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Derivations and generating degrees in the ring of arithmetical functions

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Abstract. In this paper we study a family of derivations in the ring of arithmetical functions of several variables over an integral domain, and compute the generating degrees of the ring of arithmetical functions over the kernel of these derivations.

Keywords. Derivations; generating degrees; arithmetical functions.

1. Introduction

Cashwell and Everett [5] proved that the ring $(A, +, \cdot)$ of complex-valued arithmetical functions with Dirichlet convolution is a unique factorization domain. Yokom [10] studied the prime factorization of arithmetical functions in a certain subring of the regular convolution ring, and determined a discrete valuation subring of the unitary ring of arithmetical functions. Recently, Alkan *et al* [4] investigated a class of derivations and norms in the ring of arithmetical functions in several variables with unitary convolution. Schwab and Silberberg [8] constructed an extension of $(A, +, \cdot)$ which is a discrete valuation ring. In [9], they showed that A is a quasi-noetherian ring. Further results, in a more abstract categorical setting, have been obtained by Schwab in [6] and [7]. In [3], a class of absolute values and a family of derivations on the ring of arithmetical functions in several variables, with the analogue of Dirichlet convolution as multiplication, are defined and studied. Let R be an integral domain, r a positive integer, and $A_r(R) = \{f\colon \mathbb{N}^r \to R\}$. For any $f, g \in A_r(R)$, the convolution $f * g$ of f and g is defined by

$$(f * g)(n_1, \ldots, n_r) = \sum_{d_1 | n_1} \cdots \sum_{d_r | n_r} f(d_1, \ldots, d_r) g\left(\frac{n_1}{d_1}, \ldots, \frac{n_r}{d_r}\right).$$

The topologies obtained from the valuation constructed in [8] and its generalizations defined in [3] play an important role in our present investigation. We consider a natural family of subrings $B_{r,k,p}(R)$ of $A_r(R)$, which are closed in each of these topologies. For any $k \in \{1, \ldots, r\}$, and any prime number p, $B_{r,k,p}(R)$ consists of all the functions $f \in A_r(R)$ with the property that for all $n_1, \ldots, n_r \in \mathbb{N}$ with p dividing n_k, one has $f(n_1, \ldots, n_r) = 0$. In order to understand the algebraic and topological structure of the extensions $A_r(R)/B_{r,k,p}(R)$, we proceed to find the generating degrees of $A_r(R)$ with respect to each of the subrings $B_{r,k,p}(R)$. For two commutative topological rings $A \subseteq B$, a subset $M \subseteq B$ is said to be a (topologically) generating set of B over A provided that the ring $A[M]$ is dense in B. Suppose B has a finite generating set M over A. Then the smallest nonnegative integer m for which there exists a generating set M of B over A having exactly m elements, is called the generating degree of B over A, and is denoted by $\mathrm{gdeg}(B/A)$ [2]. Thus A is dense in B if and only if $\mathrm{gdeg}(B/A) = 0$. As an example of a nonzero, finite generating degree, if $\mathbb{C}_p$ denotes the topological closure of the algebraic closure of the field $\mathbb{Q}_p$ of p-adic numbers, then $\mathrm{gdeg}(\mathbb{C}_p/\mathbb{Q}_p) = 1$ [1, 2]. If B has no finite generating set over A, one writes $\mathrm{gdeg}(B/A) = \infty$. For some general connections between generating degrees, continuous derivations, and topological Krull dimensions, the reader is referred to [2]. In the present paper we are also lead to consider a family of continuous derivations $\theta_{k,p}$: $A_r(R) \to A_r(R)$, and obtain the subrings $B_{r,k,p}(R)$ as kernels of these derivations. We also make use of a topological isomorphism that generalizes the one constructed in [5]. We find that the generating degree of $A_r(R)$ over each of the subrings $B_{r,k,p}(R)$ is equal to 1.

