Proc. Indian Acad. Sci. (Math. Sci.) Vol. 117, No. 3, August 2007, pp. 301–306.

Classification of framed links in 3-manifolds

MATIJA CENCELJ, DUŠAN REPOVŠ and
MIKHAIL B SKOPENKOV*

Institute for Mathematics, Physics and Mechanics and Faculty of Education, University of Ljubljana, P.O. Box 2964, 1001 Ljubljana, Slovenia
*Department of Differential Geometry, Faculty of Mechanics and Mathematics, Moscow State University, Moscow 119992, Russia
E-mail: matija.cencelj@guest.arnes.si; dusan.repovs@guest.arnes.si; skopenkov@rambler.ru

MS received 4 November 2005

Abstract. We present a short and complete proof of the following Pontryagin theorem, whose original proof was complicated and has never been published in detail. *Let M be a connected oriented closed smooth* 3*-manifold*, $L_1(M)$ *be the set of framed links in M up to a framed cobordism*, *and* deg: $L_1(M) \to H_1(M;\mathbb{Z})$ *be the map taking a framed link to its homology class. Then for each $\alpha \in H_1(M;\mathbb{Z})$ there is a one-to-one correspondence between the set* $\deg^{-1}\alpha$ *and the group $\mathbb{Z}_{2d(\alpha)}$, where $d(\alpha)$ is the divisibility of the projection of α to the free part of $H_1(M;\mathbb{Z})$.*

Keywords. Framed link; framed cobordism; framing; normal bundle; normal Euler class; homotopy classification of maps; cohomotopy set; degree of a map; Pontryagin–Thom construction.

1. Introduction

Throughout this paper let M be a connected oriented closed smooth 3-dimensional manifold. Denote by $L_1(M)$ the set of 1-dimensional framed links in M up to framed cobordism. The main purpose of this paper is to describe the set $L_1(M)$.

This classification problem appeared in Pontryagin's investigations connected with the calculation of the homotopy groups of spheres and, in a more general situation, of cohomotopy sets. *The cohomotopy set* $\pi^2(M) = [M; S^2]$ is the set of continuous maps $M \to S^2$ up to homotopy. By the Pontryagin–Thom construction this set is in one-to-one correspondence with the set $L_1(M)$. Notice that the set of all nonzero vector fields on M up to homotopy, as well as the set of all oriented plane fields on M up to homotopy, is also in one-to-one correspondence with the set $L_1(M)$, because every orientable 3-manifold is paralellizable.

To state the main result we need the notions of *the natural orientation* on a framed link and *the degree* of a framed link, defined as follows. The link L is *naturally oriented* if for each point $x \in L$ the tangent vector of the orientation together with the two vectors of the framing gives a positive basis of M. The *degree* $\deg L$ of L is the homology class (with integral coefficients) of naturally oriented L. So we have a map

$$\deg\colon L_1(M) \to H_1(M;\mathbb{Z}).$$

The classical Hopf–Whitney theorem (1932–35) asserts that this map is always surjective.

Theorem 1 [Po]. *Let M^3 be a connected oriented closed smooth* 3*-manifold. Then for each $\alpha \in H_1(M^3;\mathbb{Z})$ there is a one-to-one correspondence between the sets* $\deg^{-1}\alpha$ *and $\mathbb{Z}_{2d(\alpha)}$, where $d(\alpha)$ is the divisibility of the projection of α to the free part of $H_1(M^3;\mathbb{Z})$.*

Example. The set of all maps $f\colon S^1 \times S^1 \times S^1 \to S^2$ is up to homotopy in a bijective correspondence with the set of all 4-tuples (p, q, r, t), where $p, q, r \in \mathbb{Z}$ are the degrees of the restrictions of f to the 2-dimensional subtori, $t \in \mathbb{Z}$, for $p = q = r = 0$, and $t \in \mathbb{Z}_{2\,\mathrm{gcd}(p,q,r)}$, otherwise.

Recall that the divisibility of zero is zero and the divisibility of a nonzero element $\alpha \in G$ is $\max\{\, d \in \mathbb{Z} \mid \exists\beta \in G\colon \alpha = \mathrm{d}\beta\,\}$. We denote $\mathbb{Z}_d = \mathbb{Z}/\mathrm{d}\mathbb{Z}$, in particular, $\mathbb{Z}_0 = \mathbb{Z}$.

In this paper we give a short and direct proof of Theorem 1 (which was stated without proof in [Po]). In fact, Theorem 1 was not even properly stated in the paper [Po] itself (the paper was written in English), but only in the abstract (written in Russian) without any indication of the proof. The statement in the abstract is not clear, so we have borrowed it from [St].

