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On the Schwartz space isomorphism theorem for rank one symmetric space

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Abstract. In this paper we give a simpler proof of the L^p-Schwartz space isomorphism $(0 < p \leq 2)$ under the Fourier transform for the class of functions of left δ-type on a Riemannian symmetric space of rank one. Our treatment rests on Anker's [2] proof of the corresponding result in the case of left K-invariant functions on X. Thus we give a proof which relies only on the Paley–Wiener theorem.

Keywords. δ Spherical transform; Helgason Fourier transform.

1. Introduction

Let X be a rank one Riemannian symmetric space of noncompact type. We recall that such a space can be realized as G/K, where G is a connected noncompact semisimple Lie group of real rank one with finite center and K is a maximal compact subgroup of G. Anker [2], in his paper gave a remarkably short and elegant proof of the L^p-Schwartz space isomorphism theorem for K bi-invariant functions on G under the spherical Fourier transform for $(0 < p \leq 2)$. The result for K bi-invariant functions was first proved by Harish-Chandra [6–8] (for $p = 2$) and Trombi and Varadarajan [12] (for $0 < p < 2$). Eguchi and Kowata [4] addressed the isomorphism problem for the L^p-Schwartz spaces on X. In [2], Anker has successfully avoided the involved asymptotic expansion of the elementary spherical functions, which has a crucial role in all the earlier works. In this paper, we have exploited Anker's technique to obtain the isomorphism of the L^p-Schwartz space $(0 < p \leq 2)$ under Fourier transform for functions on X of a fixed K-type.

Let (δ, V_δ) be an unitary irreducible representation of K of dimension δ. Our basic L^p-Schwartz space $S^p_\delta(X)$ is a space of $\mathrm{Hom}(V_\delta, V_\delta)$-valued C^∞ functions, the Eisenstein integral $\Phi_{\lambda,\delta}(x)$ is a $\mathrm{Hom}(V_\delta, V_\delta)$-valued entire function on $\mathbb{C}$ and $S_\delta(\mathfrak{a}^*_{\mathbb{C}})$ consists of analytic functions on the strip $\mathfrak{a}^*_\epsilon = \{\lambda \in \mathbb{C} \,|\, |\mathrm{Im}\,\lambda| \leq \epsilon\}$. Anticipating these and other notations and definitions developed in §§2 and 3, we state the main result of the paper.

Theorem 1.1. *For* $0 < p \leq 2$ *and* $\epsilon = 2/p - 1$ *the* δ*-spherical transform* $f \mapsto \tilde{f}$, *where*

$$\tilde{f}(\lambda) = d(\delta) \int_X \mathrm{tr}\, f(x)\, \Phi_{\bar{\lambda},\delta}(x)^* \mathrm{d}x, \tag{1.1}$$

is a topological vector space isomorphism between the spaces $S_\delta^p(X)$ *and* $S_\delta(\mathfrak{a}_\epsilon^*)$; *with the inverse*

$$f(x) = \omega^{-1} \int_{\mathfrak{a}^*} \Phi_{\lambda,\delta}(x) \tilde{f}(\lambda) |c(\lambda)|^{-2} \, \mathrm{d}\lambda. \tag{1.2}$$

2. Preliminaries

The pair (G, K) and X are as described in the introduction. We let $G = KAN$ denote a fixed Iwasawa decomposition of G. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ denote the Lie algebras of G, K, A and N respectively. We recall that dimension of $\mathfrak{a} = 1$.

Let $\mathfrak{a}^*$ be the real dual of $\mathfrak{a}$ and $\mathfrak{a}_{\mathbb{C}}^*$ be its complexification. We identify $\mathfrak{a}$, $\mathfrak{a}^*$ with $\mathbb{R}$ and $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}$ using a normalization explained below. Let H: $g \mapsto H(g)$ and A: $g \mapsto A(g)$ be projections of $g \in G$ in $\mathfrak{a}$ in Iwasawa KAN and NAK decompositions respectively, that is any $g \in G$ can be written as $g = k \exp H(g) n = n_1 \exp A(g) k_1$. These two are related by $A(g) = -H(g^{-1})$ for all $g \in G$. Let M' and M respectively be the normalizer and centralizer of A in K. M also normalizes N. Let $W = M'/M$ be the Weyl group of G. Here $W = \{\pm 1\}$. Let us choose and fix a system of positive restricted roots which we denote by Σ^+. The real number ρ corresponds to $\frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$ where m_α is the multiplicity of the root α. With a suitable normalization of the basis of A we can identify ρ with 1. The positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ ($\mathfrak{a}^{*+} \subset \mathfrak{a}^*$) is identified with the positive real numbers. We denote x^+ to be the $\overline{\mathfrak{a}^+}$ component of $x \in G$ for the Cartan decomposition $G = K\overline{A^+}K = K(\exp \overline{\mathfrak{a}^+})K$ and let $|x| = x^+$. We have a basic estimate (Proposition 4.6.11 of [5]): there is a constant $c > 0$ such that

$$|H(x)| \leq c|x| \quad \text{for } x \in G\,. \tag{2.1}$$

We note that, any function f on X can also be considered as a function on the group G with the property $f(gk) = f(k)$, where $g \in G$ and $k \in K$. Let $x = ka_t k'$ where $a_t = \exp t \in A$, $t \in \mathfrak{a} \cong \mathbb{R}$. The Haar measure of G for the Cartan decomposition is given by

$$\int_G f(x) \mathrm{d}x = \text{const} \int_K \mathrm{d}k \int_{\mathfrak{a}^+} \Delta(t) \, \mathrm{d}t \int_K \mathrm{d}k' f(ka_t k'), \tag{2.2}$$

where $\Delta(t) = \prod_{\alpha \in \Sigma^+} \sinh^{m_\alpha} \alpha(t)$. In the Iwasawa decomposition, $x = ka_t n$, the Haar measure is

$$\int_G f(x) \mathrm{d}x = \text{const} \int_K \mathrm{d}k \int_{\mathfrak{a}^+} \mathrm{e}^{2t} \mathrm{d}t \int_N \mathrm{d}n f(ka_t n). \tag{2.3}$$

In both (2.2) and (2.3) 'const' stands for positive normalizing constants for the respective cases.

Let (δ, V_δ) be an unitary irreducible representation of K. Let $d(\delta)$ and χ_δ stand for the dimension and character of the representation δ. Let V_δ^M be the subspace of V_δ fixed under $\delta|_M$; i.e $V_\delta^M = \{v \in V_\delta \,|\, \delta(m)v = v, \forall m \in M\}$. Recall that as G is of real rank one, the dimension of V_δ^M is 0 or 1 (see [11]). Let $\hat{K}_M$ be the set of all equivalence classes of irreducible unitary representation δ of K for which $V_\delta^M \neq \{0\}$. For our result we choose $\delta \in \hat{K}_M$. We shall also fix an orthonormal basis $\{v_1, v_2, \ldots, v_{d(\delta)}\}$ of V_δ such that v_1 spans V_δ^M.