2. Absolute values

Let r be a positive integer, R be an integral domain with identity 1_R, and $A_r(R) = \{f\colon \mathbb{N}^r \to R\}$. Note that R has a natural embedding in the ring $A_r(R)$, and $A_r(R)$ with addition and convolution defined as in the Introduction naturally becomes an R-algebra. We now recall the construction from [3] of a class of absolute values on $A_r(R)$. Fix $\underline{t} = (t_1, \ldots, t_r) \in \mathbb{R}^r$ with $t_1, \ldots, t_r$ linearly independent over $\mathbb{Q}$, and $t_i > 0$, $(i = 1, 2, \ldots, r)$. Given $n \in \mathbb{N}$, denote by $\Omega(n)$ the total number of prime factors of n, counting multiplicities. Thus, if $n = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$ is the prime factorization of n, then $\Omega(n) = \alpha_1 + \cdots + \alpha_k$. Define also $\Omega_r\colon \mathbb{N}^r \to \mathbb{N}^r$ by

$$\Omega_r(n_1, \ldots, n_r) = (\Omega(n_1), \ldots, \Omega(n_r)).$$

For any $f \in A_r(R)$ denote $\mathrm{supp}(f) = \{\underline{n} \in \mathbb{N}^r \,|\, f(\underline{n}) \neq 0\}$, and let

$$V_{\underline{t}}(f) = \min_{\underline{n} \in \mathrm{supp}(f)} \underline{t} \cdot \Omega_r(\underline{n}),$$

with the convention $\min(\emptyset) = \infty$, where $\underline{t} \cdot \Omega_r(\underline{n}) = t_1\Omega(n_1) + \cdots + t_r\Omega(n_r)$. It is shown in [3] that for any $f, g \in A_r(R)$ one has

$$V_{\underline{t}}(f + g) \geq \min(\{V_{\underline{t}}(f), V_{\underline{t}}(g)\})$$

and

$$V_{\underline{t}}(f * g) = V_{\underline{t}}(f) + V_{\underline{t}}(g).$$

Next, one extends the valuation $V_{\underline{t}}$ to a valuation $\bar{V}_{\underline{t}}$ on the field of fractions $\mathbb{K} = \{\frac{f}{g} \,|\, f, g \in A_r(R), g \neq 0\}$ of $A_r(R)$ by letting $V_{\underline{t}}(\frac{f}{g}) = V_{\underline{t}}(f) - V_{\underline{t}}(g)$. Further, to obtain an absolute value on $\mathbb{K}$, fix a real number $\rho \in (0, 1)$ and define $|\cdot| = |\cdot|_{\underline{t}}\colon \mathbb{K} \to \mathbb{R}$ by

$$|x|_{\underline{t}} = \rho^{\bar{V}_{\underline{t}}(x)} \text{ if } x \neq 0, \quad \text{and} \quad |x|_{\underline{t}} = 0 \text{ if } x = 0.$$

The above absolute value is nonarchimedean. In [3] it is proved that for any integral domain R, any positive integer r, and any $\underline{t}$ as above, $A_r(R)$ is complete with respect to the absolute value $|\cdot|_{\underline{t}}$.

3. Derivations

For each integer i with $1 \leq i \leq r$ and each prime number p we define a derivation $\theta_{i,p} = \theta_{r,i,p}(R)$ on $A_r(R)$ as follows. For $n \in \mathbb{N}$, let $v_p(n)$ denote the largest integer m for which p^m divides n. Given an $f \in A_r(R)$, define $\theta_{i,p} f \in A_r(R)$ by

$$\theta_{i,p} f(n_1, \ldots, n_r) = (1 + v_p(n_i)) f(n_1, \ldots, n_{i-1}, pn_i, n_{i+1}, \ldots, n_r),$$

for all $(n_1, \ldots, n_r) \in \mathbb{N}^r$.

Lemma 1. *For any* $f, g \in A_r(R)$,

$$\theta_{i,p}(f + g) = \theta_{i,p} f + \theta_{i,p} g, \tag{3.1}$$

and

$$\theta_{i,p}(f * g) = \theta_{i,p} f * g + f * \theta_{i,p} g. \tag{3.2}$$

Proof. The equality (3.1) is clear from the definitions. In order to prove (3.2), fix $f, g \in A_r(R)$ and let $(n_1, \ldots, n_r) \in \mathbb{N}^r$. Write n_i as $n_i = p^k m$, with m, p relatively prime. Then,