The statement from [St] asserts that there is a one-to-one correspondence between

$$\deg^{-1}\alpha \quad \text{and} \quad \frac{\mathbb{Z}}{2\alpha \cap H_2(M;\mathbb{Z})},$$

which by the Poincaré duality, is equivalent to our statement of Theorem 1. There are reasons to believe that our proof is the same as the proof which Pontryagin had in mind, but never published it going straight to the general case when M is an arbitrary polyhedron.

In the stable codimension $n \geq 4$ there is an analogous theorem describing the set of framed links in n-manifolds [RSS].

Theorem 2 [Po,St,Wu]. *Let M be a connected oriented closed smooth n-manifold, $n \geq 4$. Then the degree map* $\deg\colon L_1(M) \to H_1(M;\mathbb{Z})$ *is a bijection, if there exists $\beta \in H_2(M,\mathbb{Z})$ such that $\rho_2\beta \cdot w_2(M) = 1 \pmod 2$. If such β does not exist, then* deg *is a* 2-1 *map (that is, each $\alpha \in H_1(M;\mathbb{Z})$ has exactly two preimages).*

Here $w_2(M)$ is the Stiefel–Whitney class and $\rho_2\colon H_1(M;\mathbb{Z}) \to H_1(M;\mathbb{Z}_2)$ is the reduction modulo 2. In this paper we use an extension of the ideas of [RSS]. (Note that there is a misprint in the statement of this theorem in Theorem 1a of [RSS]).

The results stated above remain of sufficient interest up to the present – see [AK], [BP], [Du], [Go], [GG] and [Ka]. Note that Proposition 4.1 of [Go] is equivalent to our Theorem 1. However, even though [Go] gives the proof of this proposition, it is not written in detail – in the notations of our paper (see §3 below), the statement (1) that the invariant h is well-defined and the surjectivity of h are indeed verified, whereas the proof of the *injectivity*, which is not evident, is absent from [Go].

Notice also that the statements of this result in Theorem 6.2.7 of [BP] and Proposition 1 of [Du] are erroneous, because of a different definition of the number $d(\alpha)$. (In these papers $d(\alpha)$ is defined to be 0 if α is a torsion element, otherwise it is the divisibility of α in $H_1(M^3;\mathbb{Z})$. This is not equivalent to our definition.) An alternative approach to Theorem 1, by different methods, can be found in [AuKa]. A sketch of alternative proof can be found in Proposition 2.1 of [Ku]. A more general result appears in Theorem 2.4.7 of [Ka] and Proposition 7 of [GG].

In §2 we first recall a nice geometric definition of the normal Euler class and we then prove another Pontryagin's classification theorem (see Remark after Lemma 3). In §3 we prove Theorem 1.

2. Preliminaries

We are going to use the following *geometric definition* of the normal Euler class, which is equivalent to other definitions. Let M^4 be a closed oriented connected 4-manifold. Let L^2 be a connected oriented manifold immersed in M^4. Let $\nu(L)$ be the normal bundle of L. Identify L with the zero section of $\nu(L)$. Fix a natural orientation of $\nu(L)$. Take a general position section L' of $\nu(L)$. *The normal Euler class* $\bar{e}(L) = \bar{e}(\nu(L)) \in \mathbb{Z}$ is the difference between the number of positive and negative intersection points of L and L'. Further denote by $X \cap Y$ the difference between the number of positive and negative intersection points of X and Y.

We are also going to use the following *geometric definition* of the relative normal Euler class. Fix an orientation of $M^3 \times [0; 1]$. Let $L_1 \subset M \times 1$ and $L_2 \subset M \times 0$ be a pair of framed links, let $L \subset M \times [0; 1]$ be a (unframed) cobordism between them. Fix a natural orientation of L, i.e. an orientation that induces natural orientations of L_1 and L_2. Fix a natural orientation of $\nu(L)$. The first vector field of the framings of L_1 and L_2 can be considered as a section of $\partial\nu(L)$. Let L' be a general position extension of this section to a section of $\nu(L)$. *The relative normal Euler class* $\bar{e}(L) \in \mathbb{Z}$ is the difference between the number of positive and negative intersection points of L and L'. If we reverse the orientation of $M^3 \times [0; 1]$ (and, consequently, of L, because L is naturally oriented), then the sign of the integer $\bar{e}(L)$ changes.

It can be shown that the class $\bar{e}(L)$ is the complete obstruction to the extension of the framing of ∂L to a framing of L.

