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Ergodic theory of amenable semigroup actions

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Abstract. In this paper, among other things, we state and prove the mean ergodic theorem for amenable semigroup algebras.

Keywords. Asymptotically invariance property; Banach algebras; locally compact semigroup; mean ergodic theorem; topologically left invariant mean; topologically right invariant mean; weakly almost periodic.

1. Introduction

Actions of amenable semigroups in connection with fixed point conditions were considered in [8]. The classical mean ergodic and pointwise erogdic theorems are concerned with the convergence of the sequence of Cesaro means [14]. The mean ergodic theorem for amenable locally compact groups was proved by Greenleaf [9, 10] (for more on mean ergodic theorem on locally compact group, the reader is referred to [10] and [14]). For the pointwise ergodic theorem, refer [14].

Our primary concern in this paper is to prove a theorem for an amenable locally compact semigroup S. Other aspects of the relationship between amenability and ergodic theory are mentioned in Corollaries 1, 2. A detailed historical account on the notion of ergodicity is given in [13].

Let S be a locally compact Hausdorff semitopological semigroup. Let $M(S)$ be the Banach algebra of all bounded, regular Borel measures on S with total variation norm and convolution as multiplication, and let $M_0(S)$ be the semigroup of all probability measures in $M(S)$. Let $M(S)^*$ be the continuous dual of $M(S)$, and 1 the linear functional in $M(S)^*$ such that $\langle 1, \mu\rangle = \mu(S)$ for all μ in $M(S)$.

Recall that on $M(S)^{**}$ we define the first Arens product by

$$\langle f\mu, \nu\rangle = \langle f, \mu * \nu\rangle, \quad \langle Ff, \mu\rangle = \langle F, f\mu\rangle, \quad \langle GF, f\rangle = \langle G, Ff\rangle,$$

where $\mu, \nu \in M(S)$, $f \in M(S)^*$ and $F, G \in M(S)^{**}$. A linear functional $M \in M(S)^{**}$ is called a *mean* if $\langle M, f\rangle \geq 0$ whenever $f \geq 0$ and if $\langle M, 1\rangle = 1$. A mean M is *topological left* (*right*) *invariant* if $\langle M, f\mu\rangle = \langle M, f\rangle$ ($\langle M, \mu f\rangle = \langle M, f\rangle$) for any $\mu \in M_0(S)$ and $f \in M(S)^*$. We know that topologically left invariant means on $M(S)^*$ have been studied by Riazi and Wong [16] and by Wong [18–20]. We shall follow [8] and [16] for definitions and terminologies not explained here.

2. Main results

Throughout the paper, S is a locally compact Hausdorff semitopological semigroup. A semigroup S is said to *act* on a set X (from the right) if there exists a mapping

$$(\xi, s) \mapsto \xi s,\ X \times S \to X$$

such that $\xi(st) = (\xi s)t$ whenever $s, t \in S$ and $\xi \in X$; S is a *transformation semigroup* on X. We say that the complex Banach space X with a mapping $(\xi, s) \mapsto \xi s$ constitutes a *right Banach S-module* if the mapping has the following properties:

(a) for each fixed $s \in S$, the mapping $\xi \mapsto \xi s$ is linear on X;
(b) there exists a positive constant k such that $\|\xi s\| \leq k\|\xi\|$ $(\xi \in X,\ s \in S)$ and, for every $\xi \in X$, the mapping $s \mapsto \xi s$ is continuous.

We define similarly a *left dual S-module structure* on X^* by putting $\langle sf, \xi\rangle = \langle f, \xi s\rangle$ whenever $s \in S$, $\xi \in X$ and $f \in X^*$. For every $f \in X^*$, $s \mapsto sf$ is a continuous mapping of S into X^* when X^* is equipped with the weak*-topology.

Let X be a right Banach S-module. We may give X a right Banach $M(S)$-module structure via a vector-valued integral. We put

$$\xi\mu = \int \xi s \mathrm{d}\mu(s),$$

for $\mu \in M(S)$, $\xi \in X$. Notice also that if $s \in S$ and $\xi \in X$, then $\xi\delta_s = \xi s$. Finally, for every $\mu \in M(S)$, $\xi \in X$ and $f \in X^*$, we define $\langle \mu, f\xi\rangle = \langle f, \xi\mu\rangle$.