$$\begin{aligned}
&\theta_{i,p}(f * g)(n_1, \ldots, n_r)\\
&\quad = (1 + v_p(n_i))(f * g)(n_1, \ldots, n_{i-1}, pn_i, n_{i+1}, \ldots, n_r)\\
&\quad = (k+1) \sum_{d_1 | n_1} \cdots \sum_{d_{i-1} | n_{i-1}} \sum_{d_i | p^{k+1} m} \sum_{d_{i+1} | n_{i+1}} \cdots \sum_{d_r | n_r}\\
&\qquad f(d_1, \ldots, d_{i-1}, d_i, d_{i+1}, \ldots, d_r) g\left(\frac{n_1}{d_1}, \ldots, \frac{n_{i-1}}{d_{i-1}}, \frac{p^{k+1} m}{d_i}, \ldots, \frac{n_r}{d_r}\right)\\
&\quad = \sum_{d_1 | n_1} \cdots \sum_{d_{i-1} | n_{i-1}} \sum_{d_{i+1} | n_{i+1}} \cdots \sum_{d_r | n_r} \sum_{d_i | p^{k+1} m}\\
&\qquad (k+1) f(d_1, \ldots, d_{i-1}, d_i, d_{i+1}, \ldots, d_r) g\left(\frac{n_1}{d_1}, \ldots, \frac{n_{i-1}}{d_{i-1}}, \frac{p^{k+1} m}{d_i}, \ldots, \frac{n_r}{d_r}\right).
\end{aligned}$$

The innermost sum above equals

$$\begin{aligned}
&(k+1) \sum_{d_i | m} f(d_1, \ldots, d_{i-1}, d_i, d_{i+1}, \ldots, d_r) g\left(\frac{n_1}{d_1}, \ldots, \frac{n_{i-1}}{d_{i-1}}, \frac{p^{k+1} m}{d_i}, \ldots, \frac{n_r}{d_r}\right)\\
&\quad + (k+1) \sum_{d_i | m} f(d_1, \ldots, d_{i-1}, pd_i, d_{i+1}, \ldots, d_r) g\left(\frac{n_1}{d_1}, \ldots, \frac{n_{i-1}}{d_{i-1}}, \frac{p^k m}{d_i}, \ldots, \frac{n_r}{d_r}\right)\\
&\quad + \cdots\\
&\quad + (k+1) \sum_{d_i | m} f(d_1, \ldots, d_{i-1}, p^{k+1} d_i, d_{i+1}, \ldots, d_r) g\left(\frac{n_1}{d_1}, \ldots, \frac{n_{i-1}}{d_{i-1}}, \frac{m}{d_i}, \ldots, \frac{n_r}{d_r}\right).
\end{aligned}$$

Also,

$$
\begin{aligned}
&(\theta_{i,p} f * g)(n_1, \dots, n_r) \\
&\quad = \sum_{d_1 \mid n_1} \cdots \sum_{d_{i-1} \mid n_{i-1}} \sum_{d_i \mid p^k m} \sum_{d_{i+1} \mid n_{i+1}} \cdots \sum_{d_r \mid n_r} \\
&\qquad \theta_{i,p} f(d_1, \dots, d_{i-1}, d_i, d_{i+1}, \dots, d_r) g\left(\frac{n_1}{d_1}, \dots, \frac{n_{i-1}}{d_{i-1}}, \frac{p^k m}{d_i}, \dots, \frac{n_r}{d_r}\right) \\
&\quad = \sum_{d_1 \mid n_1} \cdots \sum_{d_{i-1} \mid n_{i-1}} \sum_{d_{i+1} \mid n_{i+1}} \cdots \sum_{d_r \mid n_r} \sum_{d_i \mid p^k m} \\
&\qquad (1 + v_p(d_i)) f(d_1, \dots, d_{i-1}, p d_i, d_{i+1}, \dots, d_r) g\left(\frac{n_1}{d_1}, \dots, \frac{n_{i-1}}{d_{i-1}}, \frac{p^k m}{d_i}, \dots, \frac{n_r}{d_r}\right).
\end{aligned}
$$