We shall denote $\mathcal{D}(X)$ for the space of all $\mathbb{C}$-valued C^∞ functions on X with compact support. For any function $f \in \mathcal{D}(X)$, the Helgason Fourier transform (HFT) (III, §1 of [9]) $\mathcal{F}f$ is defined by

$$\mathcal{F}f(\lambda, kM) = \int_X f(x) \mathrm{e}^{(i\lambda - 1)H(x^{-1}k)}\, \mathrm{d}x. \tag{2.4}$$

Let us fix the notation $\mathcal{F}f(\lambda, kM) = \mathcal{F}f(\lambda, k)$. The inversion formula for HFT for $f \in \mathcal{D}(X)$ is given by

$$f(x) = \frac{1}{\omega} \int_{\mathfrak{a}^*} \int_K \mathcal{F}f(\lambda, k) \mathrm{e}^{-(i\lambda + 1)H(x^{-1}k)} |\mathbf{c}(\lambda)|^{-2} \mathrm{d}\lambda \mathrm{d}k. \tag{2.5}$$

Here, $\omega = |W|$ is the cardinality of the Weyl group and $\mathbf{c}(\lambda)$ is the Harish-Chandra $\mathbf{c}$-function. For our purpose we shall need the following simple estimate on $\mathbf{c}(\lambda)$: there exist constants $c, b > 0$ such that

$$|\mathbf{c}(\lambda)|^{-2} \ \le\ c(|\lambda| + 1)^b \quad \text{for } \lambda \in \mathfrak{a}^* \tag{2.6}$$

(see, [IV, Proposition 7.2 of [10]).

Let $\mathcal{D}(X, \mathrm{Hom}(V_\delta, V_\delta))$ be the space of all C^∞ functions on X taking values in $\mathrm{Hom}(V_\delta, V_\delta)$ and with compact support.

Let $\mathcal{D}^\delta(X) = \{f \in \mathcal{D}(X, \mathrm{Hom}(V_\delta, V_\delta)) \mid f(k \cdot x) = \delta(k) f(x)\}$. We topologize $\mathcal{D}^\delta(X)$ by the inductive limit topology of the spaces $\mathcal{D}_R(X, \mathrm{Hom}(V_\delta, V_\delta))$, where $R = 0, 1, 2, \ldots$. These are the spaces of functions on X with support lying in the geodesic R-balls. Let $\check{\delta}$ be the contragradient representation of δ. The class of functions $\mathcal{D}_{\check{\delta}}(X) = \{f \in \mathcal{D}(X) \mid f \equiv d(\delta)\chi_\delta * f\}$ is the space of all left $\check{\delta}$ type functions on X. Being a subspace of $\mathcal{D}(X)$, $\mathcal{D}_{\check{\delta}}(X)$ inherits the subspace topology of $\mathcal{D}(X)$. We also notice that, for $f \in C^\infty(X)$ the function

$$f^\delta(x) = d(\delta) \int_K f(k \cdot x) \delta(k^{-1}) \mathrm{d}k \tag{2.7}$$

is a C^∞ map from X to $\mathrm{Hom}(V_\delta, V_\delta)$ satisfying

$$f^\delta(k \cdot x) = \delta(k) f^\delta(x).$$

The following lemma (III, Proposition 5.10 of [9]) shows that the two function spaces $\mathcal{D}^\delta(X)$ and $\mathcal{D}_{\check{\delta}}(X)$ are topologically isomorphic.

Lemma 2.1 [9]. *The map* $Q\colon \mathcal{D}^\delta(X) \longrightarrow \mathcal{D}_{\check{\delta}}(X)$ *given by*

$$Q\colon f \longmapsto \mathrm{tr}\, f$$

is a homeomorphism with the inverse given by $Q^{-1}(g) = g^\delta$ *for* $g \in \mathcal{D}_{\check{\delta}}(X)$.

3. The δ-spherical transform

Most of the material in this section can be retrieved from [9]. Here we will restructure the results in a form which is suitable for our purpose. In particular we will transfer the results from $D_{\check{\delta}}(X)$ to $\mathcal{D}^\delta(X)$ using the homomorphism Q, defined in Lemma 2.1.

DEFINITION 3.1

For $f \in \mathcal{D}^{\delta}(X)$ the δ-spherical transform $\tilde{f}$ is given by

$$\tilde{f}(\lambda) = d(\delta) \int_X \operatorname{tr} f(x) \Phi_{\bar{\lambda},\delta}(x)^* \mathrm{d}x, \quad \lambda \in \mathbb{C} \tag{3.1}$$

where, $\Phi_{\lambda,\delta}(x)$ is the generalized spherical function (Eisenstein integral). Precisely,

$$\Phi_{\lambda,\delta}(x) = \int_K \mathrm{e}^{-(i\lambda+1)H(x^{-1}k)} \delta(k) \mathrm{d}k \tag{3.2}$$

and therefore, the adjoint of $\Phi_{\lambda,\delta}(x)$ is

$$\Phi_{\bar{\lambda},\delta}(x)^* = \int_K \mathrm{e}^{(i\lambda-1)H(x^{-1}k)} \delta(k^{-1}) \mathrm{d}k. \tag{3.3}$$

The following is a list of some basic properties of the generalized spherical functions.

1. For $k \in K$, $\Phi_{\lambda,\delta}(kx) = \delta(k)\Phi_{\lambda,\delta}(x)$ and $\Phi_{\lambda,\delta}(kx)^* = \Phi_{\lambda,\delta}(x)^* \delta(k^{-1})$. For $v \in V_\delta$ and $m \in M$, $\delta(m)(\Phi_{\bar{\lambda},\delta}(x)^* v) = \Phi_{\bar{\lambda},\delta}(x)^* v$. This shows that $\Phi^*_{\bar{\lambda},\delta}$ is a $\mathrm{Hom}(V_\delta, V_\delta^M)$-valued function on X.
2. Let $\mathbf{L}$ be the Laplace–Beltrami operator of X. Then $\mathbf{L}\, \Phi_{\lambda,\delta} = -(\lambda^2 + 1)\Phi_{\lambda,\delta}$ (§1(6) of [9]).
3. Let $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ be the the universal enveloping algebra of G. For any $g_1, g_2 \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ there exist constants $c_\delta = c_\delta(g_1, g_2, \delta)$, $c_0 > 0$, $b = b(g_1, g_2)$ so that (see [3])

$$\|\Phi_{\lambda,\delta}(g_1, x, g_2)\| \le c_\delta (1 + |\lambda|)^b \varphi_0(x) \mathrm{e}^{c_0 |\mathrm{Im}\, \lambda|(1+|x|)}, \quad x \in X. \tag{3.4}$$

   Here $\|\cdot\|$ is the Hilbert–Schmidt norm.
4. If δ is the trivial representation of K then $\Phi_{\lambda,\delta}(x)$ reduces to the elementary spherical function

$$\varphi_\lambda(x) = \int_K \mathrm{e}^{-(i\lambda+1)H(x^{-1}k)} \mathrm{d}k. \tag{3.5}$$

   It satisfies the following estimates:

   (i) For each $H \in \overline{\mathfrak{a}^+}$ and $\lambda \in \overline{\mathfrak{a}^{*+}}$,

$$0 < \varphi_{-i\lambda}(\exp H) \le \mathrm{e}^{\lambda H} \varphi_0(\exp H), \tag{3.6}$$

   where, $\varphi_0(\cdot)$ is the elementary spherical function at $\lambda = 0$ (see Proposition 4.6.1 of [5]).

   (ii) For all $g \in G$, $0 < \varphi_0(g) \le 1$ (Proposition 4.6.3 of [5]) and for $t \in \overline{\mathfrak{a}^+}$,

$$\mathrm{e}^{-t} \le \varphi_0(\exp t) \le q(1+t)\mathrm{e}^{-t} \tag{3.7}$$

   for some $q > 0$ (see [1] for a sharper estimate).

5. We have already noticed that V_δ^M is 1-dimensional. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, $\delta \in \hat{K}_M$ and $x \in X$, the linear functional $\Phi_{\lambda,\delta}(x)|_{V_\delta^M}$ is a scalar multiplication. The elementary spherical function ϕ_λ is related to $\Phi_{\lambda,\delta}$ in the following way (see III, Corollary 5.17 of [9]):

$$\Phi_{\lambda,\delta}(x)\,|_{V_\delta^M} = Q^\delta(\lambda)^{-1}(\mathbf{D}^\delta \varphi_\lambda)(x), \tag{3.8}$$

where $\mathbf{D}^\delta$ is a certain constant coefficient differential operator and $Q^\delta(\lambda)$ is a constant real coefficient polynomial in $i\lambda$. An explicit expression for the polynomial Q^δ is available in III, §2 of [9].
6. For each $a \in A$, the functions $\lambda \mapsto Q^\delta(\lambda)\Phi_{\lambda,\delta}(a)$ and $\lambda \mapsto Q^\delta(\lambda)^{-1}\Phi_{\bar{\lambda},\delta}(a)^*$ are even holomorphic functions on $\mathfrak{a}_{\mathbb{C}}^*$ (see III, Theorem 5.15 of [9]).

