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On an inequality concerning the polar derivative of a polynomial

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Abstract. In this paper, we present a correct proof of an L_p-inequality concerning the polar derivative of a polynomial with restricted zeros. We also extend Zygmund's inequality to the polar derivative of a polynomial.

Keywords. Zygmund's inequality; polar derivative; L_p-norm inequalities.

1. Introduction and statement of results

Let $P(z)$ be a polynomial of degree$n$ and let $P'(z)$ be its derivative. Then according to the famous result known as Bernstein's inequality (see [7] or [10])

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1}$$

Inequality (1) is sharp and equality in (1) holds for$P(z) = az^n$, $a \neq 0$. Inequality (1) was extended to L_p-norm by Zygmund [11] who proved that if $P(z)$ is a polynomial of degree n, then for $p \geq 1$,

$$\left\{\int_0^{2\pi} |P'(e^{i\theta})|^p d\theta\right\}^{1/p} \leq n \left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{1/p}. \tag{2}$$

The result is sharp and equality in (2) holds for $P(z) = az^n$, $a \neq 0$. If we let $p \to \infty$ in (2), we get inequality (1).

Let $D_\alpha P(z)$ denote the polar differentiation of polynomial $P(z)$ with respect to a real or complex number α. Then

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$$

uniformly on compact subsets of C.

As an extension of (1) to the polar derivative, Aziz and Shah (Theorem 4 with $k = 1$, [3]) have shown that if $P(z)$ is a polynomial of degree n, then for every complex number α with $|\alpha| \geq 1$,

$$|D_\alpha P(z)| \leq n|\alpha| \max_{|z|=1} |P(z)| \quad \text{for } |z| = 1. \tag{3}$$

Inequality (3) becomes equality for $P(z) = az^n, a \neq 0$.

If we divide the two sides of (3) by $|\alpha|$ and let $|\alpha| \to \infty$, we get inequality (1).

It is natural to seek L_p-norm analog of inequality (3). In view of the L_p-norm extension (2) of inequality (1), one would expect that if $P(z)$ is a polynomial of degree n, then

$$\left\{\int_0^{2\pi} |D_\alpha P(\mathrm{e}^{i\theta})|^p \mathrm{d}\theta\right\}^{1/p} \leq n|\alpha| \left\{\int_0^{2\pi} |P(\mathrm{e}^{i\theta})|^p\right\}^{1/p} \tag{4}$$

will be L_p-norm extension of (3) analogous to (2). But unfortunately inequality (4) is not, in general, true for every real or complex number α. To see this, we take in particular $p = 2$, $P(z) = (1 - iz)^n$ and $\alpha = i\beta$ where β is any positive real number such that

$$1 \leq \beta < \frac{n + \sqrt{2n(2n-1)}}{3n-2}. \tag{5}$$

Now

$$\begin{aligned} D_\alpha P(z) &= n(1 - iz)^n - ni(\alpha - z)(1 - iz)^{n-1} \\ &= n(1 - iz)^{n-1}(1 - i\alpha) \end{aligned}$$

so that

$$\begin{aligned} \int_0^{2\pi} |D_\alpha P(\mathrm{e}^{i\theta})|^p \mathrm{d}\theta &= n^2|1 - i\alpha|^2 \int_0^{2\pi} |1 - i\mathrm{e}^{i\theta}|^{2(n-1)} \mathrm{d}\theta \\ &= n^2|1 - i\alpha|^2 \int_0^{2\pi} |(1 - i\mathrm{e}^{i\theta})^{n-1}|^2 \mathrm{d}\theta \\ &= n^2|1 - i\alpha|^2 \int_0^{2\pi} \left|\binom{n-1}{0} - \binom{n-1}{1}(i\mathrm{e}^{i\theta}) \right. \\ &\quad + \binom{n-1}{2}(i\mathrm{e}^{i\theta})^2 - \cdots \\ &\quad \left. + (-1)^{n-1}\binom{n-1}{n-1}(i\mathrm{e}^{i\theta})^{n-1}\right|^2 \mathrm{d}\theta \\ &= 2\pi n^2|1 - i\alpha|^2 \left(\binom{n-1}{0}^2 + \binom{n-1}{1}^2 \right. \\ &\quad \left. + \binom{n-1}{2}^2 + \cdots + \binom{n-1}{n-1}^2\right) \\ &= 2\pi n^2|1 - i\alpha|^2 \binom{2(n-1)}{n-1}. \end{aligned} \tag{6}$$