Here the innermost sum equals

$$
\begin{aligned}
&\sum_{d_i \mid m} f(d_1, \dots, d_{i-1}, p d_i, d_{i+1}, \dots, d_r) g\left(\frac{n_1}{d_1}, \dots, \frac{n_{i-1}}{d_{i-1}}, \frac{p^k m}{d_i}, \dots, \frac{n_r}{d_r}\right) \\
&\quad + \sum_{d_i \mid m} 2 f(d_1, \dots, d_{i-1}, p^2 d_i, d_{i+1}, \dots, d_r) g\left(\frac{n_1}{d_1}, \dots, \frac{n_{i-1}}{d_{i-1}}, \frac{p^{k-1} m}{d_i}, \dots, \frac{n_r}{d_r}\right) \\
&\quad + \dots \\
&\quad + \sum_{d_i \mid m} (k+1) f(d_1, \dots, d_{i-1}, p^{k+1} d_i, d_{i+1}, \dots, d_r) g\left(\frac{n_1}{d_1}, \dots, \frac{n_{i-1}}{d_{i-1}}, \frac{m}{d_i}, \dots, \frac{n_r}{d_r}\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&(f * \theta_{i,p} g)(n_1, \dots, n_r) \\
&\quad = \sum_{d_1 \mid n_1} \cdots \sum_{d_{i-1} \mid n_{i-1}} \sum_{d_{i+1} \mid n_{i+1}} \cdots \sum_{d_r \mid n_r} \sum_{d_i \mid p^k m} \\
&\qquad f(d_1, \dots, d_{i-1}, p d_i, d_{i+1}, \dots, d_r) \left(1 + v_p\left(\frac{p^k m}{d_i}\right)\right) g \\
&\qquad \times \left(\frac{n_1}{d_1}, \dots, \frac{n_{i-1}}{d_{i-1}}, \frac{p^{k+1} m}{d_i}, \dots, \frac{n_r}{d_r}\right),
\end{aligned}
$$

and the innermost sum equals

$$
\begin{aligned}
&\sum_{d_i \mid m} (k+1) f(d_1, \dots, d_{i-1}, d_i, d_{i+1}, \dots, d_r) g\left(\frac{n_1}{d_1}, \dots, \frac{n_{i-1}}{d_{i-1}}, \frac{p^{k+1} m}{d_i}, \dots, \frac{n_r}{d_r}\right) \\
&\quad + \sum_{d_i \mid m} k f(d_1, \dots, d_{i-1}, p d_i, d_{i+1}, \dots, d_r) g\left(\frac{n_1}{d_1}, \dots, \frac{n_{i-1}}{d_{i-1}}, \frac{p^k m}{d_i}, \dots, \frac{n_r}{d_r}\right) \\
&\quad + \dots \\
&\quad + \sum_{d_i \mid m} f(d_1, \dots, d_{i-1}, p^k d_i, d_{i+1}, \dots, d_r) g\left(\frac{n_1}{d_1}, \dots, \frac{n_{i-1}}{d_{i-1}}, \frac{p m}{d_i}, \dots, \frac{n_r}{d_r}\right).
\end{aligned}
$$

By comparing the innermost sums in the above expressions one finds that $\theta_{i,p}(f * g) = \theta_{i,p} f * g + f * \theta_{i,p} g$.

Theorem 1. *Let R be an integral domain and let r be a positive integer. For any prime number p and any $k \in \{1, \ldots, r\}$, $\theta_{k,p}$ is a derivation on $A_r(R)$ which is continuous with respect to each absolute value of the form $|\cdot|_{\underline{t}}$ on $A_r(R)$.*

Proof. By Lemma 1 we know that $\theta_{k,p}$ is a derivation on $A_r(R)$. In order to show that $\theta_{k,p}$ is continuous with respect to each absolute value of the form $|\cdot|_{\underline{t}}$ on $A_r(R)$, note first that for any $f \in A_r(R)$,

$$\frac{|\theta_{k,p} f|_{\underline{t}}}{|f|_{\underline{t}}} = \frac{\rho^{V_{\underline{t}}(\theta_{k,p} f)}}{\rho^{V_{\underline{t}}(f)}} = \rho^{V_{\underline{t}}(\theta_{k,p} f) - V_{\underline{t}}(f)}.$$

Let $n_1, \ldots, n_r$ be such that $V_{\underline{t}}(\theta_{k,p} f) = t_1 \Omega(n_1) + \cdots + t_r \Omega(n_r)$. So in particular $(n_1, \ldots, n_r) \in \text{supp}(\theta_{k,p} f)$. By the definition of $\theta_{k,p} f$ it follows that

$$(n_1, \ldots, n_{k-1}, pn_k, n_{k+1}, \ldots, n_r) \in \text{supp}(f).$$

Thus,

$$V_{\underline{t}}(f) \leq t_1 \Omega(n_1) + \cdots + t_{k-1} \Omega(n_{k-1}) + t_k \Omega(n_k) + t_{k+1} \Omega(n_{k+1}) + \cdots + t_r \Omega(n_r) = V_{\underline{t}}(\theta_{k,p} f) + t_k.$$

Therefore $t_k \geq V_{\underline{t}}(f) - V_{\underline{t}}(\theta_{k,p} f)$, which in turn implies that $\rho^{V_{\underline{t}}(\theta_{k,p} f) - V_{\underline{t}}(f)} \leq \frac{1}{\rho^{t_k}}$. We deduce that for all $f \in A_r(R)$,

$$|\theta_{k,p} f|_{\underline{t}} \leq \frac{1}{\rho^{t_k}} |f|_{\underline{t}}.$$

This shows that $\theta_{k,p}$ is a bounded linear operator on $A_r(R)$, and hence it is continuous.