It follows from 1 and 6 above that for $f \in \mathcal{D}^\delta(X)$, $\lambda \mapsto Q^\delta(\lambda)^{-1}\tilde{f}(\lambda)$ is a Hom(V_δ, V_δ^M)-valued even entire function on $\mathbb{C}$.

The HFT and the δ-spherical transform of a function $f \in \mathcal{D}^\delta(X)$ are related in the following manner.

DEFINITION 3.2

Let $\delta \in \hat{K}_M$, $f \in \mathcal{D}(X)$ and $\mathcal{F}f$ be its HFT. Then let us define the δ-projection $(\mathcal{F}f)^\delta$ of $\mathcal{F}f$ by

$$(\mathcal{F}f)^\delta(\lambda, k) = d(\delta)\int_K \mathcal{F}f(\lambda, k_1k)\delta(k_1^{-1})\mathrm{d}k_1. \tag{3.9}$$

As noted earlier for $f \in \mathcal{D}(X)$, its δ-projection $f^\delta \in \mathcal{D}^\delta(X)$. Each of its matrix entry is a member of $\mathcal{D}(X)$. We define the HFT of f^δ by

$$\mathcal{F}(f^\delta)(\lambda, k) = \int_X f^\delta(x)\mathrm{e}^{(i\lambda-1)H(x^{-1}k)}\mathrm{d}x. \tag{3.10}$$

This is nothing but the usual HFT at each matrix entry of f^δ.

PROPOSITION 3.3

For $f \in \mathcal{D}(X)$ and $\delta \in \hat{K}_M$ the following are true:

(1) $(\mathcal{F}f)^\delta(\lambda, k) = \delta(k)(\mathcal{F}f)^\delta(\lambda, e)$.
(2) $\mathcal{F}(f^\delta)(\lambda, k) = (\mathcal{F}f)^\delta(\lambda, k)$.

Proof. It is clear from the definition that $(\mathcal{F}f)^\delta(\lambda, k) = \delta(k)(\mathcal{F}f)^\delta(\lambda, e)$.

The following straightforward calculation using Fubini's theorem proves the second assertion.

$$\begin{aligned}\mathcal{F}(f^\delta)(\lambda, kM) &= \int_X f^\delta(x)\mathrm{e}^{(i\lambda-1)H(x^{-1}k)}\mathrm{d}x,\\ &= d(\delta)\int_X \left\{\int_K f(k_1x)\delta(k_1^{-1})\mathrm{d}k_1\right\}\mathrm{e}^{(i\lambda-1)H(x^{-1}k)}\mathrm{d}x,\end{aligned}$$

$$
\begin{aligned}
&= d(\delta) \int_K \int_X f(y) \mathrm{e}^{(i\lambda-1)H(y^{-1}k_1 k)} \delta(k_1^{-1}) \mathrm{d}y \mathrm{d}k_1, \\
&= d(\delta) \int_X \int_K f(y) \mathrm{e}^{(i\lambda-1)H(y^{-1}k_2)} \delta(k) \delta(k_2^{-1}) \mathrm{d}k_2 \mathrm{d}y, \\
&= d(\delta) \int_K \mathcal{F} f(\lambda, k_2 M) \delta(k) \delta(k_2^{-1}) \mathrm{d}k_2, \\
&= d(\delta) \int_K \mathcal{F} f(\lambda, k_3 k M) \delta(k_3^{-1}) \mathrm{d}k_3, \\
&= (\mathcal{F} f)^{\delta}(\lambda, kM).
\end{aligned}
$$

The next lemma relates the δ-spherical transform defined in (3.1) with the HFT.

Lemma 3.4. *If $f \in \mathcal{D}^{\delta}(X)$ and $\delta \in \hat{K}_M$, then* $\mathcal{F} f(\lambda, e) = \tilde{f}(\lambda)$.

Proof. For any $f \in \mathcal{D}^{\delta}(X)$, by Lemma 2.1, $f(x) = d(\delta) \int_K \operatorname{tr} f(kx) \delta(k^{-1})\, \mathrm{d}k$. From the definition of HFT (2.4) we get

$$
\begin{aligned}
\mathcal{F} f(\lambda, e) &= \int_X f(x) \mathrm{e}^{(i\lambda-1)H(x^{-1})} \mathrm{d}x, \\
&= \int_X d(\delta) \int_K \operatorname{tr} f(kx) \delta(k^{-1})\, \mathrm{d}k\, \mathrm{e}^{(i\lambda-1)H(x^{-1})}\, \mathrm{d}x.
\end{aligned}
$$

Substituting $kx = y$ we have

$$
\begin{aligned}
F f(\lambda, e) &= d(\delta) \int_X \operatorname{tr} f(y) \int_K \mathrm{e}^{(i\lambda-1)H(y^{-1}k)} \delta(k^{-1}) \mathrm{d}k, \\
&= d(\delta) \int_X \operatorname{tr} f(y) \Phi_{\bar{\lambda},\delta}(y)^* \mathrm{d}y, \\
&= \tilde{f}(\lambda).
\end{aligned}
$$

Lemma 3.5. *The inversion formula for the δ-spherical transform $f \mapsto \tilde{f}$ is given by the following*: *For each $f \in \mathcal{D}^{\delta}(X)$*,

$$
f(x) = \frac{1}{\omega} \int_{\mathfrak{a}^*} \Phi_{\lambda,\delta}(x) \tilde{f}(\lambda) |\mathbf{c}(\lambda)|^{-2}\, \mathrm{d}\lambda. \tag{3.11}
$$

Moreover,

$$
\int_X \|f(x)\|^2\, \mathrm{d}x = \frac{1}{w} \int_{\mathfrak{a}^*} \|\tilde{f}(\lambda)\|^2 |\mathbf{c}(\lambda)|^{-2}\, \mathrm{d}\lambda. \tag{3.12}
$$

Here, *the norm $\|\cdot\|$ is the Hilbert–Schmidt norm.*

Proof. We use the inversion formula for the HFT (2.5), Proposition 3.3 and Lemma 3.4 to obtain

$$\begin{aligned} f(x) &= \frac{1}{\omega}\int_{\mathfrak{a}^*}\int_K \mathcal{F}f(\lambda,k)\mathrm{e}^{-(i\lambda+1)H(x^{-1}k)}|\mathbf{c}(\lambda)|^{-2}\,\mathrm{d}\lambda\mathrm{d}k \\ &= \frac{1}{\omega}\int_{\mathfrak{a}^*}\int_K \delta(k)\mathcal{F}f(\lambda,e)\mathrm{e}^{-(i\lambda+1)H(x^{-1}k)}|\mathbf{c}(\lambda)|^{-2}\mathrm{d}\lambda\mathrm{d}k \\ &= \frac{1}{\omega}\int_{\mathfrak{a}^*}\left(\int_K \mathrm{e}^{-(i\lambda+1)H(x^{-1}k)}\delta(k)\mathrm{d}k\right)\mathcal{F}f(\lambda,e)|\mathbf{c}(\lambda)|^{-2}\,\mathrm{d}\lambda \\ &= \frac{1}{\omega}\int_{\mathfrak{a}^*}\Phi_{\lambda,\delta}(x)\tilde{f}(\lambda)|\mathbf{c}(\lambda)|^{-2}\,\mathrm{d}\lambda. \end{aligned}$$

As the HFT (2.4) of a function $f \in \mathcal{D}^\delta(X)$ is defined entry-wise, it is clear that the Plancherel formula for Helgason Fourier transform is as follows:

$$\int_X \|f(x)\|^2\mathrm{d}x = \frac{1}{w}\int_{\mathfrak{a}^*}\int_K \|\tilde{f}(\lambda,k)\|^2\ |\mathbf{c}(\lambda)|^{-2}\mathrm{d}k\mathrm{d}\lambda. \tag{3.13}$$

Using the relation $\tilde{f}(\lambda,k) = \delta(k)\tilde{f}(\lambda)$ together with the Schur's orthogonality relation, the formula (3.12) can be deduced from (3.13).