Also,

$$\begin{aligned}
n^2|\alpha|^2 \int_0^{2\pi} |P(e^{i\theta})|^p d\theta &= n^2|\alpha|^2 \int_0^{2\pi} |1 - ie^{i\theta}|^{2n} d\theta \\
&= n^2|\alpha|^2 \int_0^{2\pi} |(1 - ie^{i\theta})^n|^2 d\theta \\
&= n^2|\alpha|^2 \int_0^{2\pi} \binom{n}{0} - \binom{n}{1}(ie^{i\theta}) + \binom{n}{2}(ie^{i\theta})^2 \\
&\quad - \cdots + (-1)^n \binom{n}{n}(ie^{i\theta})^n|^2 d\theta \\
&= 2\pi n^2|\alpha|^2 \left(\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2\right) \\
&= 2\pi n^2|\alpha|^2 \binom{2n}{n}. \qquad (7)
\end{aligned}$$

Using (6) and (7) in (4), we get

$$2\pi n^2 \binom{2(n-1)}{n-1} |1 - i\alpha|^2 \leq 2\pi n^2|\alpha|^2 \binom{2n}{n}.$$

This implies

$$n|1 - i\alpha|^2 \leq 2(2n-1)|\alpha|^2. \qquad (8)$$

Setting $\alpha = i\beta$ in (8), we get

$$n(1+\beta)^2 \leq 2(2n-1)\beta^2.$$

This inequality can be written as

$$\left(\beta - \frac{n + \sqrt{2n(2n-1)}}{3n-2}\right)\left(\beta - \frac{n - \sqrt{2n(2n-1)}}{3n-2}\right) \geq 0. \qquad (9)$$

Since $\beta \geq 1$, we have

$$\begin{aligned}
\left(\beta - \frac{n - \sqrt{2n(2n-1)}}{3n-2}\right) &\geq 1 - \frac{n - \sqrt{2n(2n-1)}}{3n-2} \\
&= \frac{2(n-1) + \sqrt{2n(2n-1)}}{3n-2} > 0
\end{aligned}$$

and hence from (9), it follows that

$$\left(\beta - \frac{n + \sqrt{2n(2n-1)}}{3n-2}\right) \geq 0.$$

This gives

$$\beta \geq \frac{n + \sqrt{2n(2n-1)}}{3n-2},$$

which clearly contradicts (5). Hence inequality (4) is not, in general, true for all polynomials $P(z)$ of degree $n \geq 1$.

However, we have been able to prove the following generalization of (2) to the polar derivatives.

Theorem 1. *If $P(z)$ is a polynomial of degree n, then for every complex number α and $p \geq 1$,*

$$\left\{\int_0^{2\pi} |D_\alpha P(\mathrm{e}^{i\theta})|^p \mathrm{d}\theta\right\}^{1/p} \leq n(|\alpha|+1)\left\{\int_0^{2\pi} |P(\mathrm{e}^{i\theta})|^p \mathrm{d}\theta\right\}^{1/p}. \tag{10}$$

Remark. If we divide both sides of (10) by $|\alpha|$ and make $|\alpha| \to \infty$, we get inequality (2) due to Zygmund [11].

For polynomials $P(z)$ which does not vanish in the unit disk, the right-hand side of (2) can be improved. In fact , in this direction, it was shown by De-Bruijn [4] that if $P(z)$ does not vanish in $|z| < 1$, then for $p \geq 1$,

$$\left\{\int_0^{2\pi} |P'(\mathrm{e}^{i\theta})|^p \mathrm{d}\theta\right\}^{1/p} \leq nC_p\left\{\int_0^{2\pi} |P(\mathrm{e}^{i\theta})|^p \mathrm{d}\theta\right\}^{1/p}, \tag{11}$$

where

$$C_p = \left\{\frac{1}{2\pi}\int_0^{2\pi} |1+\mathrm{e}^{i\beta}|^p \mathrm{d}\beta\right\}^{-1/p}. \tag{12}$$