Lemma 3. *Let L^2 and M^4 be a pair of connected oriented manifolds* (M *may have boundary*). *Suppose that L is immersed into M. Denote by $[L] \in H_2(M; \mathbb{Z})$ the class of L. Denote by σ the difference between the number of positive and negative self-intersection points of L. Then*

$$\bar{e}(L) = [L] \cdot [L] - 2\sigma,$$

where we identify the group $H_0(M; \mathbb{Z})$ with $\mathbb{Z}$. In particular, if $M = N^3 \times I$ for some 3-*manifold N^3*, *then $\bar{e}(L) = -2\sigma$.*

Remark. In particular, this well-known lemma implies Theorem 1.2b of [RSS].

Proof of Lemma 3. Let π be the natural projection of a neighbourhood of L in $\nu(L)$ to a small neighbourhood of L in M. Take a general position section L' of $\nu(L)$ close to zero. The lemma now follows from

$$\bar{e}(L) = L \cap L' = \pi L \cap \pi L' - 2\sigma = [L] \cdot [L] - 2\sigma. \qquad \square$$

3. Proof of Theorem 1

In order to construct a bijection $h\colon \deg^{-1}\alpha \to \mathbb{Z}_{2d(\alpha)}$, fix a framed circle L_1 such that $\deg L_1 = \alpha$ (clearly, such a circle exists). Take an arbitrary framed link L_2 such that

$\deg L_2 = \alpha$. Since L_1 and L_2 are homologous, it follows that there is a (not framed) cobordism L between them. By definition, put $h(L_2) = \bar{e}(L) \bmod 2d(\alpha)$. (One can see that this is the Hopf invariant if $\alpha = 0$ and L_1 is a null framed cobordant.)

It will follow from (1) and (2) below that h is well-defined:

(1) $h(L_2)$ does not depend on the choice of L; and
(2) if L_2 and L_2' are framed cobordant, then $h(L_2) = h(L_2')$.

Let us first prove (2). Assume that $L_1 \subset M \times 1$, $L_2 \subset M \times 0$, $L_2' \subset M \times (-1)$, $L \subset M \times [0, 1]$. Let $L' \subset M \times [-1, 0]$ be a framed cobordism between L_2 and L_2'. By the geometric definition of the relative normal Euler class it follows that $\bar{e}(L \cup L') = \bar{e}(L) + \bar{e}(L')$. Since the cobordism L' is framed, it follows that $\bar{e}(L') = 0$. Thus $\bar{e}(L \cup L') = \bar{e}(L)$, and we obtain the required equality $h(L_2) = h(L_2')$.

Let us prove (1). Take another general position cobordism L' between L_1 and L_2. Assume that $L_2 \subset M \times 0$, two copies of L_1 are contained in $M \times (\pm 1)$ and $L, L' \subset M \times [0, 1]$. Let $-L' \subset M \times [-1, 0]$ be the cobordism symmetric to L' (we consider the symmetry $x \times t \to x \times (-t)$ on $M \times \mathbb{R}$). Take a general position framed circle $-L_1' \subset M$ such that $L_1 \cup L_1'$ is a framed cobordant to zero, i.e. to an empty submanifold. Denote by Δ the corresponding framed cobordism. Assume that two copies of L_1' are contained in $M \times (\pm 1)$, and $\Delta \subset [1; +\infty)$. Let $-\Delta \subset (-\infty, -1]$ be the cobordism, symmetric to Δ. Denote

$$K = (-L') \cup L \cup \Delta \cup (L_1' \times [-1, 1]) \cup (-\Delta).$$

By the geometric definition of the relative normal Euler class we obtain

$$\bar{e}(K) = \bar{e}(-L') + \bar{e}(L) + \bar{e}(\Delta) + \bar{e}(L_1' \times [-1, 1]) + \bar{e}(-\Delta).$$

Here Δ, $-\Delta$ and $L_1' \times [-1, 1]$ can be framed, so $\bar{e}(\Delta) = \bar{e}(-\Delta) = \bar{e}(L_1 \times [-1, 1]) = 0$. Since the symmetry $x \times t \to x \times (-t)$ reverses the orientation of $M \times [-1; 1]$, it follows by the geometric definition of the relative normal Euler class that $\bar{e}(-L') = -\bar{e}(L')$. Thus $\bar{e}(K) = \bar{e}(L) - \bar{e}(L')$. Now (1) follows from

$$\begin{aligned} \bar{e}(K) &= -2\sigma = 2(-L' \cup L) \cap (L_1' \times [-1, 1]) \\ &= 2K \cap (L_1'' \times \mathbb{R}) = 2[pK] \cdot \alpha = 0 \bmod 2d(\alpha). \end{aligned}$$

Here σ is the difference between the numbers of positive and negative self-intersections of K, and the first equality follows from Lemma 3. The second equality follows from the construction of K. Then, $L_1'' \subset M$ is a general position circle close to L_1' and homologic to it. By the general position L_1' and L_1'' are disjoint, so $(-L' \cup L) \cap (L_1' \times [-1, 1]) = K \cap (L_1'' \times [-1, 1])$.