DEFINITION 3.6

A C^∞ function ψ on $\mathfrak{a}_{\mathbb{C}}^*$, with values in $\mathrm{Hom}(V_\delta, V_\delta^M)$, is said to be of *exponential type* R if there exists a constant $R \geq 0$ such that for each $N \in \mathbb{Z}^+$,

$$\sup_{\lambda\in\mathfrak{a}_{\mathbb{C}}^*} \mathrm{e}^{-R|\mathrm{Im}\,\lambda|}(1+|\lambda|)^N\|\psi(\lambda)\| < +\infty.$$

We denote the space of C^∞ function from $\mathfrak{a}_{\mathbb{C}}^* \longrightarrow \mathrm{Hom}(V_\delta, V_\delta^M)$ of exponential type R by $\mathcal{H}^R(\mathfrak{a}_{\mathbb{C}}^*)$. Let $\mathcal{H}(\mathfrak{a}_{\mathbb{C}}^*) = \bigcup_{R>0}\mathcal{H}^R(\mathfrak{a}_{\mathbb{C}}^*)$. We state the following topological Paley–Wiener theorem for the K-types. The proof of this theorem follows from III, Theorem 5.11 of [9] and Lemma 2.1.

Theorem 3.7. *The δ-spherical transform defined in Definition* 3.1 *is a homeomorphism between the spaces $\mathcal{D}^\delta(X)$ and $\mathcal{P}_\delta(\mathfrak{a}_{\mathbb{C}}^*)$, where*

$$\mathcal{P}_\delta(\mathfrak{a}_{\mathbb{C}}^*) = \{F \in \mathcal{H}(\mathfrak{a}_{\mathbb{C}}^*)|(Q^\delta)^{-1}\cdot F \textit{is an even entire function}\}.$$

Here $Q^\delta(\lambda)$ is the polynomial in $i\lambda$ with real coefficients introduced in (3.8).

Let $\mathcal{P}_0(\mathfrak{a}_{\mathbb{C}}^*)$ denote the set of all even functions in $\mathcal{H}(\mathfrak{a}_{\mathbb{C}}^*)$, with the relative topology. Let $h \in \mathcal{P}_0(\mathfrak{a}_{\mathbb{C}}^*)$. By definition, h is a $\mathrm{Hom}(V_\delta, V_\delta^M)$-valued function. As V_δ^M is of dimension 1 so we can write $h = (h_1, \ldots, h_{d(\delta)})$, where each of h_i satisfies the following conditions:

(i) it is of exponential type,
(ii) it is entire,
(iii) it is an even function.

Let $\mathcal{D}(K\backslash X)$ and $\mathcal{D}(K\backslash X, \mathrm{Hom}(V_\delta, V_\delta^M))$ denote the left K-invariant, compactly supported, C^∞ functions on X taking values respectively in $\mathbb{C}$ and $\mathrm{Hom}(V_\delta, V_\delta^M)$. The spherical transform of $\phi \in \mathcal{D}(K\backslash X)$ is defined by $\phi \mapsto \int_X \phi(x)\varphi_\lambda(x^{-1})\mathrm{d}x$. For the class $\mathcal{D}(K\backslash X, \mathrm{Hom}(V_\delta, V_\delta^M))$ we define it entry-wise. From the Paley–Wiener theorem for the spherical transform [5], there exists one $f_i \in \mathcal{D}(K\backslash X)$ so that $h_i(\lambda) = \int_G f_i(x)\varphi_\lambda(x^{-1})\mathrm{d}x$. Therefore $\mathcal{P}_0(\mathfrak{a}_\mathbb{C}^*)$ is the image of $\mathcal{D}(K\backslash X, \mathrm{Hom}(V_\delta, V_\delta^M))$ under the spherical transform. The following lemma shows that the Paley–Wiener (PW) spaces $\mathcal{P}_\delta(\mathfrak{a}_\mathbb{C}^*)$ and $\mathcal{P}_0(\mathfrak{a}_\mathbb{C}^*)$ are homeomorphic.

Lemma 3.8 (III, *Lemma* 5.12 *of* [9]). *The mapping*

$$\psi(\lambda) \mapsto Q^\delta(\lambda)\psi(\lambda) \tag{3.14}$$

is a homeomorphism of $\mathcal{P}_0(\mathfrak{a}_\mathbb{C}^*)$ *onto* $\mathcal{P}_\delta(\mathfrak{a}_\mathbb{C}^*)$.

Lemma 3.9. *Any* $f \in \mathcal{D}^\delta(X)$ *can be written as* $f(x) = \mathbf{D}^\delta\phi(x)$, *where* $\phi \in \mathcal{D}(K\backslash X, \mathrm{Hom}(V_\delta, V_\delta^M))$ *and* $\mathbf{D}^\delta$ *is a certain constant coefficient differential operator.*

Proof. Let $f \in \mathcal{D}^\delta(X)$. Then $\tilde{f} \in \mathcal{P}_\delta(\mathfrak{a}_\mathbb{C}^*)$. Therefore by Lemma 3.8, the map $\lambda \mapsto \Phi(\lambda) = Q^\delta(\lambda)^{-1}\tilde{f}(\lambda)$ is in $\mathcal{P}_0(\mathfrak{a}_\mathbb{C}^*)$. By the PW theorem for the spherical function we get one $\phi \in \mathcal{D}(K\backslash X, \mathrm{Hom}(V_\delta, V_\delta^M))$ such that

$$\phi(x) = \frac{1}{w}\int_{\mathfrak{a}^*} \varphi_\lambda(x)\Phi(\lambda)|\mathbf{c}(\lambda)|^{-2}\,\mathrm{d}\lambda, \tag{3.15}$$

where $\varphi_\lambda(\cdot)$ is an elementary spherical function. Now applying the differential operator $\mathbf{D}^\delta$ (see 3.8)) on both sides of (3.15) we get

$$\begin{aligned}(\mathbf{D}^\delta\phi)(x) &= \frac{1}{w}\int_{\mathfrak{a}^*} \Phi_{\lambda,\delta}(x)Q^\delta(\lambda)\Phi(\lambda)|\mathbf{c}(\lambda)|^{-2}\,\mathrm{d}\lambda,\\ &= \frac{1}{w}\int_{\mathfrak{a}^*} \Phi_{\lambda,\delta}(x)\tilde{f}(\lambda)|\mathbf{c}(\lambda)|^{-2}\,\mathrm{d}\lambda,\\ &= f(x).\end{aligned} \tag{3.16}$$

□

We shall denote the Hilbert–Schmidt norm of an operator by $\|H\|$.