PROPOSITION 1

Let $\xi \in X$ be such that $C_\xi = \overline{co}\{\xi\delta_s; s \in S\}$ is weakly compact. Then $C_\xi = \overline{\{\xi\mu; \mu \in M_0(S)\}}$.

Proof. It is clear that $C_\xi \subseteq \overline{\{\xi\mu;\ \mu \in M_0(S)\}}$. We will see that this inclusion is actually an equality. To reach a contradiction, assume that some $\xi\mu \in \{\xi\mu; \mu \in M_0(S)\}$ is not in C_ξ. Theorem 3.4(b) of [17] shows that there exist $f \in X^*$ and $\gamma \in \mathbb{R}$ such that, for every $s \in S$,

$$\mathrm{Re}\langle f, \xi s\rangle < \gamma < \mathrm{Re}\langle f, \xi\mu\rangle.$$

Therefore

$$\mathrm{Re}\langle f, \xi\mu\rangle = \int \mathrm{Re}\langle f, \xi s\rangle \mathrm{d}\mu(s) < \mathrm{Re}\langle f, \xi\mu\rangle,$$

which is contradiction. □

Let Ω be a subset of $M(S)$. We say that a net (μ_α) in $M_0(S)$ is *asymptotically Ω-invariant* if $\|\mu * \mu_\alpha - \mu_\alpha\| \to 0$ whenever $\mu \in \Omega$. Asymptotical invariance property was introduced by Day [4]. Wong [18] demonstrated that $M(S)^*$ has a topologically left invariant mean if and only if there exists a net (μ_α) in $M_0(S)$ such that $\|\mu * \mu_\alpha - \mu_\alpha\| \to 0$ whenever $\mu \in M_0(S)$. If S is a semi-foundation semigroup, the following conditions [7] are equivalent:

(a) $M(S)^*$ has a topologically left invariant mean;
(b) there is a net (μ_α) in $M_0(S)$ such that, for every compact subset K of S, $\|\mu * \mu_\alpha - \mu_\alpha\| \to 0$ uniformly over all μ in $M_0(S)$ which are supported on K.

Let $M(S)^*$ have a topologically left invariant mean, and let (μ_α) be a net in $M_0(S)$. It is easy to see that (μ_α) is asymptotically $M_0(S)$-invariant if and only if

$$\lim_\alpha \|\mu * \mu_\alpha\| = |\mu(S)|$$

whenever $\mu \in M(S)$.

Let X be a right Banach S-module. Let (μ_α) be an asymptotically $M_0(S)$-invariant in $M_0(S)$. For each α, we define $E_\alpha[\xi] \in X$ by

$$E_\alpha[\xi] = \int \xi s \mathrm{d}\mu_\alpha(s), \quad \xi \in X.$$

We now concentrate on the mean ergodic theorem for a locally compact semigroup. The following result can be referred to as 'the mean ergodic theorem' for amenable locally compact semigroups (see [10, 14, 15], for details).

Theorem 1. *Let $M(S)^*$ have a topologically right invariant mean. Suppose (μ_α) is an asymptotically $M_0(S)$-invariant in $M_0(S)$.*

(1) *If $\xi \in X$ is such that $C_\xi = \overline{co}\{\xi\delta_s;\, s \in S\}$ is weakly compact, then $(E_\alpha[\xi])$ converges to a fixed point of C_ξ that is the unique fixed point in C_ξ, and is therefore independent of the choice of (μ_α).*
(2) *Suppose that C_ξ is weakly compact for each $\xi \in X$. Let $Y = \{\xi \in X;\ \xi s = \xi \text{ for all } s \in S\}$, and let Z be the closed vector subspace of X generated by $\{\xi - \xi s;\ s \in S,\ \xi \in X\}$. Then $X = Y \bigoplus Z$ and (E_α) converges strongly to the projection of X onto Y.*
(3) *Let C_ξ be weakly compact for any $\xi \in X$. If $P\colon X \to Y$ is the projection onto Y associated with the direct sum decomposition of* (2)*, then $P(\xi) = Y \bigcap C_\xi$ for all $\xi \in X$.*

Proof.

