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Law of iterated logarithm for NA sequences with non-identical distributions

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Abstract. Based on a law of the iterated logarithm for independent random variables sequences, an iterated logarithm theorem for NA sequences with non-identical distributions is obtained. The proof is based on a Kolmogrov-type exponential inequality.

Keywords. Law of iterated logarithm; non-identical distribution; NA sequences.

1. Introduction

There are many results about law of iterated logarithm for independent random variables sequences. The following theorem is one of them.

Theorem A (Theorem 7.2 of [6]). *Let* $\{X_n, n \geq 1\}$ *be independent random variables sequences*, $EX_n = 0, \sigma_n^2 = EX_n^2 < \infty, B_n' = \sum_{k=1}^n \sigma_k^2, S_n = \sum_{k=1}^n X_k$, $\triangle_n = \sup_x |P(S_n < x\sqrt{B_n'}) - \Phi(x)|$, *where* $\Phi(x)$ *is a standard normal distribution function. If*

(i) $B_n' \longrightarrow \infty$, *when* $n \longrightarrow \infty$,
(ii) $B_{n+1}'/B_n' \longrightarrow 1$, *when* $n \longrightarrow \infty$,
(iii) $\triangle_n = O[(\log B_n')^{-1-\delta}], \delta > 0$,

hold, *then*

$$\limsup_{n\longrightarrow\infty} \frac{S_n}{(2B_n' \operatorname{loglog} B_n')^{1/2}} = 1 \text{ a.s.}$$

As for negatively associated (NA) random variables, Joag [2] gave the following definition.

DEFINITION [2]

A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets T_1 and T_2 of $\{1, 2, \ldots, n\}$, we have

$$\operatorname{Cov}(f_1(X_i, i \in T_1), f_2(X_j, j \in T_2)) \leq 0,$$

whenever f_1 and f_2 are coordinatewise increasing and the covariance exists. An infinite family is negatively associated if every finite subfamily is negatively associated.

Recently, some authors focused on the problem of limiting behavior of partial sums of NA sequences. Su *et al* [9] derived some moment inequalities of partial sums and a weak convergence for a strong stationary NA sequence. Lin [5] set up an invariance principal for NA sequences. Su and Qin [10] also studied some limiting results for NA sequences. More recently, Liang [3, 4] considered some complete convergence for weighted sums of NA sequences. Those results, especially some moment inequality by Huang and Xu [1], Shao [7] and Yang [11], undoubtedly propose important theory guide in further apply for the NA sequence. Shao [8] obtained a law of iterated logarithm for NA sequences with identical distributions. Zhang [12] got a law of iterated logarithm for NA vector with identical distributions.

Based on a law of the iterated logarithm for independent random variables sequences, the main purpose of this paper is to establish an iterated logarithm theorem for NA sequences with non-identical distributions. The proof is based on a Kolmogrov type exponential inequality. Hence the following theorem.

Theorem. *Let* $\{X_n, n \geq 1\}$ *be NA random variables sequences*, $EX_n = 0$, $\sup_{n\geq 1} EX_n^2(\log|X_n|)^{1+\delta} < \infty$, *for some* $\delta > 0$. *Let* $S_n = \sum_{k=1}^n X_k$, $B_n = \operatorname{Var} S_n > 0$, $B_n' = \sum_{k=1}^n EX_k^2$, $\triangle_n = \sup_x |P(S_n < x\sqrt{B_n}) - \Phi(x)|$, *where* $\Phi(x)$ *is a standard normal distribution function. If*

(i) $B_n = O(n)$,
(ii) $B_{n+1}/B_n \longrightarrow 1$, *when* $n \longrightarrow \infty$,
(iii) $\triangle_n = O[(\log B_n)^{-1}]$,
(iv) $B_n/B_n' \longrightarrow 1$, *when* $n \longrightarrow \infty$,

hold, *then*

$$\limsup_{n\longrightarrow\infty} \frac{S_n}{(2B_n \operatorname{loglog} B_n)^{1/2}} = 1 \text{ a.s.} \tag{1}$$

2. The proof of theorem

Throughout this paper, C will represent a positive constant though its value may change from one appearance to the next, and $a_n = O(b_n)$ will mean $a_n \leq Cb_n$, and $a_n \ll b_n$ will mean $a_n = O(b_n)$.