DEFINITION 3.10 (The L^p-Schwartz space on X)

For every $0 < p \leq 2$, $\mathbf{D}$, $\mathbf{E} \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ and $q \in \mathbb{N} \cup \{0\}$ we define a semi-norm on $f \in C^\infty(X, \mathrm{Hom}(V_\delta, V_\delta))$ by

$$\nu_{\mathbf{D},\mathbf{E},q}(f) = \sup_{x\in G} \|f(\mathbf{D}, x, \mathbf{E})\|\ \varphi_0(x)^{-2/p}(1+|x|)^q. \tag{3.17}$$

Let $S^p(X)$ be the space of all functions in $C^\infty(X, \mathrm{Hom}(V_\delta, V_\delta))$ such that $\nu_{\mathbf{D},\mathbf{E},q}(f) < \infty$ for all $\mathbf{D}, \mathbf{E} \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ and $q \in \mathbb{N} \cup \{0\}$. We topologize $S^p(X)$ by means of the seminorms $\nu_{\mathbf{D},\mathbf{E},q}, \mathbf{D}, \mathbf{E} \in \mathcal{U}(\mathfrak{g}_\mathbb{C}), q \in \mathbb{N} \cup \{0\}$.

Then $S^p(X)$ is a Frechet space and $\mathcal{D}(X, \mathrm{Hom}(V_\delta, V_\delta))$ is a dense subspace of $S^p(X)$. Let $S^p_\delta(X)$ be the subspace of $S^p(X)$ consisting of the left δ type $\mathrm{Hom}(V_\delta, V_\delta)$-valued functions in $S^p(X)$. Then clearly $\mathcal{D}^\delta(X)$ is a dense subspace in $S^p_\delta(X)$.

Remark 3.11. Let $S^p(X)_{\check{\delta}}$ be the Schwartz space of scalar-valued $\check{\delta}$ type functions. Recall that $\mathcal{D}_{\check{\delta}}(X)$ is dense in $S^p(X)_{\check{\delta}}$. Therefore the homeomorphism Q defined in Lemma 2.1 between $\mathcal{D}^\delta(X)$ and $\mathcal{D}_{\check{\delta}}(X)$ extends to a homeomorphism between the corresponding Schwartz spaces $S^p_\delta(X)$ and $S^p(X)_{\check{\delta}}$.

We shall now define the Schwartz space $S_\delta(\mathfrak{a}^*_\epsilon)$ containing the Paley–Wiener space $\mathcal{P}_\delta(\mathfrak{a}^*_{\mathbb{C}})$ as follows.

DEFINITION 3.12

Let $S_\delta(\mathfrak{a}^*_\epsilon)$ be the class of functions on $\mathfrak{a}^*_\epsilon$ taking values in $\mathrm{Hom}(V_\delta, V^M_\delta)$ and satisfying the following conditions:

(1) h is analytic in the interior of the strip $\mathfrak{a}^*_\epsilon$.
(2) h extends continuously to the boundary of the strip $\mathfrak{a}^*_\epsilon$.
(3) $(Q^\delta)^{-1}$ h is even and analytic in the interior of the strip $\mathfrak{a}^*_\epsilon$.
(4) For each positive integer r and for each symmetric polynomial P on $\mathfrak{a}^*$,

$$\tau_{r,P}(h) = \sup_{\lambda\ \mathrm{Int}\ \in \mathfrak{a}^*_\epsilon} \|P(\partial\lambda)h(\lambda)\|(1+|\lambda|)^r < +\infty. \tag{3.18}$$

$P(\partial\lambda)$ is the differential operator obtained by replacing the variable λ by $\mathrm{d}/\mathrm{d}\lambda$.

The topology given by the countable family of seminorms $\tau_{r,P}$ makes $S_\delta(\mathfrak{a}^*_\epsilon)$ a Frechet space.

The condition (3.18) can also be written in the form

$$\tau_{n,t}(h) = \sup_{\lambda\in \mathrm{Int}\ \mathfrak{a}^*_\epsilon} \left\| \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^t \{(1+\lambda^2)^n h(\lambda)\} \right\| < +\infty.$$

Let $S_0(\mathfrak{a}^*_\epsilon)$ be the class of all even functions on $\mathfrak{a}^*_{\mathbb{C}}$ taking values in $\mathrm{Hom}(V_\delta, V^M_\delta)$ satisfying conditions (1), (2) and (4) of Definition 3.12. Then $S_0(\mathfrak{a}^*_\epsilon)$ becomes a Frechet space with the seminorms $\tau_{r,P}$. Clearly, $\mathcal{P}_0(\mathfrak{a}^*_{\mathbb{C}}) \subset S_0(\mathfrak{a}^*_\epsilon)$.

Lemma 3.13. *The map*

$$h(\lambda) \mapsto Q^\delta(\lambda)h(\lambda) \tag{3.19}$$

is a homeomorphism from $S_0(\mathfrak{a}^*_\epsilon)$ *onto* $S_\delta(\mathfrak{a}^*_\epsilon)$

Proof. Let $h \in S_0(\mathfrak{a}^*_\epsilon)$. Then

$$\begin{aligned}
&\sup_{\lambda\in \mathrm{Int}\ \mathfrak{a}^*_\epsilon} \left\| \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^t Q^\delta(\lambda)h(\lambda)\right\| (1+|\lambda|)^m \\
&\quad \le \sum_{t_i=0}^{t} c_i \sup_{\lambda\in \mathrm{Int}\ \mathfrak{a}^*_\epsilon} \left\| \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^{t_i} Q^\delta(\lambda)\cdot \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^{t-t_i} h(\lambda)\right\| (1+|\lambda|)^m, \\
&\quad \le \sum c^\delta_i \sup_{\lambda\in \mathrm{Int}\ \mathfrak{a}^*_\epsilon} \left\| \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^{t-t_i} h(\lambda)\right\| (1+|\lambda|)^{m_i}.
\end{aligned}$$

The constants c_i^δ and the positive integers m_i are dependent on δ. On the other hand, if $g \in S_\delta(\mathfrak{a}_\epsilon^*)$ then $\psi(\lambda) = g(\lambda)/Q^\delta(\lambda)$ satisfies the conditions (1) and (2) of Definition 3.12. As $g \in S_\delta(\mathfrak{a}_\epsilon^*)$, by (3), ψ is an even function. We need to establish (4) of Definition 3.12 to conclude $\psi \in S_0(\mathfrak{a}_\epsilon^*)$.

Let us choose a compact subset $\mathbf{C}$ of $\mathfrak{a}_\epsilon^*$ containing all the zeros of $Q^\delta(\lambda)$ in the strip $\mathfrak{a}_\epsilon^*$ such that $|Q^\delta(\lambda)| \geq \alpha$ for all $\lambda \in \mathfrak{a}_\epsilon^* \setminus \mathbf{C}$, where α is a positive constant.

$$
\begin{aligned}
&\sup_{\lambda \in \operatorname{Int} \mathfrak{a}_\epsilon^*} \left\| \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^t \psi(\lambda) \right\| (1+|\lambda|)^m \\
&\leq \sup_{\lambda \in \mathbf{C}} \left\| \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^t \frac{g(\lambda)}{Q^\delta(\lambda)} \right\| (1+|\lambda|)^m + \sup_{\lambda \in \operatorname{Int} \mathfrak{a}_\epsilon^* \setminus \mathbf{C}} \frac{\|\beta(\lambda)\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^{t_1} g(\lambda)\|}{|Q^\delta(\lambda)|^{t_2}} \\
&\leq k_1 + \frac{k_2}{\alpha} \sup_{\lambda \in \operatorname{Int} \mathfrak{a}_\epsilon^*} \left\| \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^{t_1} g(\lambda) \right\| (1+|\lambda|)^{m_1} < +\infty,
\end{aligned}
$$

where $\beta(\lambda)$ is a polynomial in λ. This concludes the proof. □

It follows from above that any h in $S_\delta(\mathfrak{a}_\epsilon^*)$ can be written as $Q^\delta(\lambda)g(\lambda)$ where $g \in S_0(\mathfrak{a}_\epsilon^*)$ and vice-versa.

