ such that $\|x\chi_{G_{n_2}\setminus G_{n_1}}\| > \frac{d}{2}$; etc. Thus, we get a sequence $\{G_{n_i}\}_{i=1}^{\infty}$ of subsets of G such that

$$G_{n_i} \subset G_{n_{i+1}}, \|x\chi_{G_{n_{i+1}}\setminus G_{n_i}}\| > \frac{d}{2} \quad (i = 1, 2, \dots).$$

Obviously, $\lim_{i\to\infty} \|x\chi_{G_{n_i}}\| = \|x\| = 1$, and

$$\underline{\lim}_{i\to\infty} \|x + x\chi_{G_{n_i}}\| \geq 2 \lim_{i\to\infty} \|x\chi_{G_{n_i}}\| = 2.$$

Consequently, $\|x + x\chi_{G_{n_i}}\| \to 2$ as $i \to \infty$. Defining $x_i = x\chi_{G_{n_i}}$, we have

$$\|x_i - x_j\| = \|x\chi_{G_{n_i}} - x\chi_{G_{n_j}}\| \geq \|x\chi_{G_{n_j}\setminus G_{n_{j-1}}}\| > \frac{d}{2} \text{ for any } j > i.$$

This shows that x is a CLUR point.
(b) $x \in E_\Phi$. Take $y \notin E_\Phi$ such that $I_\Phi(y) < \infty$ and $d(y, E_\Phi) = d > 0$. Repeating the procedure of the above method, we can get a sequence $\{G_{n_i}\}_{i=1}^{\infty}$ of subsets of G such that

$$G_{n_i} \subset G_{n_{i+1}}, \|y\chi_{G_{n_{i+1}}\setminus G_{n_i}}\| > \frac{d}{2} \quad (i = 1, 2, \dots).$$

Let

$$x_i(t) = \begin{cases} x(t), & t \in G\setminus(G_{n_{i+1}}\setminus G_{n_i}) \\ y(t), & t \in G_{n_{i+1}}\setminus G_{n_i} \end{cases}, \tag{2.5}$$

for $i = 1, 2, \dots$. Then $\|x_i\| \geq \|x\chi_{G_{n_i}}\| \to \|x\| = 1$. So we have $\underline{\lim}_{i\to\infty} \|x_i\| \geq 1$. Moreover,

$$\begin{aligned} I_\Phi(x_i) &= I_\Phi(x\chi_{G\setminus(G_{n_{i+1}}\setminus G_{n_i})}) + I_\Phi(y\chi_{G_{n_{i+1}}\setminus G_{n_i}}) \\ &\leq I_\Phi(x) + I_\Phi(y\chi_{G\setminus G_{n_i}}) \to I_\Phi(x) = 1, i \to \infty. \end{aligned}$$

Consequently, $\overline{\lim}_{i\to\infty}\|x_i\| \leq 1$. Hence, we have $\lim_{i\to\infty}\|x_i\| = 1$. Because of $x \in E_\Phi$, we can take $i_0 \in N$ such that $\|x\chi_{G\setminus G_{n_{i_0}}}\| < \frac{d}{4}$, when $i \geq i_0$. Hence,

$$\begin{aligned}\|x_i - x_j\| \geq \|(x-y)\chi_{G_{n_{i+1}}\setminus G_{n_i}}\| &\geq \frac{1}{k}\|y\chi_{G_{n_{i+1}}\setminus G_{n_i}}\| - \|x\chi_{G_{n_{i+1}}\setminus G_{n_i}}\| \\ &> \frac{d}{4k} \quad (j > i \geq i_0).\end{aligned}$$

Since $\|x + x_i\| \geq 2\|x\chi_{G_{n_i}}\| \to 2(i \to \infty)$, we have $\lim_{i\to\infty}\|x + x_i\| = 2$. We get a contradiction since x is a CLUR point.