Since $-\Delta$ is obtained from Δ by the symmetry $x \times t \to x \times (-t)$, it follows that $K \cap (L_1'' \times [1, +\infty)) = -K \cap (L_1'' \times (-\infty, -1])$, and the third equality follows. Denote by $p\colon M \times I \to M$ the projection. Then by the general position we obtain the fourth equality, because the homological class of L_1'' is α. The last equality follows from the definition of $d(\alpha)$. So the proof of (1) is completed.

Injectivity of h. Let L_2 and L_2' be a pair of framed 1-submanifolds such that $h(L_2) = h(L_2')$. Let us prove that L_2 and L_2' are framed cobordant. Assume that $L_2 \subset M \times 1$, $L_1 \subset M \times 0$ and $L_2' \subset M \times (-1)$. Let $L \subset M \times [0, 1]$ and $-L' \subset M \times [-1, 0]$ be the cobordisms between L_1 and L_2, L_1 and L'_2 respectively. Since $h(L_2) = h(L'_2)$, it follows that $\bar{e}(L) = -\bar{e}(-L') \mod 2d(\alpha)$. Then $\bar{e}(-L' \cup L) = 2d(\alpha)y$ for some $y \in \mathbb{Z}$.

By the Poincaré duality there exists an element $\beta \in H_2(M; \mathbb{Z})$ such that $\alpha \cap \beta = d(\alpha)$. Let $K \subset M \times 0$ be a general position connected submanifold realizing the class $y\beta$. Notice that $\bar{e}(K) = 0$ by Lemma 3. Denote by K' the connected sum of $(-L' \cup L)$ and K in $M \times [-1; 1]$. By the geometric definition of the relative normal Euler class it follows that $\bar{e}(K') = \bar{e}(-L' \cup L) + \bar{e}(K) = 2d(\alpha)y$.

The difference between the number of positive and negative self-intersection points of the manifold K' is equal to $y\beta \cap \alpha = d(\alpha)y$. Let K'' be a new cobordism between L_2 and L'_2 obtained from K' by elimination of the self-intersection points. (Here we use a move in a neighbourhood of each self-intersection point analogous to the move taking the pair of the planes $x = 0$, $y = 0$ and $z = 0$, $t = 0$ to the surface

$$\begin{cases} x(\tau, \varphi) = \tau \cos\varphi, \\ y(\tau, \varphi) = \tau \sin\varphi, \\ z(\tau, \varphi) = (1 - \tau)\cos\varphi, \\ t(\tau, \varphi) = (1 - \tau)\sin\varphi; \end{cases}$$

in $\mathbb{R}^4$ with coordinates (x, y, z, t).) By the geometric definition of the normal Euler class it can be proved easily that removing of each self-intersection point decreases $\bar{e}(K')$ by ± 2, depending on the sign of the point (since our move is local, it suffices to prove it for a closed submanifold K', and this latter case follows from Lemma 3). So $\bar{e}(K'') = \bar{e}(K') - 2d(\alpha)y = 0$. Thus K'' can be framed and hence L_2 and L'_2 are framed cobordant.

Surjectivity of h. Let us construct a sequence $L_1, L_2, \dots, L_{2d(\alpha)}$ of framed 1-submanifolds such that for $j = 1, \dots, 2d(\alpha)$ we have $h(L_j) = j - 1$. Fix a homeomorphism $L_1 \cong S^1$. Denote by $f_1(x)$ the basis vector of the fixed framing of L_1 at the point $x \in S^1$. Take a map $\varphi: S^1 \to SO(2)$ realizing the generator $\pi_1(SO(2)) \cong \mathbb{Z}$. For $j = 2, \dots, 2d(\alpha)$ define the framing f_j of L_1 by the formula $f_j(x) = \varphi^{j-1}(x) f_1(x)$. Let L_j be the submanifold L_1 with framing f_j. Without loss of generality we may assume that $h(L_2) \geq 0$.