(1) Let M be a topologically right invariant mean on $M(S)^*$. If $\mathcal{M}$ is the convex set of all means on $M(S)^*$, it is well known that $\mathcal{M}$ is weak* compact and that $M_0(S)$ is weak* dense in $\mathcal{M}$ (see [3, 11] for details). Therefore there is a net (μ_α) in $M_0(S)$ such that $\mu_\alpha \to M$ in the weak* topology. A subnet $(\xi\mu_\beta)$ of $(\xi\mu_\alpha)$ converges in the weak topology to an element ζ in the compact subset $C_\xi = \overline{\{\xi\mu;\, \mu \in M_0(S)\}}$. For every $s \in S$ and $f \in X^*$, we have

$$\begin{aligned}\langle f, \zeta s\rangle &= \langle sf, \zeta\rangle = \lim_\beta \langle sf, \xi\mu_\beta\rangle = \lim_\beta \langle \mu_\beta(sf), \xi\rangle = \lim_\beta \langle (\mu_\beta * \delta_s) f, \xi\rangle \\ &= \lim_\beta \langle \mu_\beta(\delta_s f), \xi\rangle = \lim_\beta \langle \mu_\beta, (\delta_s f)\xi\rangle = \lim_\beta \langle \mu_\beta, \delta_s(f\xi)\rangle \\ &= \langle M, \delta_s(f\xi)\rangle = \langle M, f\xi\rangle = \lim_\beta \langle \mu_\beta, f\xi\rangle \\ &= \lim_\beta \langle \mu_\beta f, \xi\rangle = \lim_\beta \langle f, \xi\mu_\beta\rangle = \langle f, \zeta\rangle.\end{aligned}$$

So $\zeta s = \zeta$ for every $s \in S$, that is, ζ is a fixed point under the action of S. If $\epsilon > 0$, we determine $\alpha_1, \dots, \alpha_m$ and $s_1, \dots, s_m \in S$ such that $\sum_{i=1}^m \alpha_i = 1$ and $\|\xi' - \zeta\| < \epsilon$, where $\xi' = \sum_{i=1}^m \alpha_i \xi s_i$. We have

$$\begin{aligned}\|E_\alpha[\xi] - \zeta\| &= \|\xi\mu_\alpha - \zeta\| \leq \|\xi\mu_\alpha - \xi'\mu_\alpha\| + \|\xi'\mu_\alpha - \zeta\| \\ &= \|\xi \sum_{i=1}^m (\alpha_i\ \mu_\alpha - \alpha_i \delta_{s_i} * \mu_\alpha)\| + \|\xi'\mu_\alpha - \zeta\mu_\alpha\| \\ &\leq k\|\xi\| \sum_{i=1}^m \alpha_i \|\mu_\alpha - \delta_{s_i} * \mu_\alpha\| + k\|\xi' - \zeta\| \\ &< k\|\xi\| \sum_{i=1}^m \|\mu_\alpha - \delta_{s_i} * \mu_\alpha\| + k\epsilon.\end{aligned}$$

Hence

$$\lim_\alpha \|E_\alpha[\xi] - \zeta\| = 0.$$

For every $\mu \in M_0(S)$, we have $E_\alpha[\xi\mu] = \xi\mu * \mu_\alpha$. Hence

$$\begin{aligned}\lim_\alpha \|E_\alpha[\xi\mu] - \zeta\| &= \lim_\alpha \|\xi\mu * \mu_\alpha - \xi\mu_\alpha\| \\ &\leq k \lim_\alpha \|\xi\|\|\mu * \mu_\alpha - \mu_\alpha\| = 0\end{aligned}$$

and $\lim_\alpha \|E_\alpha[\varsigma] - \zeta\| = 0$ for every $\varsigma \in C_\xi$. Now if there exists another fixed point ζ' in C_ξ, then for every α we would have

$$E_\alpha[\zeta'] = \int \zeta' s \mathrm{d}\mu_\alpha(s) = \zeta'$$

and

$$\zeta' = \lim_\alpha E_\alpha[\zeta'] = \zeta.$$

This shows that $(E_\alpha[\xi])$ converges to the unique fixed point ζ in C_ξ. Now if there exists another asymptotically $M_0(S)$-invariant (ν_β) in $M_0(S)$, it is easy to see that $\lim_\beta E_\beta[\xi] = \zeta$.