(ii) If condition (2) does not hold, then there exists an affine interval $[a, b]$ of Φ such that $\mu\{t \in G\colon |x(t)| \in (a, b)\} > 0$. Take $\varepsilon > 0$ small enough such that $\mu\{t \in G\colon |x(t)| \in [a+\varepsilon, b-\varepsilon]\} > 0$ and let $E = \{t \in G\colon k|x(t)| \in [a+\varepsilon, b-\varepsilon]\}$, and $\Phi(u) = Au+B$ for $u \in E$. Divide E into two subsets E_1^1 and E_2^1 such that $\mu E_1^1 = \mu E_2^1$, $E_1^1 \cap E_2^1 = \emptyset$, $E_1^1 \cup E_2^1 = E$. Divide E_1^1 into two subsets E_1^2 and E_2^2 such that $\mu E_1^2 = \mu E_2^2$, $E_1^2 \cap E_2^2 = \emptyset$, $E_1^2 \cup E_2^2 = E_1^1$. Divide E_2^1 into two subsets E_3^2 and E_4^2 such that $\mu E_3^2 = \mu E_4^2$, $E_3^2 \cap E_4^2 = \emptyset$, $E_3^2 \cup E_4^2 = E_2^1$; etc. Thus, we get two sequences $\{E_{2k-1}^n\}$, $\{E_{2k}^n\}$ of subsets of E with $\mu E_{2k-1}^n = \mu E_{2k}^n$, $E_{2k-1}^n \cup E_{2k}^n = E_k^{n-1} (n = 1, 2, \ldots; k = 1, 2, \ldots, 2^{n-1})$.

Let

$$x_n(t) = \begin{cases} x(t), & t \in G\setminus E, \\ x(t) - \varepsilon, & t \in \cup_{k=1}^{2^{n-1}} E_{2k-1}^n, \\ x(t) + \varepsilon, & t \in \cup_{k=1}^{2^{n-1}} E_{2k}^n \end{cases} \tag{2.6}$$

for $n = 1, 2, \ldots$. Then

$$\begin{aligned} I_\Phi(x_n) &= I_\Phi(x\chi_{G\setminus E}) + \sum_{i=1}^{2^{n-1}} \int_{E_{2i-1}^n} \Phi(x(t) - \varepsilon)\mathrm{d}t \\ &\quad + \sum_{i=1}^{2^{n-1}} \int_{E_{2i}^n} \Phi(x(t) + \varepsilon)\mathrm{d}t \\ &= I_\Phi(x\chi_{G\setminus E}) + \sum_{i=1}^{2^{n-1}} \left(\int_{E_{2i-1}^n} \Phi(x(t))\mathrm{d}t - A\varepsilon\mu E_{2i-1}^n\right) \\ &\quad + \sum_{i=1}^{2^{n-1}} \left(\int_{E_{2i}^n} \Phi(x(t))\mathrm{d}t + A\varepsilon\mu E_{2i}^n\right) \\ &= I_\Phi(x) = \|x\| = 1 \end{aligned}$$

for $n = 1, 2, \ldots$. So we have $\|x_n\| = 1$ for all $n \in N$.

Since

$$I_\Phi\left(\frac{x_n+x}{2}\right) = I_\Phi(x\chi_{G\setminus E}) + I_\Phi\left(\frac{x_n+x}{2}\chi_E\right)$$
$$= I_\Phi(x\chi_{G\setminus E}) + \int_E \frac{\Phi(x_n(t))+\Phi(x(t))}{2}\mathrm{d}t$$
$$= \frac{I_\Phi(x)+I_\Phi(x_n)}{2} = 1,$$

we get $\|x_n + x\| = 2$. But $I_\Phi(x_n - x_m) = \Phi(2\varepsilon)\frac{\mu E}{2}$, for any $m \neq n$.

For some $\delta > 0$, we have

$$\|x_n - x_m\| \geq \delta \text{ for any } m \neq n.$$

This shows that x is not a CLUR point which leads to a contradiction.

(iii) If condition (3) does not hold, first, we assume that there exist a left endpoint of affine interval $[a, b]$ of Φ and a right endpoint of affine interval $[c, d]$ of Φ such that $\mu\{t \in G: |x(t)| = a\} > 0$ and $\mu\{t \in G: |x(t)| = b\} > 0$.

Take $b' > a$ and $c' < d$ for which $[a, b']$ and $[c', d]$ are affine intervals of Φ with $\Phi(b') - \Phi(a) = \Phi(d) - \Phi(c)$.

Let $E = \{t \in G: |x(t)| = a\}$ and $F = \{t \in G: |x(t)| = b\}$. Without loss of generality, we can assume that $\mu E = \mu F$.