4. The ring of formal r-fold power series

In this section we discuss a generalization to r variables of the isomorphism constructed by Cashwell and Everett in [5]. Let R be an integral domain. Let $p_1, p_2, p_3, \ldots$ denote the sequence of prime numbers in increasing order. Then every integer n may be written uniquely in the form $n = p_1^{\alpha_1(n)} p_2^{\alpha_2(n)} \ldots$ and uniquely described by a vector $(\alpha_1(n), \alpha_2(n), \ldots)$ with non-negative integral components, finitely many of which are nonzero, all such vectors being realized as n ranges over $\mathbb{N}$. Define a 1-fold formal power series in a countable number of indeterminates $y_{p_1}, y_{p_2}, y_{p_3}, \ldots$ with coefficients in R, by

$$\sum_n c_n y_{p_1}^{\alpha_1(n)} y_{p_2}^{\alpha_2(n)} \ldots.$$

Here the summation extends over all $n = p_1^{\alpha_1(n)} p_2^{\alpha_2(n)} p_3^{\alpha_3(n)} \ldots$ in $\mathbb{N}$, and c_n belongs to R for any such n. More generally, a formal r-fold power series in r distinct countable sets of indeterminates $x_{1p_1}, x_{1p_2}, \dots, x_{2p_1}, x_{2p_2}, \dots, x_{rp_1}, x_{rp_2}, \dots$ with coefficients in R has the form

$$\sum_{n_1} \cdots \sum_{n_r} c_{n_1,\dots,n_r} x_{1p_1}^{\alpha_1(n_1)} x_{2p_1}^{\alpha_1(n_2)} \dots x_{rp_1}^{\alpha_1(n_r)} x_{1p_2}^{\alpha_2(n_1)} x_{2p_2}^{\alpha_2(n_2)} \dots x_{rp_2}^{\alpha_2(n_r)} \dots,$$

where the summation extends over all

$$n_1 = p_1^{\alpha_1(n_1)} p_2^{\alpha_2(n_1)} p_3^{\alpha_3(n_1)} \dots,$$
$$n_2 = p_1^{\alpha_1(n_2)} p_2^{\alpha_2(n_2)} p_3^{\alpha_3(n_2)} \dots,$$
$$\dots$$
$$n_r = p_1^{\alpha_1(n_r)} p_2^{\alpha_2(n_r)} p_3^{\alpha_3(n_r)} \dots$$

in $\mathbb{N}$, and $c_{n_1,\dots,n_r} \in R$. For a fixed positive integer r, we denote the ring of all such r-fold power series by $R\{\dots, x_{ip_j}, \dots\} = R\{x_{1p_1}, x_{1p_2}, \dots\}\{x_{2p_1}, x_{2p_2}, \dots\} \dots \{x_{rp_1}, x_{rp_2}, \dots\}$. The multiplicative operation in this ring is the usual formal operation on power series involving multiplication and collection of finite number of like terms. We emphasize that only a finite number of x_{ip_j} actually appear (in the sense that $\alpha_j(n_i) > 0$) in any term. However, infinitely many x_{ip_j} may occur in the same series. For any positive integer l, any $k \in \{1, \dots, r\}$, and any power series $Q \in R\{\dots, x_{ip_j}, \dots\}$, denote by $\deg_{x_{kp_l}}(Q)$ the supremum of the set of exponents of x_{kp_l} that appear in the terms of Q with nonzero coefficients. Also, for a positive integer l, and $k \in \{1, \dots, r\}$, denote by $R\{\dots, x_{ip_j}, \dots\}_{(i,p_j)\neq(k,p_l)}$ the subring of $R\{\dots, x_{ip_j}, \dots\}$ which consists of all power series Q in $R\{\dots, x_{ip_j}, \dots\}$ such that $\deg_{x_{kp_l}}(Q)$ is zero. Consider now the following class of absolute values on $R\{\dots, x_{ip_j}, \dots\}$. Fix a real number $0 < \rho < 1$ and a $\underline{t} = (t_1, \dots, t_r)$ with $t_1, \dots, t_r > 0$, and $t_1, \dots, t_r$ linearly independent over $\mathbb{Q}$. For any $Q \in R\{\dots, x_{ip_j}, \dots\}$,

