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Solutions for a class of iterated singular equations

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Abstract. Some fundamental solutions of radial type for a class of iterated elliptic singular equations including the iterated Euler equation are given.

Keywords. Euler equation; elliptic equation; iterated equation; radial type solutions.

1. Introduction

Consider the class of equations

$$Lu = \sum_{i=1}^{n} \left(\frac{r}{x_i}\right)^p \left[x_i^2 \frac{\partial^2 u}{\partial x_i^2} + \alpha_i x_i \frac{\partial u}{\partial x_i}\right] + \lambda u = 0, \tag{1}$$

where λ, α_i $(i = 1, 2, \ldots, n)$ are real parameters, $p\,(> 0)$ is a real constant and r is defined by

$$r^p = x_1^p + x_2^p + \cdots + x_n^p. \tag{2}$$

The domain of the operator L is the set of all real-valued functions $u(x)$ of the class $C^2(D)$, where $x = (x_1, x_2, \ldots, x_n)$ denotes points in R^n and D is the regularity domain of u in R^n. Note that (1) includes the Laplace equation and an equidimensional (Euler) equation as special cases.

In [1] and [2], Altın studied radial type solutions of a class of singular partial differential equations of even order and obtained Lord Kelvin principle for this class of equations.

In [5], all radial type solutions of eq. (1) are obtained by showing that for all solutions of the form $u = f(r^m)$, $f \in C^2$, the function f satisfies

$$f(r^m) = r^{cm},$$

where c is a root of the equation

$$m^2c^2 + m\left(-p + n(p-1) + \sum_{i=1}^{n} \alpha_i\right)c + \lambda = 0.$$

In [4, 6], Özalp and Çetinkaya obtained expansion formulas and Kelvin principle for the iterates of eq. (1). Lyakhov and Ryzhkov [3] obtained Almansi's expansions for B-polyharmonic equation i.e. obtained the solutions of the equation

$$\Delta_B^m f = 0,$$

where

$$\Delta_B = \sum_{j=1}^{n} B_j + \sum_{i=n+1}^{N} \frac{\partial^2}{\partial x_i^2}, \qquad B_j = \partial^2/\partial x_j^2 + \frac{\gamma_j}{x_j}\frac{\partial}{\partial x_j}.$$

In this paper, as a continuation of [4], we consider the class of equations

$$\left(\prod_{j=1}^{q} L_j^{k_j}\right) u = (L_1^{k_1} L_2^{k_2} \dots L_q^{k_q}) u = 0, \tag{3}$$

where $q,\ k_1, \dots, k_q$ are positive integers, $\lambda_j,\ \alpha_i^{(j)}$ $(j = 1, 2, \dots, q;\ i = 1, 2, \dots, n)$ are real constants,

$$L_j = \sum_{i=1}^{n} \left(\frac{r}{x_i}\right)^p \left[x_i^2 \frac{\partial^2}{\partial x_i^2} + \alpha_i^{(j)} x_i \frac{\partial}{\partial x_i}\right] + \lambda_j$$

and the operator $L_j^{k_j}$ denotes, as usual, the successive applications of the operator L_j onto itself, that is $L_j^{k_j} u = L_j(L_j^{k_j-1} u)$.

2. Solutions for the iterated equation

We first give some properties of the operator L_j (see [4, 5]). By direct computation, it can be shown that

$$L_j(r^m) = \beta_j(m) r^m, \tag{4}$$

where

$$\beta_j(m) = [m(m + 2\phi_j) + \lambda_j] \tag{5}$$

and

$$2\phi_j = -p + n(p-1) + \sum_{i=1}^{n} \alpha_i^{(j)}. \tag{6}$$

The proof of the following lemma can be done easily by using induction argument on k_j. For a special case of the lemma, see [5].

Lemma 1. *For any real parameter m,*

$$L_j^{k_j}(r^m) = \beta_j^{k_j}(m) r^m,$$

where the integer k_j is the iteration number.

By the linearity of the operators L_j and by Lemma 1, we have the following result.

Lemma 2.

$$\left(\prod_{j=1}^{q} L_j^{k_j}\right)(r^m) = \left(\prod_{j=1}^{q} \beta_j^{k_j}(m)\right) r^m. \tag{7}$$

The following theorem, states a class of solutions for the iterated equations which is our main result.