Let $\{E_i^n\}_{i=1}^{2^n}$ and $\{F_i^n\}_{i=1}^{2^n}$ be a partition of E and F as above. Put

$$x_n(t) = \begin{cases} x(t), & t \in G\setminus(E \cup F) \\ a, & t \in \cup_{k=1}^{2^{n-1}} E_{2k-1}^n \\ b', & t \in \cup_{k=1}^{2^{n-1}} E_{2k}^n \\ c', & t \in \cup_{k=1}^{2^{n-1}} F_{2k-1}^n \\ d, & t \in \cup_{k=1}^{2^{n-1}} F_{2k}^n \end{cases}. \tag{2.7}$$

Then

$$I_\Phi(x_n) = I_\Phi(x\chi_{G\setminus(E\cup F)}) + (\Phi(a)+\Phi(b'))\frac{\mu E}{2} + (\Phi(c')+\Phi(d))\frac{\mu F}{2}$$
$$= I_\Phi(x\chi_{G\setminus(E\cup F)}) + ((\Phi(a)+\Phi(d)-\Phi(c')$$
$$+\Phi(a)+\Phi(c')+\Phi(d))\frac{\mu E}{2}$$
$$= I_\Phi(x\chi_{G\setminus(E\cup F)}) + (\Phi(a)+\Phi(d))\mu E$$
$$= I_\Phi(x\chi_{G\setminus(E\cup F)}) + \Phi(a)\mu E + \Phi(d)\mu F = I_\Phi(x) = 1$$

for $n = 1, 2, \ldots$. Consequently, $\|x_n\| = 1$ for all $n \in N$. Moreover,

$$I_\Phi\left(\frac{x_n + x}{2}\right) = I_\Phi(x\chi_{G\setminus(E\cup F)}) + \Phi(a)\frac{\mu E}{2} + \Phi\left(\frac{a + b'}{2}\right)\frac{\mu E}{2}$$
$$+ \Phi\left(\frac{c' + d}{2}\right)\frac{\mu F}{2} + \Phi(d)\frac{\mu F}{2}$$
$$= I_\Phi(x\chi_{G\setminus(E\cup F)})$$
$$+ \left(\frac{3}{2}(\Phi(a) + \Phi(d)) + \frac{1}{2}(\Phi(c') + \Phi(b'))\right)\frac{\mu E}{2}$$
$$= I_\Phi(x) = 1.$$

Hence, we have $\|x_n + x\| = 2$. Since $\Phi \in \Delta_2$, there exists $\delta > 0$ such that

$$\|x_n - x_m\| \geq \delta \text{ for any } m \neq n.$$

However, $I_\Phi(x_n - x_m) = \Phi(b - a)\frac{\mu E}{2}(m \neq n)$ which contradicts the fact that x is a CLUR point.

Now suppose that there is a right endpoint of affine interval $[c, d]$ such that $\mu\{t \in G\colon |x(t)| = d\} > 0$ and $\Phi \notin \nabla_2$.

Since $\Phi \notin \nabla_2$, there exists a sequence $u_n \nearrow \infty$ such that

$$\Phi\left(\frac{u_n}{2}\right) > \left(1 - \frac{1}{n}\right)\frac{\Phi(u_n)}{2}, \quad n = 1, 2, \ldots.$$

Let $E = \{t \in G\colon |x(t)| = d\}$ and take $\varepsilon > 0$ with $d - \varepsilon \in (c, d)$ and a sequence $\{E_n\}$ of subsets of E such that $E_n \cap E_m = \emptyset$ $(n = 1, 2, \ldots; n \neq m)$, and let $\Phi(u) = Au + B$ for $u \in R^+$.

Take a subsequence of $\{u_n\}$, and denote by $\{u_n\}$, for which $\Phi(u_n - d)\mu E_n > A\varepsilon\mu E$. Choose a subset F_n of E_n such that $\Phi(u_n - d)\mu F_n = A\varepsilon\mu E$ $(n = 1, 2, \ldots)$. Obviously, $\lim_{n\to\infty} \mu F_n = 0$. Let

$$x_n(t) = \begin{cases} x(t), & t \in G\setminus E, \\ d - \varepsilon, & t \in E\setminus F_n, \\ u_n - d, & t \in F_n. \end{cases} \tag{2.8}$$