Inequality (11) is best possible with equality for $P(z) = az^n + b$, $|a| = |b|$. If we let $p \to \infty$ in (11), it follows that if $P(z) \neq 0$ for $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{13}$$

Inequality (13) was conjectured by Erdös and later verified by Lax [6]. Aziz [1] extended (13) to the polar derivative of a polynomial and proved that if $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for every complex number α with $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha P(z)| \leq \frac{n}{2}(|\alpha|+1) \max_{|z|=1} |P(z)|. \tag{14}$$

The estimate (14) is best possible with equality for $P(z) = z^n + 1$. If we divide both sides of (14) by $|\alpha|$ and make $|\alpha| \to \infty$, we get inequality (13) due to Lax [6].

While seeking the desired extension to the polar derivatives, recently Govil *et al* [5] have made an incomplete attempt by claiming to have proved the following generalization of (11) and (14).

Theorem 2. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for every complex number α with $|\alpha| \geq 1$ and $p \geq 1$,*

$$\left\{\int_0^{2\pi} |D_\alpha P(\mathrm{e}^{i\theta})|^p \mathrm{d}\theta\right\}^{1/p} \leq n(|\alpha|+1)C_p\left\{\int_0^{2\pi} |P(\mathrm{e}^{i\theta})|^p \mathrm{d}\theta\right\}^{1/p}, \tag{15}$$

where C_p is defined by (12).

Unfortunately the proof of this theorem, which is the main result (Theorem 1.1 of [5]) given by Govil, Nyuydinkong and Tameru is not correct, because the claim made by the authors on page 624 in lines 12 to 16 by using Lemma 2.3 is incorrect. The reason being that their polynomial

$$D_\alpha P_n(z) + e^{i\gamma}\{n\bar{\alpha}zP_n(z) + (1-\bar{\alpha}z)zP_n'(z)\}, \quad z = e^{i\theta},$$

in general does not take the form

$$\sum_{k=0}^{n} l_k a_k z^k, \quad z = e^{i\theta}$$

where

$$P_n(z) = \sum_{k=0}^{n} a_k z^k$$

and the complex numbers l_k defined by them on page 624, line 10, by

$$L(P_n(e^{i\theta})) = [\Lambda P_n(e^{i\theta})]_{\theta=0} = \sum_{k=0}^{n} l_k a_k$$

along with the equation (24) of [5].

It is worthwhile to note here that if we take

$$L(P_n(e^{i\theta})) = [nP_n(e^{i\theta}) + (\alpha - e^{i\theta})P_n'(e^{i\theta})]_{\theta=0}$$

and use the same argument as used by Govil *et al* (page 624, line 10 of [5]), then in view of the inequality

$$|D_\alpha P(z)| \le n|\alpha| \max_{|z|=1} |P(z)| \quad \text{for } |z| = 1$$

(see Theorem 4 with $k = 1$ of [3]), the above bounded functional has norm $N \le n|\alpha|$. Therefore, if we use Lemma 2.3 of [5] which is due to Rahman (Lemma 3 of [8]), it would follow that

$$\left\{\int_0^{2\pi} |D_\alpha P(e^{i\theta})|^p d\theta\right\}^{1/p} \le n|\alpha| \left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{1/p}$$

for every $p \ge 1$ and $|\alpha| \ge 1$, which is not true in general as shown above.

Here we shall also present a correct proof of Theorem 2, which shall validate Theorems 1.2 and 1.3 of Govil *et al* [5] as well. Finally we shall also present a short proof of Theorem 1.3 of [5]. That is, we prove the following.

Theorem 3. *If $P(z)$ is a self-inversive polynomial of degree n, then for every complex number α and $p \ge 1$,*

$$\left\{\int_0^{2\pi} |D_\alpha P(e^{i\theta})|^p d\theta\right\}^{1/p} \le n(|\alpha|+1)C_p \left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{1/p}, \quad (16)$$

where C_p is the same as in Theorem 2.

2. Lemmas

For the proofs of these theorems, we need the following lemmas.