Let $g = (g_1, \ldots, g_{d(\delta)}) \in S_0(\mathfrak{a}_\epsilon^*)$. Then each scalar-valued function g_i belongs to the Schwartz space $S(\mathfrak{a}_\epsilon^*)$ containing the Paley–Wiener space $\mathcal{P}(\mathfrak{a}_\mathbb{C}^*)$ under the spherical Fourier transform.

PROPOSITION 3.14

The Paley–Wiener space $\mathcal{P}_\delta(\mathfrak{a}_\mathbb{C}^*)$ is a dense subspace of $S_\delta(\mathfrak{a}_\epsilon^*)$.

Proof. We have seen in Lemma 3.13 that any $h = (h_1, \ldots, h_{d(\delta)}) \in \mathcal{P}_\delta(\mathfrak{a}_\mathbb{C}^*)$ can be written as $Q^\delta \cdot (g_1, \ldots, g_{d(\delta)})$, where each g_i belongs to the Paley–Wiener space $\mathcal{P}(\mathfrak{a}_\mathbb{C}^*)$ under the spherical Fourier transform. We recall that $\mathcal{P}(\mathfrak{a}_\mathbb{C}^*)$ is dense in $S(\mathfrak{a}_\epsilon^*)$. Let $H = (H_1, \ldots, H_{d(\delta)}) \in S_\delta(\mathfrak{a}_\epsilon^*)$. Then $H = (Q^\delta G_1, \ldots, Q^\delta G_{d(\delta)})$ where $G = (G_1, \ldots, G_{d(\delta)}) \in S_0(\mathfrak{a}_\epsilon^*)$, i.e., each $G_i \in S(\mathfrak{a}_\epsilon^*)$. Then there exists a sequence $\{g_{i_n}\}$ in $\mathcal{P}_0(\mathfrak{a}_\mathbb{C}^*)$ converging to G_i. Hence $\{Q^\delta \cdot g_{i_n}\}$ converges to $Q^\delta \cdot G_i$ in the topology of $S(\mathfrak{a}_\epsilon^*)$. Therefore, the sequence $\{Q^\delta \cdot (g_{1_n}, \ldots, g_{d(\delta)_n})\} \subset \mathcal{P}_\delta(\mathfrak{a}_\mathbb{C}^*)$ converges to $(Q^\delta G_1, \ldots, Q^\delta G_{d(\delta)})$ in $S_\delta(\mathfrak{a}_\epsilon^*)$. This completes the proof. □

4. Proof of Theorem 1.1

Lemma 4.1. *Let $f \in S_\delta^p(X)$. Then its δ-spherical transform $\tilde{f}$ is an analytic function in the interior of the strip $\mathfrak{a}_\epsilon^*$.*

Proof. For any function $f \colon X \mapsto \operatorname{Hom}(V_\delta, V_\delta^M)$, it is easy to show that $|\operatorname{tr} f(x)| \leq \|f(x)\|$ for all $x \in X$. As $f \in S_\delta^p(X)$, from (3.17), we conclude that for each $\mathbf{D}, \mathbf{E} \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ and $n \in \mathbb{Z}^+ \cup \{0\}$,

$$\sup_{x \in X} \|\operatorname{tr} f(\mathbf{D}, x, \mathbf{E})\| (1+|x|)^n \varphi_0^{-2/p}(x) < +\infty. \tag{4.1}$$

Using (4.1) and the estimate (3.4) one can show that the integral in Definition 3.1 of the δ-spherical transform converges absolutely for $\lambda \in \mathfrak{a}_\epsilon^*$.

A standard application of Morera's theorem together with Fubini's theorem shows that $\lambda \mapsto \tilde{f}(\lambda)$ is analytic in the interior of the strip $\mathfrak{a}_\epsilon^*$. □

Lemma 4.2. *For $f \in S_\delta^p(X)$ and for each $t, n \in \mathbb{Z}^+ \cup \{0\}$, there exists a positive integer m and n such that*

$$\sup_{\lambda \in \operatorname{Int} \mathfrak{a}_\epsilon^*} \left\| \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^t \{(1+\lambda^2)^n \tilde{f}(\lambda)\} \right\| \leq c \sup_{x \in X} \|\mathbf{L}^n f(x)\| (1+|x|)^m \varphi_0^{-2/p}(x),$$

where c is a positive constant.

Proof. From (3.1) we have

$$\begin{aligned} \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^t \{(1+\lambda^2)^n \tilde{f}(\lambda)\} &= \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^t \left\{ d(\delta) \int_X \operatorname{tr} f(x)(1+\lambda^2)^n \Phi_{\bar{\lambda},\delta}(x)^* \mathrm{d}x \right\} \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^t \left\{ d(\delta) \int_X \operatorname{tr} f(x)(-\mathbf{L})^n \Phi_{\bar{\lambda},\delta}(x)^* \mathrm{d}x \right\}, \end{aligned} \tag{4.2}$$

where the last equality follows from (2) of the discussion following Definition 3.1. Using integration by parts we get from above that

$$\begin{aligned} &\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^t \{(1+\lambda^2)^n \tilde{f}(\lambda)\} \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^t \left\{ d(\delta) \int_X (-\mathbf{L})^n \operatorname{tr} f(x) \Phi_{\bar{\lambda},\delta}(x)^* \mathrm{d}x \right\} \\ &= d(\delta) \int_X (-\mathbf{L})^n \operatorname{tr} f(x) \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^t \int_K \mathrm{e}^{(i\lambda-1)H(x^{-1}k)} \delta(k^{-1}) \mathrm{d}k \mathrm{d}x \\ &= d(\delta) \int_X (-\mathbf{L})^n \operatorname{tr} f(x) \int_K (iH(x^{-1}k))^t \, \mathrm{e}^{(i\lambda-1)H(x^{-1}k)} \delta(k^{-1}) \mathrm{d}k \mathrm{d}x \\ &= (i)^{t'} d(\delta) \int_X \int_K (H(x^{-1}k))^t \, \mathbf{L}^n \operatorname{tr} f(x) \, \mathrm{e}^{(i\lambda-1)H(x^{-1}k)} \delta(k^{-1}) \mathrm{d}k \mathrm{d}x \\ &= (i)^{t'} d(\delta) \int_K \int_X (H(x^{-1}k))^t \, \mathbf{L}^n \operatorname{tr} f(x) \, \mathrm{e}^{(i\lambda-1)H(x^{-1}k)} \delta(k^{-1}) \mathrm{d}x \mathrm{d}k. \end{aligned}$$

We substitute $x^{-1}k = y^{-1}$ and use $\mathbf{L} \operatorname{tr} f(y) = \operatorname{tr}(\mathbf{L} f)(y)$ to obtain

$$\begin{aligned} &\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^t \{(1+\lambda^2)^n \tilde{f}(\lambda)\} \\ &= (i)^{t'} d(\delta) \int_X \int_K (H(y^{-1}))^t \operatorname{tr}(\mathbf{L}^n f)(ky) \, \mathrm{e}^{(i\lambda-1)H(y^{-1})} \delta(k^{-1}) \mathrm{d}k \mathrm{d}y. \end{aligned}$$

Note that $\mathbf{L}^n f$ is again a function of left δ type. Therefore from above we get

$$\begin{aligned}
&\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^t \{(1+\lambda^2)^n \tilde{f}(\lambda)\} \\
&= (i)^{t'} \int_X H(y^{-1})^t \, \mathrm{e}^{(i\lambda-1)H(y^{-1})} \left\{ d(\delta) \int_K \operatorname{tr}(\mathbf{L}^n f)(ky)\delta(k^{-1})\mathrm{d}k \right\} \mathrm{d}y, \\
&= (i)^{t'} \int_X (H(y^{-1}))^t \mathbf{L}^n f(y) \, \mathrm{e}^{(i\lambda-1)H(y^{-1})} \mathrm{d}y \quad \text{(by (2.7))} \\
&= (i)^{t'} \int_X (H(y))^t \mathbf{L}^n f(y^{-1}) \, \mathrm{e}^{(i\lambda-1)H(y)} \mathrm{d}y.
\end{aligned} \tag{4.3}$$