In order to prove our results, we need the following lemma.

Lemma 1[7]. *Let* $\{X_n, n \geq 1\}$ *be NA random variables sequences*, $EX_n = 0$, $EX_n^2 < \infty$, $B_n = \sum_{k=1}^n EX_k^2$, $S_n = \sum_{k=1}^n X_k$. *Then for all* $x > 0$, $a > 0$ *and* $0 < \alpha < 1$, *we have*

$$P\Big(\max_{1\leq i\leq n} |S_i| \geq x\Big) \leq 2P\Big(\max_{1\leq i\leq n} |X_i| > a\Big) + \frac{2}{1-\alpha} \exp\left\{-\frac{x^2\alpha}{2(ax+B_n)}\right\}, \quad n \geq 1.$$

Proof of Theorem. For all $i \geq 1$, let $X_i^{(n)} = X_i I(|X_i| \leq a_i) + a_i I(X_i > a_i) - a_i I(X_i < -a_i)$, $\forall\varepsilon \in \big(0, \frac{1}{20}\big)$, where $a_i = \varepsilon\big(\frac{B_i}{\operatorname{loglog} B_i}\big)^{\frac{1}{2}}$. Let $S_{n,1} = \sum_{i=1}^n (X_i^{(n)} - EX_i^{(n)})$, $S_{n,2} = \sum_{i=1}^n [X_i - X_i^{(n)} - E(X_i - X_i^{(n)})]$. By $\sup_{n\geq 1} EX_n^2(\log|X_n|)^{1+\delta} < \infty$ and $B_n = O(n)$, we have

$$
\begin{aligned}
&\sum_{i=1}^{\infty} E|X_i|I(|X_i|>a_i)/(B_i \operatorname{loglog} B_i)^{1/2} \\
&\quad \le \sum_{i=1}^{\infty} EX_i^2(\log|X_i|)^{1+\delta} I(|X_i|>a_i)\frac{a_i^{-1}(\log a_i)^{-1-\delta}}{(B_i \operatorname{loglog} B_i)^{\frac{1}{2}}} \\
&\quad \ll \sum_{i=1}^{\infty} \frac{1}{(B_i \operatorname{loglog} B_i)^{1/2}\cdot\left(\frac{B_i}{\operatorname{loglog} B_i}\right)^{1/2}}\cdot\left[\log\left(\frac{B_i}{\operatorname{loglog} B_i}\right)^{1/2}\right]^{-1-\delta} \\
&\quad \ll \sum_{i=1}^{\infty} \frac{1}{i(\log i)^{1+\frac{\delta}{2}}} < \infty.
\end{aligned}
$$

By Kronecker lemma, we have

$$
\frac{\sum_{i=1}^{\infty} E|X_i|I(|X_i|>a_i) + \sum_{i=1}^{\infty}|X_i|I(|X_i|>a_i)}{(B_n \operatorname{loglog} B_n)^{1/2}} \longrightarrow 0 \text{ a.s.}
$$

Then we have

$$
\frac{S_{n,2}}{(B_n \operatorname{loglog} B_n)^{1/2}} \longrightarrow 0 \text{ a.s.} \tag{2}
$$

By Lemma 1 and $\alpha = 1-\varepsilon$, $a = 2a_n$, we have

$$
\begin{aligned}
&P\Big(\max_{1\le i\le n}|S_{i,1}| > (1+4\varepsilon)(2B_n \operatorname{loglog} B_n)^{1/2}\Big) \\
&\quad \le \frac{2}{\varepsilon}\exp\left(-\frac{(1+4\varepsilon)^2(2B_n \operatorname{loglog} B_n)(1-\varepsilon)}{4a_n(1+4\varepsilon)(2B_n \operatorname{loglog} B_n)^{1/2}+2B_n'}\right).
\end{aligned}
$$