Lemma 1. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, and $Q(z) = z^n \overline{P(1/\bar{z})}$, then for every complex number α with $|\alpha| \geq 1$,*

$$|D_\alpha P(z)| \leq |D_\alpha Q(z)| \quad \textit{for } |z| \geq 1.$$

Lemma 1 is due to Aziz (p. 190 of [1]).

Lemma 2. *If $P(z)$ is a polynomial of degree n and $Q(z) = z^n \overline{P(1/\bar{z})}$, then for every $p \geq 0$ and β real,*

$$\int_0^{2\pi} \int_0^{2\pi} |Q'(\mathrm{e}^{i\theta}) + \mathrm{e}^{i\beta} P'(\mathrm{e}^{i\theta})|^p \mathrm{d}\theta \mathrm{d}\beta \leq 2\pi n^p \int_0^{2\pi} |P(\mathrm{e}^{i\theta})|^p \mathrm{d}\theta.$$

Lemma 2 is due to Aziz [2] (see also [8]). We also need the following lemma.

Lemma 3. *If $P(z)$ is a polynomial of degree n, $P(0) \neq 0$ and $Q(z) = z^n \overline{P(1/\bar{z})}$, then for every complex number α, $p \geq 1$ and β real,*

$$\int_0^{2\pi} \int_0^{2\pi} |D_\alpha Q(\mathrm{e}^{i\theta}) + \mathrm{e}^{i\beta} D_\alpha P(\mathrm{e}^{i\theta})|^p \mathrm{d}\theta \mathrm{d}\beta \leq 2\pi n^p (|\alpha| + 1)^p \int_0^{2\pi} |P(\mathrm{e}^{i\theta})|^p \mathrm{d}\theta.$$

Proof of Lemma 3. We have by Minkowski's inequality for every $p \geq 1$ and β real,

$$\begin{aligned}
&\left\{\int_0^{2\pi} \int_0^{2\pi} |D_\alpha Q(\mathrm{e}^{i\theta}) + \mathrm{e}^{i\beta} D_\alpha P(\mathrm{e}^{i\theta})|^p \mathrm{d}\theta \mathrm{d}\beta\right\}^{1/p} \\
&\quad = \left\{\int_0^{2\pi} \int_0^{2\pi} |(nQ(\mathrm{e}^{i\theta}) + (\alpha - \mathrm{e}^{i\theta}) Q'(\mathrm{e}^{i\theta})) + \mathrm{e}^{i\beta}(nP(\mathrm{e}^{i\theta}) \right. \\
&\quad \left. + (\alpha - \mathrm{e}^{i\theta}) P'(\mathrm{e}^{i\theta}))|^p \mathrm{d}\theta \mathrm{d}\beta\right\}^{1/p} \\
&\quad = \left\{\int_0^{2\pi} \int_0^{2\pi} |(nQ(\mathrm{e}^{i\theta}) - \mathrm{e}^{i\theta} Q'(\mathrm{e}^{i\theta})) + \mathrm{e}^{i\beta}(nP(\mathrm{e}^{i\theta}) - \mathrm{e}^{i\theta} P'(\mathrm{e}^{i\theta})) \right. \\
&\quad \left. + \alpha(Q'(\mathrm{e}^{i\theta}) + \mathrm{e}^{i\beta} P'(\mathrm{e}^{i\theta}))|^p \mathrm{d}\theta \mathrm{d}\beta\right\}^{1/p} \\
&\quad \leq \left\{\int_0^{2\pi} \int_0^{2\pi} |(nQ(\mathrm{e}^{i\theta}) - \mathrm{e}^{i\theta} Q'(\mathrm{e}^{i\theta})) \right. \\
&\quad \left. + \mathrm{e}^{i\beta}(nP(\mathrm{e}^{i\theta}) - \mathrm{e}^{i\theta} P'(\mathrm{e}^{i\theta}))|^p \mathrm{d}\theta \mathrm{d}\beta\right\}^{1/p} \\
&\quad + |\alpha| \left\{\int_0^{2\pi} \int_0^{2\pi} |Q'(\mathrm{e}^{i\theta}) + \mathrm{e}^{i\beta} P'(\mathrm{e}^{i\theta})|^p \mathrm{d}\theta \mathrm{d}\beta\right\}^{1/p}. \qquad (17)
\end{aligned}$$