We use the Iwasawa decomposition $G = KAN$ and write $y = ka_r n$, where $r \in \mathfrak{a}$ and $\exp r = a_r$ to obtain

$$\begin{aligned}
&\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^t \{(1+\lambda^2)^n \tilde{f}(\lambda)\} \\
&= c(i)^{t'} \int_K \int_{\mathfrak{a}} \int_N \mathbf{L}^n f(n^{-1}a_r^{-1}k^{-1})(H(ka_r n))^t \mathrm{e}^{(i\lambda-1)H(ka_r n)} \mathrm{d}k \mathrm{e}^{2r} \mathrm{d}r \mathrm{d}n \\
&= (i)^{t'} \int_{\mathfrak{a}} \int_N \mathbf{L}^n f((a_r n)^{-1}) \, r^t \mathrm{e}^{(i\lambda+1)r} \mathrm{d}r \mathrm{d}n.
\end{aligned} \tag{4.4}$$

From (4.4) we get the following norm inequality.

$$\left\| \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^t \{(1+\lambda^2)^n \tilde{f}(\lambda)\} \right\| \leq c \int_{\mathfrak{a}} \int_N \|\mathbf{L}^n f((a_r n)^{-1})\| \, |r|^t \mathrm{e}^{(|\mathrm{Im}\,\lambda|+1)r} \mathrm{d}r \mathrm{d}n. \tag{4.5}$$

As $f \in S_\delta^p(X)$, for each $m \in \mathbb{Z}^+$ we have $\|\mathbf{L}^n f((a_r n)^{-1})\| \leq \nu_{\mathbf{L}^n,m}(f)(1+|(a_r n)^{-1}|)^{-m}$ $\varphi_0^{2/p}((a_r n)^{-1})$ where ν is as defined in (3.17). Using (2.1) we get from above

$$\begin{aligned}
&\left\| \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^t \{(1+\lambda^2)^n \tilde{f}(\lambda)\} \right\| \\
&\leq c\nu_{\mathbf{L}^n,m}(f) \int_{\mathfrak{a}} \int_N (1+|(a_r n)|)^{-m} \varphi_0^{2/p}((a_r n)^{-1})(1+|r|)^t \mathrm{e}^{(|\mathrm{Im}\,\lambda|+1)r} \mathrm{d}r \mathrm{d}n \\
&\leq c_1 \nu_{\mathbf{L}^n,m}(f) \int_{\mathfrak{a}} \int_N (1+|(a_r n)|)^{-m+t} \varphi_0^{2/p}((a_r n)^{-1}) \mathrm{e}^{(|\mathrm{Im}\,\lambda|+1)H(a_r n)} \mathrm{d}r \mathrm{d}n \\
&= c_1 \nu_{\mathbf{L}^n,m}(f) \int_G (1+|x|)^{t-m} \varphi_0^{2/p}(x) \mathrm{e}^{(|\mathrm{Im}\,\lambda|-1)H(x)} \mathrm{d}x.
\end{aligned} \tag{4.6}$$

For convenience we denote $c_1 \nu_{\mathbf{L}^n,m}(f)$ by c_ν. We use the Cartan decomposition $G = K\overline{A^+}K$ and write $x = k_1 \exp|x| k_2$ and decompose the integral (4.6) as follows:

$$c_\nu \int_K \int_{\mathfrak{a}^+} \int_K (1 + |k_1 \exp|x| k_2|)^{-m+t} \varphi_0^{2/p}(\exp|x^{-1}|)$$
$$\times\, \mathrm{e}^{(|\mathrm{Im}\,\lambda|-1)H(\exp|x|k_2)} \mathrm{d}k_1 \Delta(|x|)\mathrm{d}|x|\mathrm{d}k_2$$
$$= c_\nu \int_{\mathfrak{a}^+} \int_K (1+|x|)^{-m+t} \varphi_0^{2/p}(\exp|x|)$$
$$\times\, \mathrm{e}^{(|\mathrm{Im}\,\lambda|-1)H(\exp|x|k_2)} \Delta(|x|)\mathrm{d}|x|\mathrm{d}k_2,$$

as $|x^{-1}| = |x|$ and $|k_1 \exp|x|k_2| = |x|$. Using (3.6) the expression above is

$$= c_\nu \int_{\mathfrak{a}^+} (1+|x|)^{-m+t} \varphi_0^{2/p}(\exp|x|)$$
$$\times \left\{ \int_K \mathrm{e}^{(-i(i|\mathrm{Im}\,\lambda|)-1)H(\exp|x|k_2)} \mathrm{d}k_2 \right\} \Delta(|x|)\mathrm{d}|x|$$
$$= c_\nu \int_{\mathfrak{a}^+} (1+|x|)^{-m+t} \varphi_0^{2/p}(\exp|x|) \varphi_{-i|\mathrm{Im}\,\lambda|}(\exp|x|)\ \Delta(|x|)\mathrm{d}|x|$$
$$\le c_\nu \int_{\mathfrak{a}^+} (1+|x|)^{-m+t} \varphi_0^{(2/p)+1}(\exp|x|)\ \mathrm{e}^{|\mathrm{Im}\,\lambda||x|}\ \Delta(|x|)\mathrm{d}|x|. \tag{4.7}$$

We take $\lambda \in \mathrm{Int}\,\mathfrak{a}_\epsilon^*$, i.e., $|\mathrm{Im}\,\lambda| < \epsilon = \left(\frac{2}{p} - 1\right)$. Using the estimate (3.7) we get

$$\le c_\nu \int_{\mathfrak{a}^+} (1+|x|)^{(-m+t+\frac{2}{p}-1)} \varphi_0^2(\exp|x|)\ \Delta(|x|)\mathrm{d}|x|$$
$$\le c_\nu \int_G (1+|x|)^{(-m+t+\frac{2}{p}-1)} \varphi_0^2(x)\mathrm{d}x, \quad \text{(see (2.2))}. \tag{4.8}$$

Choosing a suitably large m, we see that the integral in (4.8) converges (Lemma 11 of [8]). Hence, we conclude that

$$\sup_{\lambda \in \mathrm{Int}\,\mathfrak{a}_\epsilon^*} \left\| \left(\frac{\mathrm{d}}{\mathrm{d}\lambda} \right)^t \{(1+\lambda^2)^n \tilde{f}(\lambda)\} \right\| \le \text{const}\ \nu_{\mathbf{L}^n,m}(f). \tag{4.9}$$

This completes the proof of the lemma. □

Lemma 4.3. *The δ-spherical transform $f \mapsto \tilde{f}$ is a continuous injection of $S_\delta^p(X)$ into $S_\delta(\mathfrak{a}_\epsilon^*)$.*

Proof. From Lemma 4.1, Lemma 4.2 and (6) of the discussion below Definition 3.1, we conclude that if $f \in S_\delta^p(X)$ then $\tilde{f} \in S_\delta(\mathfrak{a}_\epsilon^*)$. Also the transform $f \mapsto \tilde{f}$ is continuous. The fact that $f \mapsto \tilde{f}$ is injective is a consequence of the Plancherel formula for the HFT (III, Theorem 1.5 of [9]). □

The next lemma is an extension of the inversion formula given in Lemma 3.5 for the Schwartz class functions.