(2) Define an operator $P\colon X \to X$ by $P(\xi) = \lim_\alpha E_\alpha[\xi]$. Since X is a right S-module, it is easy to see that $\|P\| \leq k$. For every $\xi \in X$, $P(\xi) \in Y$ and $P(P(\xi)) = \lim_\alpha P(\xi)\mu_\alpha = P(\xi)$, $P^2 = P$. If $\xi \in Y$, then $P(\xi) = \lim_\alpha E_\alpha[\xi] = \xi \in Y$. We conclude that Y is the closed subspace $P(X)$ of X. Let $\xi \in X$ and $s \in S$. We have

$$\begin{aligned}\|P(\xi - \xi s)\| &= \|\lim_\alpha E_\alpha[\xi - \xi s]\| = \lim_\alpha \|\xi\mu_\alpha - \xi\delta_s * \mu_\alpha\| \\ &\leq k\|\xi\| \lim_\alpha \|\delta_s * \mu_\alpha - \mu_\alpha\| = 0.\end{aligned}$$

So $\xi - \xi\delta_s \in Y^\perp$. Conversely if $\zeta \in Y^\perp$, $P(\zeta) = 0$ is the unique fixed point in C_ζ and, for every $\epsilon > 0$, there exist $\alpha_1, \dots, \alpha_m$ and $s_1, \dots, s_m$ in S such that $\sum_{i=1}^m \alpha_i = 1$ and $\|\sum_{i=1}^m \alpha_i \zeta\delta_{s_i}\| < \epsilon$. Hence $\|\zeta - \sum_{i=1}^m \alpha_i(\zeta - \zeta\delta_{s_i})\| = \|\sum_{i=1}^m \alpha_i \zeta\delta_{s_i}\| < \epsilon$, such that $\zeta \in Z$.

(3) This is trivial. □

COROLLARY 1

Let $M(S)^*$ have a topologically right invariant mean. Suppose further that (μ_α) is an asymptotically $M_0(S)$-invariant in $M_0(S)$. Suppose that S acts from the right on a reflexive Banach space X. Then $(E_\alpha[\xi])$ converges to an element ζ in X that is S-invariant.

Proof. For every $\xi \in X$, the set $C_\xi = \overline{co}\{\xi s; s \in S\}$ is a weak*-closed subset of the closed ball of radius $k\|\xi\|$ in $(X^*)^*$. This ball is weak* compact by Alaoglu's theorem and hence C_ξ is weak compact in the reflexive space X (p. 111 of [17]). The statement is an immediate consequence of Theorem 1. □

We recall that, for a semigroup S, $M_a(S)$ is the set of all measures $\mu \in M(S)$ such that both mappings $s \mapsto |\mu| * \delta_s$ and $s \mapsto \delta_s * |\mu|$ from S into $M(S)$ are weakly continuous (see [1] and [5] for definition). A semigroup S is called a *foundation semigroup* if $\bigcup\{\operatorname{supp}\mu;\ \mu \in M_a(S)\}$ is dense in S. It is known that $M_a(S)$ admits a positive approximate identity with norm 1 [5].

A functional f in $M_a(S)^*$ is said to be *weakly almost periodic* if the set $\{f\mu;\ \mu \in M_a(S),\ \|\mu\| \leq 1\}$ is relatively weakly compact. We denote by $\mathrm{wap}(M_a(S))$ the closed subspace of $M_a(S)^*$ consisting of all the weakly almost periodic functionals in $M_a(S)^*$ (for more on weakly almost periodic functionals, the reader is referred to [2] and [12]).

COROLLARY 2

Let S be a foundation, locally compact Hausdorff topological semigroup with identity. Let $M(S)^*$ have a topologically right invariant mean. Suppose (μ_α) is an asymptotically $M_0(S)$-invariant in $M_0(S)$.

(1) *If $f \in \mathrm{wap}(M_a(S))$, then $E_\alpha[f]$ converges to an element f' in $\mathrm{wap}(M_a(S))$ that is S-invariant, that is, $f's = s$ for all $s \in S$; f' is independent of the choice of (μ_α).*
(2) *Let M be the topologically left invariant mean on $\mathrm{wap}(M_a(S))$. Then $Y = \mathbb{C}1$ and*

$$Z = \{f \in \mathrm{wap}(M_a(S));\ \langle M, f\rangle = 0\}.$$

Note that if (μ_α) is an asymptotically $M_0(S)$-invariant in $M_0(S)$, then

$$\|\mu * \mu_\alpha - \mu_\alpha\| \to 0$$

for all $\mu \in M_0(S)$. We choose $\nu \in M_a(S) \bigcap M_0(S)$ and let $\nu_\alpha = \mu_\alpha * \nu$, $\alpha \in I$. Clearly $\|\mu * \nu_\alpha - \nu_\alpha\| \to 0$ for all $\mu \in M_0(S)$. If M is a weak* cluster point of (ν_α), then M is a topologically left invariant mean on $\mathrm{wap}(M_a(S))$.