Since $Q(z) = z^n\overline{P(1/\bar{z})}$, we have $P(z) = z^n\overline{Q(1/\bar{z})}$ and it can be easily verified that for $0 \le \theta < 2\pi$,

$$nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta}) = e^{i(n-1)\theta}\overline{Q'(e^{i\theta})} \tag{18}$$

and

$$nQ(e^{i\theta}) - e^{i\theta}Q'(e^{i\theta}) = e^{i(n-1)\theta}\overline{P'(e^{i\theta})}. \tag{19}$$

Using (18) and (19) in (17), we obtain

$$\begin{aligned}
&\left\{\int_0^{2\pi}\int_0^{2\pi}|D_\alpha Q(e^{i\theta}) + e^{i\beta}D_\alpha P(e^{i\theta})|^p d\theta d\beta\right\}^{1/p}\\
&\quad\le \left\{\int_0^{2\pi}\int_0^{2\pi}\left|e^{i(n-1)\theta}\overline{P'(e^{i\theta})} + e^{i\beta}e^{i(n-1)\theta}\overline{Q'(e^{i\theta})}\right|^p d\theta d\beta\right\}^{1/p}\\
&\qquad + |\alpha|\left\{\int_0^{2\pi}\int_0^{2\pi}\left|Q'(e^{i\theta}) + e^{i\beta}P'(e^{i\theta})\right|^p d\theta d\beta\right\}^{1/p}\\
&\quad= \left\{\int_0^{2\pi}\int_0^{2\pi}\left|Q'(e^{i\theta}) + e^{i\beta}P'(e^{i\theta})\right|^p d\theta d\beta\right\}^{1/p}\\
&\qquad + |\alpha|\left\{\int_0^{2\pi}\int_0^{2\pi}\left|Q'(e^{i\theta}) + e^{i\beta}P'(e^{i\theta})\right|^p d\theta d\beta\right\}^{1/p}\\
&\quad= (|\alpha| + 1)\left\{\int_0^{2\pi}\int_0^{2\pi}\left|Q'(e^{i\theta}) + e^{i\beta}P'(e^{i\theta})\right|^p d\theta d\beta\right\}^{1/p}.
\end{aligned}$$

This gives with the help of Lemma 2,

$$\int_0^{2\pi}\int_0^{2\pi}|D_\alpha Q(e^{i\theta}) + e^{i\beta}D_\alpha P(e^{i\theta})|^p d\theta d\beta \le 2\pi n^p(|\alpha| + 1)^p\int_0^{2\pi}|P(e^{i\theta})|^p d\theta.$$

This completes the proof of Lemma 3.

3. Proofs of the theorems

Proof of Theorem 1. By Lemma 3, we have for every complex number α, $p \ge 1$ and β real,

$$\int_0^{2\pi}\int_0^{2\pi}|D_\alpha Q(e^{i\theta}) + e^{i\beta}D_\alpha P(e^{i\theta})|^p d\theta d\beta \le 2\pi n^p(|\alpha| + 1)^p\int_0^{2\pi}|P(e^{i\theta})|^p d\theta. \tag{20}$$

Using in (20) the fact that for any $p \ge 0$,

$$\int_0^{2\pi}|a + be^{i\beta}|^p d\beta \ge 2\pi\max\{|a|^p, |b|^p\}$$

(see inequality (19) of [4]), we obtain

$$\left\{\int_0^{2\pi} |D_\alpha P(e^{i\theta})|^p d\theta\right\}^{1/p} \le n(|\alpha|+1)\left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{1/p}.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Since $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, by Lemma 1 we have for every complex number α with $|\alpha| \ge 1$,

$$|D_\alpha P(z)| \le |D_\alpha Q(z)|, \quad \text{for } |z| = 1 \tag{21}$$

where $Q(z) = z^n \overline{P(1/\bar{z})}$. Also by Lemma 3 for every complex number α, $p \ge 1$ and β real,

$$\int_0^{2\pi}\left\{\int_0^{2\pi} |D_\alpha Q(e^{i\theta}) + e^{i\beta} D_\alpha P(e^{i\theta})|^p d\beta\right\} d\theta \le 2\pi n^p (|\alpha|+1)^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta. \tag{22}$$