Lemma 4.4. *Let $h \in S_\delta(\mathfrak{a}_\epsilon^*)$. Then the inversion $\mathcal{I}h$ given by*

$$\mathcal{I}h(x) = \frac{1}{\omega}\int_{\mathfrak{a}^*} \Phi_{\lambda,\delta}(x)h(\lambda)|\mathbf{c}(\lambda)|^{-2}\mathrm{d}\lambda$$

is a left δ-type C^∞ function on X taking values in $\mathrm{Hom}(V_\delta, V_\delta)$.

Proof. Let us take any derivative $\mathbf{D}$ of X. For any $\mathbf{D} \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$,

$$\mathcal{I}h(\mathbf{D}; x) = \frac{1}{\omega}\int_{\mathfrak{a}^*} \Phi_{\lambda,\delta}(\mathbf{D}; x)h(\lambda)|\mathbf{c}(\lambda)|^{-2}\mathrm{d}\lambda. \tag{4.10}$$

Therefore,

$$\begin{aligned}\|\mathcal{I}h(\mathbf{D}; x)\| &\le c\int_{\mathfrak{a}^*} \|\Phi_{\lambda,\delta}(\mathbf{D}; x)\|\|h(\lambda)\|(1+|\lambda|)^b\mathrm{d}\lambda \\ &\le c_\delta \int_{\mathfrak{a}^*}(1+|\lambda|)^{b_\mathbf{D}+b-n}\varphi_0(x)\mathrm{d}\lambda,\end{aligned}$$

(using estimate (3) of the discussion below Definition 3.1 and (3.18))

$$\le c_\delta \int_{\mathfrak{a}^*}(1+|\lambda|)^{b_\mathbf{D}+b-n}\mathrm{d}\lambda.$$

We choose n sufficiently large so that the last integral on the right-hand side exists. Hence, $\mathcal{I}h(\mathbf{D}; x)$ exists for every $\mathbf{D}$. Therefore $\mathcal{I}h$ is a C^∞ function on X. As $\Phi_{\lambda,\delta}(x)$ is of left δ type, so is $\mathcal{I}h$. □

Lemma 4.5. *If $h \in S_\delta(\mathfrak{a}_\epsilon^*)$, then $\mathcal{I}h \in S_\delta^p(X)$.*

Proof. We consider the spaces $\mathcal{P}_\delta(\mathfrak{a}_\mathbb{C}^*)$ and $\mathcal{D}^\delta(X)$ equipped with the topologies of the Schwartz spaces $S_\delta(\mathfrak{a}_\epsilon^*)$ and $S_\delta^p(X)$ respectively. It is clear from the Paley–Wiener theorem that $\mathcal{I}$ maps $\mathcal{P}_\delta(\mathfrak{a}_\mathbb{C}^*)$ onto $\mathcal{D}^\delta(X)$. We shall show that $\mathcal{I}$ is a continuous map from $\mathcal{P}_\delta(\mathfrak{a}_\mathbb{C}^*)$ onto $\mathcal{D}^\delta(X)$ in these topologies. Let $h \in \mathcal{P}_\delta(\mathfrak{a}_\mathbb{C}^*)$ and $\mathcal{I}h = f \in \mathcal{D}^\delta(X)$. We have to show that for any seminorm ν on $\mathcal{D}^\delta(X)$ there exists a seminorm τ on $P(\mathfrak{a}_\epsilon^*)$ so that

$$\nu(f) \le c_\delta \tau(h),$$

where c_δ is a constant depending only on δ.

Let $\mathbf{D} \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$ and $n \in \mathbb{Z}^+$. We consider f as a right K-invariant function on the group G. Let

$$\nu_{\mathbf{D},n}(f) = \sup_{x\in G}\|\mathbf{D}f(x)\|(1+|x|)^n\varphi_0^{-2/p}(x). \tag{4.11}$$

From Lemmas 3.8 and 3.9 we know that $f(x) = \mathbf{D}^\delta\phi(x)$, where ϕ is a K bi-invariant function on G and $h(\lambda) = Q^\delta(\lambda)\Phi(\lambda)$. Here Φ is the spherical Fourier transform of ϕ. Hence from (4.11) we have

$$\nu_{\mathbf{D},n}(f) = \sup_{x\in G}\|\mathbf{D}\mathbf{D}^\delta\phi(x)\|(1+|x|)^n\varphi_0^{-2/p}(x) = \nu_{\mathbf{D}\mathbf{D}^\delta,n}(\phi). \tag{4.12}$$

By the isomorphism of the K bi-invariant functions in the Schwartz space (see [2]), for each $\mathbf{D} \in \mathcal{U}(\mathfrak{g})$, $\mathbf{D}^{\delta} \in \mathcal{U}(\mathfrak{g})$ and for each $n \in \mathbb{Z}^{+}$ there exists $m_\delta, t \in \mathbb{Z}^{+}$ and a positive constant c_δ so that,

$$\sup_{x\in G} \|\mathbf{D}\mathbf{D}^{\delta}\phi(x)\|(1+|x|)^{n}\varphi_0^{-2/p}(x) \leq c_\delta \sup_{\lambda\in \operatorname{Int}\mathfrak{a}_\epsilon^*} \left\| \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^{t} \Phi(\lambda) \right\| (1+|\lambda|)^{m_\delta}. \tag{4.13}$$

Now by Lemma 3.13, for $t, m_\delta \in \mathbb{Z}^{+}$ there exists $t_1, m_1 \in \mathbb{Z}^{+}$ such that

$$\begin{aligned} \sup_{\lambda\in\mathfrak{a}_\epsilon^*} \left\| \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^{t} \Phi(\lambda) \right\| (1+|\lambda|)^{m} &\leq c'_\delta \sup_{\lambda\in \operatorname{Int}\mathfrak{a}_\epsilon^*} \left\| \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^{t_1} h(\lambda) \right\| (1+|\lambda|)^{m_1} \\ &= c'_\delta \tau_{t_1,m_1}(h) < +\infty. \end{aligned}$$

Hence, $\nu_{\mathbf{D},n}(f) \leq c'_\delta c_\delta \tau_{t_1,m_1}(h)$. The positive constants c_δ and c'_δ are dependent on $|\delta|$. The positive integer m_1 can be made independent of the $\delta \in \hat{K}_M$ chosen. This shows that the inversion $\mathcal{I}$ is a continuous linear transformation on a dense subset $\mathcal{P}_\delta(\mathfrak{a}_\mathbb{C}^*)$ of $S_\delta(\mathfrak{a}_\epsilon^*)$ onto $\mathcal{D}^{\delta}(X)$ (The surjectivity follows from Theorem 3.7.)

Let us now take $h \in S_\delta(\mathfrak{a}_\epsilon^*)$. As $\mathcal{P}_\delta(\mathfrak{a}_\mathbb{C}^*)$ is dense in $S_\delta(\mathfrak{a}_\epsilon^*)$, there exists a Cauchy sequence $\{h_n\} \subset \mathcal{P}_\delta(\mathfrak{a}_\mathbb{C}^*)$ converging to h. Then by what we have proved above, we can get a Cauchy sequence $\{f_n\} \subset \mathcal{D}^{\delta}(X)$ such that $\tilde{f}_n = h_n$. As $S_\delta^p(X)$ is a Frechet space, the sequence converge to some $f \in S_\delta^p(X)$. Clearly, $f = \mathcal{I}h$. This completes the proof. □

Finally, Lemmas 3.7, 4.3 and 4.5 together show that the δ-spherical transform is a surjection onto $S_\delta(\mathfrak{a}_\epsilon^*)$ and that $\mathcal{I}$: $S_\delta(\mathfrak{a}_\epsilon^*) \to S_\delta^p(X)$ is continuous. That is, the δ-spherical transform is a topological isomorphism between the spaces $S_\delta^p(X)$ and $S_\delta(\mathfrak{a}_\epsilon^*)$. This proves Theorem 1.1.

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