So we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n}P\Big(\max_{1\le i\le n}|S_{i,1}| > (1+4\varepsilon)(2B_n \operatorname{loglog} B_n)^{1/2}\Big) \\
&\quad \le \sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{2}{\varepsilon}\exp\left(-\frac{(1+4\varepsilon)^2(2B_n \operatorname{loglog} B_n)(1-\varepsilon)}{4a_n(1+4\varepsilon)(2B_n \operatorname{loglog} B_n)^{1/2}+2B_n'}\right) \\
&\quad \ll \sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{2}{\varepsilon}\exp\left(-\frac{(1+4\varepsilon)^2(1-\varepsilon)\operatorname{loglog} B_n}{2\sqrt{2}\varepsilon(1+4\varepsilon)+1+\varepsilon}\right) \\
&\quad \le \sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{2}{\varepsilon}\exp(-(1+\varepsilon)\operatorname{loglog} B_n)
\end{aligned}
$$

$$\ll \sum_{n=1}^{\infty} \frac{1}{n} \exp(-(1+\varepsilon) \operatorname{loglog} n)$$
$$= \sum_{n=1}^{\infty} \frac{1}{n(\log n)^{1+\varepsilon}} < \infty. \tag{3}$$

By (3), for all $\varepsilon > 0$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n} P\Big(\max_{1 \le i \le n} |S_{i,1}| > (1+4\varepsilon)(2B_n \operatorname{loglog} B_n)^{1/2} \Big) < \infty.$$

Then, we have

$$\sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} (2^{k+1}-1)^{-1} P\Big(\max_{1 \le j \le 2^k} |S_{j,1}| > (1+4\varepsilon)(2B_{2^{k+1}-1} \operatorname{loglog} B_{2^{k+1}-1})^{1/2} \Big) < \infty.$$

Then

$$\sum_{k=0}^{\infty} P\Big(\max_{1 \le j \le 2^k} |S_{j,1}| > (1+4\varepsilon)(2B_{2^{k+1}-1} \operatorname{loglog} B_{2^{k+1}-1})^{1/2} \Big) < \infty.$$

So, we have

$$\frac{\max_{1 \le j \le 2^k} |S_{j,1}|}{(2B_{2^{k+1}-1} \operatorname{loglog} B_{2^{k+1}-1})^{1/2}} \le 1 + 4\varepsilon \text{ a.s.}$$

Notice that for any positive integer n there exists a non-negative integer k_0, such that $2^{k_0} \le n < 2^{k_0+1}$. Then, we have

$$\frac{|S_{n,1}|}{(2B_n \operatorname{loglog} B_n)^{1/2}} \le \frac{\max_{1 \le j \le 2^{k_0+1}} |S_{j,1}|}{(2B_{2^{k_0+1}-1} \operatorname{loglog} B_{2^{k_0+1}-1})^{1/2}} \le 1 + 4\varepsilon \text{ a.s.}$$

So, we have

$$\limsup_{n \longrightarrow \infty} \frac{|S_{n,1}|}{(2B_n \operatorname{loglog} B_n)^{1/2}} \le 1 + 4\varepsilon \text{ a.s.} \tag{4}$$

By $S_n = S_{n,1} + S_{n,2}$, (2) and (4),

$$\limsup_{n \longrightarrow \infty} \frac{S_n}{(2B_n \operatorname{loglog} B_n)^{1/2}} \le 1 + 5\varepsilon \text{ a.s.} \tag{5}$$

Next, we prove the following:

$$\limsup_{n \longrightarrow \infty} \frac{S_n}{(2B_n \operatorname{loglog} B_n)^{1/2}} > 1 - \varepsilon \text{ a.s.} \tag{6}$$