$$Q = \sum_{n_1} \cdots \sum_{n_r} c_{n_1,\dots,n_r} x_{1p_1}^{\alpha_1(n_1)} x_{2p_1}^{\alpha_1(n_2)} \dots x_{rp_1}^{\alpha_1(n_r)} x_{1p_2}^{\alpha_2(n_1)} x_{2p_2}^{\alpha_2(n_2)} \dots x_{rp_2}^{\alpha_2(n_r)} \dots,$$

denote $\operatorname{supp}(Q) = \{\underline{n} \in \mathbb{N}^r | c_{n_1,\dots,n_r} \neq 0\}$, and let

$$W_{\underline{t}}(Q) = \min_{\underline{n} \in \operatorname{supp}(Q)} \underline{t} \cdot \Omega_r(\underline{n}).$$

Then one obtains an absolute value $|\cdot|'_{\underline{t}}$ on $R\{\dots, x_{ip_j}, \dots\}$ by letting

$$|x|'_{\underline{t}} = \rho^{W_{\underline{t}}(x)} \text{ if } x \neq 0, \quad \text{and} \quad |x|'_{\underline{t}} = 0 \text{ if } x = 0.$$

Lemma 2. *For any prime number p_{j_0} and any $k \in \{1, \dots, r\}$, $x_{kp_{j_0}}$ is a generating element of $R\{\dots, x_{ip_j}, \dots\}$ over $R\{\dots, x_{ip_j}, \dots\}_{(i,p_j)\neq(k,p_{j_0})}$ with respect to each absolute value $||'_{\underline{t}}$.*

The proof follows easily by writing each element Q of $R\{\dots, x_{ip_j}, \dots\}$ as a formal power series in the variable $x_{kp_{j_0}}$ with coefficients in $R\{\dots, x_{ip_j}, \dots\}_{(i,p_j)\neq(k,p_{j_0})}$ and then truncating this power series up to $x_{kp_{j_0}}^m$, for each positive integer m. In this way one obtains for each m a polynomial Q_m in $x_{kp_{j_0}}$ with coefficients in $R\{\dots, x_{ip_j}, \dots\}_{(i,p_j)\neq(k,p_{j_0})}$. As

m tends to infinity, Q_m converges to Q in the absolute value $|x|'_{\underline{t}}$. This follows from the fact that the tail $Q - Q_m$ consists only of terms in which $x_{kp_{j_0}}$ appears with exponents larger than m, and hence $W_{\underline{t}}(Q - Q_m) \geq mt_k$, which tends to infinity as $m \to \infty$. In conclusion the ring of polynomials in the variable $x_{kp_{j_0}}$ with coefficients in $R\{\dots, x_{ip_j}, \dots\}_{(i,p_j)\neq(k,p_{j_0})}$ is dense in $R\{\dots, x_{ip_j}, \dots\}$, which proves the lemma.

Next, following [5], we associate to each arithmetical function $a = a(n) \in A_1(R)$ in one variable a formal power series in a countable number of indeterminates $y_{p_1}, y_{p_2}, y_{p_3}, \dots$ with coefficients in R, by means of the correspondence

$$a \to P(a) = \sum_n a(n) y_{p_1}^{\alpha_1(n)} y_{p_2}^{\alpha_2(n)} \dots .$$

Here the summation extends over all $n = p_1^{\alpha_1(n)} p_2^{\alpha_2(n)} p_3^{\alpha_3(n)} \dots$ in $\mathbb{N}$. Similarly, for any $r \geq 1$, an arithmetical function $a = a(n_1, \dots, n_r) \in A_r(R)$ in r variables may be associated to a formal r-fold power series in a countable number of indeterminates $x_{1p_1}, x_{1p_2}, \dots, x_{2p_1}, x_{2p_2}, \dots, \dots, x_{rp_1}, x_{rp_2}, \dots$ with coefficients in R, by means of the correspondence

$$a \to P(a) = \sum_{n_1} \dots \sum_{n_r} a(n_1, \dots, n_r) x_{1p_1}^{\alpha_1(n_1)} x_{2p_1}^{\alpha_1(n_2)} \dots x_{rp_1}^{\alpha_1(n_r)} x_{1p_2}^{\alpha_2(n_1)} x_{2p_2}^{\alpha_2(n_2)} \dots x_{rp_2}^{\alpha_2(n_r)} \dots .$$