Theorem 1. *The function defined by*

$$\begin{aligned} u = &\sum_{v\in I_1} \sum_{l=0}^{k_v-1} r^{-\phi_v} \left[A_l\, r^{\sqrt{\phi_v^2-\lambda_v}} + B_l\, r^{-\sqrt{\phi_v^2-\lambda_v}}\right] (\ln r)^l \\ &+ \sum_{v\in I_2} \sum_{l=0}^{k_v-1} r^{-\phi_v} \left[C_l \cos\left(\sqrt{\lambda_v-\phi_v^2}\,\ln r\right) + D_l \sin\left(\sqrt{\lambda_v-\phi_v^2}\,\ln r\right)\right] (\ln r)^l \\ &+ \sum_{v\in I_3} \sum_{l=0}^{2k_v-1} E_l\, r^{-\phi_v} (\ln r)^l \end{aligned} \tag{8}$$

is the r^m type solution of the iterated equation (3). *Here, A_l, B_l, C_l, D_l, E_l are arbitrary constants, ϕ_v is as given in* (6) *and we divide the index set $I = \{v = 1, 2, \ldots, q\}$ into three parts*:

$$I_1 = \{v \in I;\ \phi_v^2 - \lambda_v > 0\},$$

$$I_2 = \{v \in I;\ \phi_v^2 - \lambda_v < 0\},$$

$$I_3 = \{v \in I;\ \phi_v^2 - \lambda_v = 0\}.$$

Proof. For any $v \in I$, we can rewrite (7) as

$$\left(\prod_{j=1}^{q} L_j^{k_j}\right)(r^m) = \left(\beta_v^{k_v}(m) \prod_{\substack{j=1\\ j\neq v}}^{q} \beta_j^{k_j}(m)\right) r^m$$

or simply

$$\left(\prod_{j=1}^{q} L_j^{k_j}\right)(r^m) = \beta_v^{k_v}(m) F(m). \tag{9}$$

Here, we let $F(m) = \left(\prod_{\substack{j=1\\ j\neq v}}^{q} \beta_j^{k_j}(m)\right) r^m$. Now, since

$$\frac{\partial}{\partial m}\left(\prod_{j=1}^{q} L_j^{k_j}\right)(r^m) = \left(\prod_{j=1}^{q} L_j^{k_j}\right)\left(\frac{\partial}{\partial m} r^m\right) = \left(\prod_{j=1}^{q} L_j^{k_j}\right)(r^m \ln r),$$

by taking the derivative with respect to m on both sides of (9), we get

$$\left(\prod_{j=1}^{q} L_j^{k_j}\right)(r^m \ln r) = \frac{\partial}{\partial m}(\beta_v^{k_v}(m)F(m))$$

$$= \beta_v^{k_v-1}(m)\{k_v\beta_v'(m)F(m) + \beta_v(m)F'(m)\}$$

or simply

$$\left(\prod_{j=1}^{q} L_j^{k_j}\right)(r^m \ln r) = \beta_v^{k_v-1}(m)\Theta_1(m). \tag{10}$$

Here, we set

$$\Theta_1(m) = k_v\beta_v'(m)F(m) + \beta_v(m)F'(m).$$

Now, by taking the derivative with respect to m on both sides of (10), we obtain

$$\left(\prod_{j=1}^{q} L_j^{k_j}\right)(r^m (\ln r)^2) = \beta_v^{k_v-2}(m)\Theta_2(m),$$

where

$$\Theta_2(m) = (k_v - 1)\beta_v'(m)\Theta_1(m) + \beta_v(m)\Theta_1'(m).$$

In a similar fashion, taking the successive derivatives $k_v - 1$ times, with respect to m on both sides of (9), we finally obtain

$$\left(\prod_{j=1}^{q} L_j^{k_j}\right)(r^m (\ln r)^{k_v-1}) = \beta_v(m)\Theta_{k_v-1}(m). \tag{11}$$

Here,

$$\Theta_{k_v-1}(m) = 2\beta_v'(m)\Theta_{k_v-2}(m) + \beta_v(m)\Theta_{k_v-2}'(m).$$

Since the roots of the equation

$$\beta_v(m) = m(m + 2\phi_v) + \lambda_v = 0$$

are

$$m_v^{(1)} = -\phi_v + \sqrt{\phi_v^2 - \lambda_v}$$

and

$$m_v^{(2)} = -\phi_v - \sqrt{\phi_v^2 - \lambda_v},$$

we conclude from (11) that the functions

$$r^{m_v^{(i)}} (\ln r)^l \qquad (i = 1, 2; \quad l = 0, 1, \ldots, k_v - 1)$$

are all solutions of eq. (3). Thus, since the equation is linear, by the superposition principle, the function

$$\sum_{v=1}^{q} \sum_{l=0}^{k_v-1} \left\{ A_l \, r^{m_v^{(1)}} + B_l \, r^{m_v^{(2)}} \right\} (\ln r)^l \tag{12}$$

is also a solution of (3).