Proof.

(1) We apply Theorem 1 to the Banach space $X = \mathrm{wap}(M_a(S))$. If $f \in \mathrm{wap}(M_a(S))$ and $s \in S$, let

$$\langle fs, \mu\rangle = \langle f, \delta_s * \mu\rangle, \quad \mu \in M_a(S).$$

We check the continuity of the mapping $s \to fs$ for any $f \in \mathrm{wap}(M_a(S))$. Let $f \in \mathrm{wap}(M_a(S))$ and (e_α) be a bounded approximate identity in $M_a(S)$. It is easy to see that $fe_\alpha \to f$ in the weak* topology. By the compactness of $\{f\mu;\ \mu \in M_a(S),\ \|\mu\| \leq 1\}$, we can suppose that $fe_\alpha \to f$ in the weak topology, passing to a subnet, if necessary. Clearly $f \in M_a(S)^* M_a(S)$, since $M_a(S)^* M_a(S)$ is a Banach space and $fe_\alpha \in M_a(S)^* M_a(S)$ for all $\alpha \in I$. By Lemma 2.2 in [6], $s \mapsto fs$ is continuous. Obviously X is a right Banach S-module.

If $f \in \text{wap}(M_a(S))$, then $\{fs;\ s \in S\}$ is relatively weakly compact. So Theorem 1 asserts that, for every $f \in X$, $(E_\alpha[f])$ converges to an element f' in X that is fixed; $f's = f'$ and $\langle f', \delta_s * \mu\rangle = \langle f', \mu\rangle$ for every $s \in S$ and every $\mu \in M_a(S)$.

(2) By Theorem 1, Y is the set of all $f \in X$ such that $fs = f$ for all $s \in S$. If $f \in Y$, then $\langle f, \delta_s * \mu\rangle = \langle f, \mu\rangle$ for all $s \in S$ and $\mu \in M_a(S)$. Consequently, $f = \langle f, \delta_e\rangle 1$.

We know that Z admits a dense subset consisting of functionals of the form $g = \Sigma_1^n f_i - f_i s_i$, where $f_i \in \text{wap}(M_a(S))$ and $s_i \in S$. By the definition of M, we have $\langle M, g\rangle = 0$ and so $Z \subseteq \{f \in \text{wap}(M_a(S));\ \langle M, f\rangle = 0\}$. For the reverse inclusion, let $f \in \text{wap}(M_a(S))$ and $\langle M, f\rangle = 0$. By Theorem 1, we can write $f = f_1 + f_2$ for some $f_1 \in Y$ and $f_2 \in Z$, and so

$$\begin{aligned} 0 &= \langle M, f\rangle = \langle M, f_1\rangle + \langle M, f_2\rangle \\ &= \langle M, f_1\rangle + 0 = \langle M, f_1\rangle, \end{aligned}$$

since $Z \subseteq \{f \in \text{wap}(M_a(S));\ \langle M, f\rangle = 0\}$. Consequently $\langle M, f_1\rangle = 0$. For each $\mu \in M_0(S)$, we have

$$\begin{aligned} 0 &= \langle M, f_1\rangle = \langle M, f_1\mu\rangle = \lim_\alpha \langle \mu_\alpha, f_1\mu\rangle \\ &= \lim_\alpha \langle f_1, \mu * \mu_\alpha\rangle = \lim_\alpha \langle f_1\mu * \mu_\alpha, \delta_e\rangle \\ &= \lim_\alpha \langle f_1\mu, \delta_e\rangle = \langle f_1, \mu\rangle. \end{aligned}$$

Therefore $f_1 = 0$. It follows that $Z = \{f \in \text{wap}(M_a(S));\ \langle M, f\rangle = 0\}$. □

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