By Theorem 1(iii), we have $\triangle_n = \sup_x |1 - \Phi(x) - P(S_n > x\sqrt{B_n})|$. Then

$$\begin{aligned} &P(S_n > (1-\varepsilon)(2B_n \operatorname{loglog} B_n)^{1/2}) \\ &\quad \geq 1 - \phi((1-\varepsilon)(2\operatorname{loglog} B_n)^{1/2}) - \triangle_n \\ &\quad \geq C \frac{1}{\sqrt{2\pi}(1-\varepsilon)(2\operatorname{loglog} B_n)^{1/2}} \\ &\qquad \times \exp\left(-\frac{(1-\varepsilon)^2(2\operatorname{loglog} B_n)}{2}\right) - C(\log B_n)^{-1} \\ &\quad \geq \frac{C}{(\log B_n)^{(1-\varepsilon)^2}(\operatorname{loglog} B_n)^{1/2}}. \end{aligned} \tag{7}$$

For all $\tau > 0$, there exists non-decreasing positive integers sequence $\{n_k\}$. We have $n_k \longrightarrow \infty$ when $k \longrightarrow \infty$, and

$$B_{n_k - 1} \leq (1+\tau)^k < B_{n_k}, \quad k = 1, 2, \ldots. \tag{8}$$

Let $\psi(n_k) = (2(B_{n_k} - B_{n_{k-1}}) \operatorname{loglog}(B_{n_k} - B_{n_{k-1}}))^{1/2}$, for all $r \in (0, 1)$. By (7), we have

$$\begin{aligned} &\sum_{k=1}^{\infty} P((S_{n_k} - S_{n_{k-1}}) > (1-r)\psi(n_k)) \\ &\quad \geq \sum_{k=1}^{\infty} \frac{C}{(\log(B_{n_k} - B_{n_{k-1}}))^{(1-r)^2}(\operatorname{loglog}(B_{n_k} - B_{n_{k-1}}))^{1/2}} \\ &\quad \geq \sum_{k=1}^{\infty} \frac{C}{k^{(1-r)^2}(\log k)^{1/2}} = \infty. \end{aligned} \tag{9}$$

By the generalized Borel–Cantelli lemma and (9), we have

$$P((S_{n_k} - S_{n_{k-1}}) > (1-r)\psi(n_k) \text{ i.o.}) = 1. \tag{10}$$

By (7), when n large enough, we have

$$|S_n| \leq 2(2B_n \operatorname{loglog} B_n)^{1/2} \text{ a.s.} \tag{11}$$

Let $\chi(n) = (2B_n \operatorname{loglog} B_n)^{1/2}$, when $k \longrightarrow \infty$,

$$(1-r)\psi(n_k) - 2\chi(n_{k-1}) \sim [(1-r)\tau^{1/2}(1+\tau)^{-1/2} - 2(1+\tau)^{-1/2}]\chi(n_k). \tag{12}$$

Because $\varepsilon \in R^+$, we can choose r and τ. Then

$$(1-r)\tau^{1/2}(1+\tau)^{-1/2} - 2(1+\tau)^{-1/2} > 1 - \varepsilon. \tag{13}$$

By (12), (13), (11) and (10), we have

$$\begin{aligned}
&P(S_{n_k} > (1-\varepsilon)\chi(n_k) \text{ i.o.}) \\
&\geq P(S_{n_k} > (1-r)\psi(n_k) - 2\chi(n_{k-1}) \text{ i.o.}) \\
&\geq P((S_{n_k} - S_{n_{k-1}}) > (1-r)\psi(n_k) \text{ i.o.}) = 1.
\end{aligned}$$

Now we complete the proof of (6). By (5) and (6), (1) holds.

Remark. Theorem 1 generalizes the Kolmogrov type law of iterated logarithm (see Theorem 7.2 of [6]) to NA sequences.

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