Here, the summation extends over all

$$\begin{aligned} n_1 &= p_1^{\alpha_1(n_1)} p_2^{\alpha_2(n_1)} p_3^{\alpha_3(n_1)} \dots, \\ n_2 &= p_1^{\alpha_1(n_2)} p_2^{\alpha_2(n_2)} p_3^{\alpha_3(n_2)} \dots, \\ &\dots \\ n_r &= p_1^{\alpha_1(n_r)} p_2^{\alpha_2(n_r)} p_3^{\alpha_3(n_r)} \dots \end{aligned}$$

in $\mathbb{N}$. The correspondence is clearly one-to-one and onto from $A_r(R)$ to the set

$$R\{\dots, x_{ip_j}, \dots\} = R\{x_{1p_1}, x_{1p_2}, \dots\}\{x_{2p_1}, x_{2p_2}, \dots\} \dots \{x_{rp_1}, x_{rp_2}, \dots\}$$

of all such power series. Moreover, addition is preserved, and, as we will see below, $P(f * g) = P(f)P(g)$. Thus the ring $A_r(R)$ is isomorphic to the ring $R\{\dots, x_{ip_j}, \dots\}$ of all r-fold formal power series. We now check that $P(f * g) = P(f)P(g)$.

$$\begin{aligned} &P(f)P(g) \\ &= \left(\sum_{n_1}\sum_{n_2} \dots \sum_{n_r} f(n_1, \dots, n_r) x_{1p_1}^{\alpha_1(n_1)} x_{2p_1}^{\alpha_1(n_2)} \dots x_{rp_1}^{\alpha_1(n_r)} x_{1p_2}^{\alpha_2(n_1)} x_{2p_2}^{\alpha_2(n_2)} \dots x_{rp_2}^{\alpha_2(n_r)} \dots\right) \\ &\quad \left(\sum_{m_1}\sum_{m_2} \dots \sum_{m_r} g(m_1, \dots, m_r) x_{1p_1}^{\alpha_1(m_1)} x_{2p_1}^{\alpha_1(m_2)} \dots x_{rp_1}^{\alpha_1(m_r)} x_{1p_2}^{\alpha_2(m_1)} x_{2p_2}^{\alpha_2(m_2)} \dots x_{rp_2}^{\alpha_2(m_r)} \dots\right) \\ &= \sum_{\substack{n_1, \dots, n_r \\ m_1, \dots, m_r}} f(n_1, \dots, n_r) g(m_1, \dots, m_r) \\ &\quad x_{1p_1}^{\alpha_1(n_1)+\alpha_1(m_1)} x_{2p_1}^{\alpha_1(n_2)+\alpha_1(m_2)} \dots x_{rp_1}^{\alpha_1(n_r)+\alpha_1(m_r)} \\ &\quad x_{1p_2}^{\alpha_2(n_1)+\alpha_2(m_1)} x_{2p_2}^{\alpha_2(n_2)+\alpha_2(m_2)} \dots x_{rp_2}^{\alpha_2(n_r)+\alpha_2(m_r)} \dots \end{aligned}$$

$$= \sum_{\substack{n_1,\dots,n_r \\ m_1,\dots,m_r}} f(n_1,\dots,n_r)g(m_1,\dots,m_r)$$

$$x_{1p_1}^{\alpha_1(n_1m_1)} x_{2p_1}^{\alpha_1(n_2m_2)} \dots x_{rp_1}^{\alpha_1(n_rm_r)} x_{1p_2}^{\alpha_2(n_1m_1)} x_{2p_2}^{\alpha_2(n_2m_2)} \dots x_{rp_2}^{\alpha_2(n_rm_r)} \dots$$

$$= \sum_{k_1,\dots,k_r} \left(\sum_{k_1=m_1n_1} \cdots \sum_{k_r=m_rn_r} f(n_1,\dots,n_r)g(m_1,\dots,m_r) \right)$$