Then

$$I_\Phi(x_n) = I_\Phi(x\chi_{G\setminus E}) + \Phi(d - \varepsilon)\mu(E\setminus F_n) + \Phi(u_n - d)\mu F_n$$
$$= I_\Phi(x\chi_{G\setminus E}) + (\Phi(d) - A\varepsilon)\mu(E\setminus F_n) + A\varepsilon\mu E$$
$$\to I_\Phi(x\chi_{G\setminus E}) + \Phi(d)\mu E = I_\Phi(x) = 1 \ (n \to \infty).$$

Hence, $\|x_n\| \to 1$ as $n \to \infty$. Similarly,

$$\begin{aligned} I_\Phi\left(\frac{x_n+x}{2}\right) &= I_\Phi(x\chi_{G\setminus E}) + \Phi\left(d-\frac{\varepsilon}{2}\right)\mu(E\setminus F_n) + \Phi\left(\frac{u_n}{2}\right)\mu F_n \\ &> I_\Phi(x\chi_{G\setminus E}) + \left(\Phi(d) - \frac{1}{2}A\varepsilon\right)\mu(E\setminus F_n) \\ &\quad + \left(1-\frac{1}{n}\right)\frac{\Phi(u_n)}{2}\mu F_n \\ &> I_\Phi(x\chi_{G\setminus E}) + \left(\Phi(d) - \frac{1}{2}A\varepsilon\right)\mu(E\setminus F_n) \\ &\quad + \left(1-\frac{1}{n}\right)\frac{\Phi(u_n-d)\mu F_n}{2} \\ &\to I_\Phi(x\chi_{G\setminus E}) + \Phi(d)\mu E = I_\Phi(x) = 1, \end{aligned}$$

whence $\|x_n + x\| \to 2$ as $n \to \infty$. Since $\Phi \in \triangle_2$, there exists $\delta > 0$ such that

$$|I_\Phi(x) - I_\Phi(x-w)| < \frac{\varepsilon}{2}A\mu E$$

whenever $I_\Phi(x) = A\varepsilon\mu E$ and $I_\Phi(w) < \delta$.

Since $\lim_{n\to\infty}\Phi(d-\varepsilon)\mu F_n = 0$, there exists $n_0 > 0$ such that

$$\Phi(d-\varepsilon)\mu F_n < \delta \ (n \geq n_0).$$

Then

$$\begin{aligned} I_\Phi(x_n - x_m) &> \Phi((u_n-d)-(d-\varepsilon))\mu F_n \\ &> \Phi(u_n-d)\mu F_n - \frac{1}{2}A\varepsilon\mu E \\ &= \frac{1}{2}A\varepsilon\mu E \end{aligned}$$

for $m > n > n_0$. For some $\delta' > 0$, we have

$$\|x_n - x_m\| \geq \delta' \text{ for any } m > n > n_0,$$

which shows that x is not CLUR point which leads to a contradiction. Thus, the proof is completed.

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References

[1] Chen S T, Geometry of Orlicz Spaces, *Dissertations Math.* (1996) (Warszawa) vol. 356

[2] Hudzik H and Kowalewski W, On some local geometric properties in Musielak-Orlicz function spaces, Zeitschrift für Analysis und ihre Anwendungen, *J. Anal. Appl.* **23** (2004) 683–712

[3] Hudzik H, Wang L and Wang T, Very smooth and strongly smooth points in Musielak-Orlicz function spaces equipped with the Luxemburg norm, *Comment. Math. Univ Carolinae.* **42** (2001) 487–497

[4] Hudzik H and Ye Y, Supporting functionals and smoothness in Musielak-Orlicz sequence spaces equipped with the Orlicz norm, *Math. Japan.* **39** (1994) 271–279

[5] Hudzik H and Maligranda L, Amemiya norm equals Orlicz norm in general, *Indagationes. Mathematicae* **11** (2000) 573–585

[6] Musielak J, Orlicz Spaces and Modular Spaces (1983) (New York: Berlin-Springer-Verlag)

[7] Panda B B and Kapoor O P, A generalization of local uniform convexity of the norm, *J. Math. Anal. Appl.* **52** (1975) 300–308

[8] Wang T, Cui Y and Meng C, Local property (β) in Orlicz spaces, *Math. Balkanica.* **14** (2000) 102–122

[9] Wang T, Ren Z and Zhang Y, On *UR*(*WUR*) point of Orlicz spaces, *J. Math.* **13** (1993) 443–452

[10] Wu C, Wang T, Chen S and Wang Y, Geometric Theory of Orlicz Spaces (1986) (Harbin)