Now for every real β and $r \ge 1$, we have

$$|r + e^{i\beta}| \ge |1 + e^{i\beta}|,$$

which implies

$$\int_0^{2\pi} |r + e^{i\beta}|^p d\beta \ge \int_0^{2\pi} |1 + e^{i\beta}|^p d\beta, \quad p \ge 0.$$

If $D_\alpha P(e^{i\theta}) \ne 0$, we take $r = |D_\alpha Q(e^{i\theta})|/|D_\alpha P(e^{i\theta})|$, and by (21) $r \ge 1$ and we get

$$\begin{aligned}
&\int_0^{2\pi} |D_\alpha Q(e^{i\theta}) + e^{i\beta} D_\alpha P(e^{i\theta})|^p d\beta \\
&\quad = |D_\alpha P(e^{i\theta})|^p \int_0^{2\pi} \left|\frac{D_\alpha Q(e^{i\theta})}{D_\alpha P(e^{i\theta})} + e^{i\beta}\right|^p d\beta \\
&\quad = |D_\alpha P(e^{i\theta})|^p \int_0^{2\pi} \left|\left|\frac{D_\alpha Q(e^{i\theta})}{D_\alpha P(e^{i\theta})}\right| + e^{i\beta}\right|^p d\beta \\
&\quad \ge |D_\alpha P(e^{i\theta})|^p \int_0^{2\pi} |1 + e^{i\beta}|^p d\beta.
\end{aligned}$$

For $D_\alpha P(e^{i\theta}) = 0$, this inequality is trivially true. Using this in (22), we conclude that for every complex number α with $|\alpha| \ge 1$ and $p \ge 1$,

$$\int_0^{2\pi} |1 + e^{i\beta}|^p d\beta \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^p d\theta \le 2\pi n^p (|\alpha|+1)^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta,$$

which immediately leads to (15) and this completes the proof of Theorem 2.

Proof of Theorem 3. Since $P(z)$ is a self-inversive polynomial of degree n, we have $P(z) = Q(z)$ where $Q(z) = z^n \overline{P(1/\bar{z})}$. Therefore, for every complex number α,

$$|D_\alpha P(z)| = |D_\alpha Q(z)| \quad \text{for all } z \in C,$$

so that

$$|D_\alpha Q(\mathrm{e}^{i\theta})/D_\alpha P(\mathrm{e}^{i\theta})| = 1.$$

Using this in (22) and proceeding similarly as in the proof of Theorem 2, we get (16) and this proves Theorem 3.

References

[1] Aziz A, Inequalities for the polar derivative of a polynomial, *J. Approx. Theory* **55** (1988) 183–193

[2] Aziz A and Rather N A, Some Zygmund type L^q inequalities for polynomials, *J. Math. Anal. Appl.* **289** (2004) 14–29

[3] Aziz A and Shah W M, Inequalities for the polar derivative of a polynomial, *Indian J. Pure Appl. Math.* **29** (1998) 163–173

[4] De-Bruijn N G, Inequalities concerning polynomials in the complex domain, *Nederl. Akad. Wetensch. Proc.* **50** (1947) 1265–1272; *Indag. Math.* **9** (1947) 591–598

[5] Govil N K, Nyuydinkong G and Tameru B, Some L^p inequalities for the polar derivative of a polynomial, *J. Math. Anal. Appl.* **254** (2001) 618–626

[6] Lax P D, Proof of a conjecture of P. Erdös on the derivative of a polynomial, *Bull. Am. Math. Soc.* **50** (1944) 509–513

[7] Milovanović G V, Mitrinović D S and Rassias Th, Topics in polynomials: Extremal properties, inequalities, zeros (Singapore: World Scientific) (1994)

[8] Rahman Q I, Functions of exponentials type, *Trans. Am. Math. Soc.* **135** (1969) 295–309

[9] Rather N A, Extremal properties and location of the zeros of polynomials, Ph.D. thesis, submitted to the University of Kashmir (1998)

[10] Schaeffer A C, Inequalities of A. Markoff and S.N. Bernstein for polynomials and related functions, *Bull. Am. Math. Soc.* **47** (1941) 565–579

[11] Zygmund A, A remark on conjugate series, *Proc. London Math. Soc.* **34** (1932) 392–400