$$x_{1p_1}^{\alpha_1(k_1)} x_{2p_1}^{\alpha_1(k_2)} \dots x_{rp_1}^{\alpha_1(k_r)} x_{1p_2}^{\alpha_2(k_1)} x_{2p_2}^{\alpha_2(k_2)} \dots x_{rp_2}^{\alpha_2(k_r)} \dots$$

$$= \sum_{k_1,\dots,k_r} f * g(k_1,\dots,k_r) x_{1p_1}^{\alpha_1(k_1)} x_{2p_1}^{\alpha_1(k_2)} \dots x_{rp_1}^{\alpha_1(k_r)} x_{1p_2}^{\alpha_2(k_1)} x_{2p_2}^{\alpha_2(k_2)} \dots x_{rp_2}^{\alpha_2(k_r)} \dots$$

$$= P(f * g).$$

Let us also remark that this isomorphism preserves the absolute value, in the sense that $|a|_{\underline{t}} = |P(a)|'_{\underline{t}}$, for any $a \in A_r(R)$.

5. Generating degrees

Recall that for two commutative topological rings $A \subset B$, a subset $M \subset B$ is said to be a generating set of B over A if the ring $A[M]$ is dense in B. The generating degree of B/A, $\text{gdeg}(B/A) \in \mathbb{N} \cup \{0, \infty\}$ is defined by

$$\text{gdeg}(B/A){:} = \min\{|M|, \quad \text{where } M \text{ is a generating set of } B/A\}.$$

$|M|$ denotes the number of elements of M if M is finite and ∞ if M is not finite.

Let R be an integral domain, and F be its field of fractions. Let $r \geq 1$ be an integer, and p be a prime number. Then $A_r(R)$ is embedded in $A_R(F)$. Let k be an integer with $1 \leq k \leq r$. Let $B_{r,k,p}(R)$ denote the set of all $f \in A_r(R)$ satisfying the condition that for all $n_1,\dots,n_k,n_{k+1},\dots,n_r \in \mathbb{N}$ for which p divides n_k one has $f(n_1,\dots,n_r) = 0$.

Lemma 3. *For each r, k, p, and R as above, $B_{r,k,p}(R)$ is a R-subalgebra of $f \in A_r(R)$. Moreover, for any $\underline{t} = (t_1,\dots,t_r)$, $B_{r,k,p}(R)$ is topologically closed in $A_r(R)$.*

Proof. Denote as usual $\ker\theta_{k,p} = \{f \in A_r(R){:}\, \theta_{k,p} f = 0\}$. We observe that $B_{r,k,p}(R) = \ker\theta_{k,p}$. From general theory it follows that $\ker\theta_{k,p}$, and hence also $B_{r,k,p}(R)$, is an R-subalgebra of $A_r(R)$. Moreover, $\theta_{k,p}$ is continuous with respect to each absolute value of the form $||_{\underline{t}}$ on $A_r(R)$ by Theorem 1. It follows that $\ker\theta_{k,p}$ is topologically closed in $A_r(R)$. Hence $B_{r,k,p}(R)$ is closed in $A_r(R)$, and the lemma is proved.

Theorem 2. *For any integral domain R, any integer $r \geq 1$, any $k \in \{1,\dots,r\}$, any prime number p, and any $\underline{t} = (t_1,\dots,t_r)$ with $t_1,\dots,t_r > 0$, $t_1,\dots,t_r$ linearly independent over $\mathbb{Q}$, one has*

$$\text{gdeg}(A_r(R)/B_{r,k,p}(R)) = 1,$$

where gdeg *is computed with respect to the topology on $A_r(R)$ defined by the absolute value $||_{\underline{t}}$.*

Proof. Since by Lemma 3 the ring $B_{r,k,p}(R)$ is topologically closed in $A_r(R)$ but it does not coincide with $A_r(R)$, it follows that $B_{r,k,p}(R)$ is not dense in $A_r(R)$. We deduce that

$$\mathrm{gdeg}\left(A_r(R)/B_{r,k,p}(R)\right) \geq 1.$$

On the other hand, by employing the topological isomorphism from §4, we derive from Lemma 2 that $A_r(R)$ has a generating element over $B_{r,k,p}(R)$. This means that

$$\mathrm{gdeg}\left(A_r(R)/B_{r,k,p}(R)\right) \leq 1.$$

Combining the above inequalities we obtain $\mathrm{gdeg}(A_r(R)/B_{r,k,p}(R)) = 1$, which completes the proof of the theorem.

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