Let us prove that $h(L_2) = 1$. Then it can be shown analogously that $h(L_j) = j - 1$. Take $L = L_1 \times I$. It suffices to construct a general position normal vector field on L extending the first field of the framing of L_1 and L_2 with a unique singular point. The normal bundle to L in $M \times \mathbb{R}$ is trivial. Identify this bundle with $\mathbb{R} \times \mathbb{R} \times L$ and denote by $p_1, p_2: \mathbb{R} \times \mathbb{R} \times L \to \mathbb{R}$ the projections to the first and the second multiples respectively. Further denote by f_2 the first vector field of the framing f_2. Clearly, $p_1 f_2(x)$, where $x \in L_2$ has exactly two zeros. Join them by an arc $A \subset L$.

Analogously, join by an arc B the pair of zeros of $p_2 f_2(x)$. Clearly, we can choose the arcs A and B intersecting transversally at a single point. Take a general position normal vector field F_1 on L extending the fields $p_1 f_2$, $p_1 f_1$ such that $p_2 F_1 = 0$, $p_1 F_1 |_A = 0$. Analogously, extend $p_2 f_2$ and $p_2 f_1$ to a normal vector field F_2 such that $p_1 F_2 = 0$, $p_2 F_2 |_B = 0$. The sum $F_1 + F_2$ with a single zero at the point $A \cap B$ is the required vector field. □

Acknowledgements

The authors would like to thank A B Skopenkov for comments and suggestions. This paper first appeared as a preprint [CRS1] in 2003. After we had posted it at the ArXives

[CRS2] in 2007, D Auckley, E Dufraine and U Kaiser kindly brought to our attention [AK], [BP], [Du], [Go], [GG] and [Ka]. The first two authors were supported in part by the Slovenian Research Agency program P1-0292-0101-04 and grants BI-RU/06-07/004,013. The third author was supported in part by Russian Foundation for Basic Research grants Nos 02-01-00014-a, 05-01-00993-a, 06-01-72551-NCNIL-a and 07-01-00648-a, INTAS grant No. 06-1000014-6277, President of the Russian Federation grant for state support of leading science schools of the Russian Federation NSh-4578.2006.1 and the Russian Education and Science Agency program RNP 2.1.1.7988.

References

[AK] Auckly D and Kapitanski L, Analysis of the Faddeev model, preprint, arXiv:math-ph/0403025

[BP] Benedetti R and Petronio C, Branched standard spines of 3-manifolds, Lect. Notes Math. 1653 (Berlin: Springer-Verlag)

[CRS1] Cencelj M, Repovš D and Skopenkov M, Classification of framed links in 3-manifolds, Preprint Series Univ. of Ljubljana **41** (2003) 906

[CRS2] Cencelj M, Repovš D and Skopenkov M, Classification of framed links in 3-manifolds, preprint, arXiv:math-gt/0705.4166v1

[Du] Dufraine E, Classes d'homotopie de champs de vecteurs Morse-Smale sans singularite sur les fibres de Seifert, *Enseign. Math. (2)* **51** (2005) 1–2, 3–30

[GG] Giroux E and Goodman N, On the stable equivalence of open books in three-manifolds, *Geom. Topol.* **10** (2006) 97–114

[Go] Gompf R, Handlebody construction of Stein surfaces, *Ann. Math. (2)* **148** (1998) 619–693

[Ka] Kaiser U, Link theory in manifolds, Lect. Notes Math. 1669 (Berlin: Springer-Verlag)

[Ku] Kuperberg G, Noninvolutary Hopf algebras and 3-manifold invariants, *Duke Math. J.* **84(1)** (1996) 83–129

[Po] Pontryagin L S, A classification of mappings of the 3-dimensional complex into the 2-dimensional sphere, *Rec. Math. (Mat. Sbornik)* **9(51)** (1941) 331–363

[RSS] Repovš D, Skopenkov M and Spaggiari F, On the Pontryagin–Steenrod–Wu theorem, *Israel J. Math.* **145** (2005) 341–348

[St] Steenrod N, Products of cocycles and extensions of mappings, *Ann. Math. (2)* **48** (1947) 290–320

[Wu] Wu W-T, On Pontrjagin classes, Part I, *Acta Math. Sinica* **3** (1953) 291–315; Part II, *Acta Math. Sinica* **4** (1954) 171–199; Part III, *Acta Math. Sinica* **4** (1954) 323–346; Part IV, *Acta Math. Sinica* **5** (1955) 37–63; Part V, *Acta Math. Sinica* **5** (1955) 401–410 (Chinese)