We have three cases for the roots:

Case 1. If $v \in I_1$, then $m_v^{(1)}$ and $m_v^{(2)}$ are both real. In this case, from (12), the function

$$\sum_{v \in I_1} \sum_{l=0}^{k_v-1} r^{-\phi_v} \left\{ A_l \, r^{\sqrt{\phi_v^2 - \lambda_v}} + B_l \, r^{-\sqrt{\phi_v^2 - \lambda_v}} \right\} (\ln r)^l. \tag{13}$$

is a real-valued solution of (3).

Case 2. If $v \in I_2$, then $m_v^{(1)}$ and $m_v^{(2)}$ are both complex and conjugate. In this case, from (12), the function

$$\sum_{v \in I_2} \sum_{l=0}^{k_v-1} r^{-\phi_v} \left\{ C_l \, \cos\left(\sqrt{\lambda_v - \phi_v^2} \ln r \right) + D_l \sin\left(\sqrt{\lambda_v - \phi_v^2} \ln r \right) \right\} (\ln r)^l \tag{14}$$

satisfies (3). Here, we use the Euler formula

$$\begin{aligned} r^{\pm i \sqrt{\lambda_v - \phi_v^2}} &= \mathrm{e}^{\pm i \sqrt{\lambda_v - \phi_v^2} \ln r} \\ &= \left[\cos\left(\sqrt{\lambda_v - \phi_v^2} \ln r \right) \pm i \sin\left(\sqrt{\lambda_v - \phi_v^2} \ln r \right) \right], \end{aligned}$$

and $C_l = A_l + B_l$, $D_l = i(A_l - B_l)$ and $i = \sqrt{-1}$ as usual.

Case 3. Finally, if $v \in I_3$, then $m_v^{(1)} = m_v^{(2)} = -\phi_v$ is a multiple root. Thus, from (12), the function

$$\sum_{v \in I_3} \sum_{l=0}^{k_v-1} \{ E_l \, r^{m_v^{(1)}} \} (\ln r)^l$$

is a solution of (3). Now, from (9), since we have

$$\left(\prod_{j=1}^{q} L_j^{k_j} \right) (r^m) = \beta_v^{k_v}(m) F(m) = (m - m_v^{(1)})^{2k_v} F(m),$$

by taking the derivatives $2k_v - 1$ times, with respect to m, on both sides of the above equality and letting $m = m_v^{(1)}$, we obtain

$$\left(\prod_{v\in I_3} L_v^{k_v}\right)(r^{m_v^{(1)}}(\ln(r))^l) = 0, \quad l = 0, 1, \ldots, 2k_v - 1.$$

Hence, we conclude that the function

$$\sum_{v\in I_3}\sum_{l=0}^{2k_v-1} E_l\, r^{-\phi_v}(\ln r)^l \tag{15}$$

satisfies (3).

Summing up the above three cases with the superposition principle we get (8), which proves the theorem.

3. General solution for the iterated Euler equations

In this section, we state the general solution of the iterated Euler equations. In [5], for the Euler equation

$$Eu = x^2\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \alpha x\frac{\mathrm{d}u}{\mathrm{d}x} + \lambda u = 0,$$

the general solutions for the iterated equations $E^k u = 0$ are given for any integer k, where α and λ are arbitrary constants. Now consider the Euler equations

$$E_v u = x^2\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \alpha_v x\frac{\mathrm{d}u}{\mathrm{d}x} + \lambda_v u = 0,$$

where α_v and λ_v $(v = 1, 2, \ldots, q)$ are arbitrary constants.

The following result gives the general solutions of the iterated Euler equations.

Theorem 2. *The general solution of the iterated Euler equations*

$$\left(\prod_{v=1}^{q} E_v^{k_v}\right)u = (E_1^{k_1}E_2^{k_2}\ldots E_q^{k_q})u = 0$$

is

$$u = \sum_{v\in I_1}\sum_{l=0}^{k_v-1} x^{-\phi_v}\left[A_l\, x^{\sqrt{\phi_v^2-\lambda_v}} + B_l\, r^{-\sqrt{\phi_v^2-\lambda_v}}\right](\ln x)^l$$
$$+\sum_{v\in I_2}\sum_{l=0}^{k_v-1} x^{-\phi_v}\left[C_l\cos\left(\sqrt{\lambda_v-\phi_v^2}\ln x\right)+D_l\sin\left(\sqrt{\lambda_v-\phi_v^2}\ln x\right)\right](\ln x)^l$$
$$+\sum_{v\in I_3}\sum_{l=0}^{2k_v-1} E_l\, x^{-\phi_v}(\ln x)^l.$$

Proof. In Theorem 1, by letting $n = 1$, and hence letting $r = x_1 = x$, $\alpha_1^{(v)} = \alpha_v$, we obtain the result for $\phi_v = \frac{1}{2}(-1+\alpha_v)$.

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