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Strong convergence of modified Ishikawa iterations for nonlinear mappings

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Abstract. In this paper, we prove a strong convergence theorem of modified Ishikawa iterations for relatively asymptotically nonexpansive mappings in Banach space. Our results extend and improve the recent results by Nakajo, Takahashi, Kim, Xu, Matsushita and some others.

Keywords. Relatively asymptotically nonexpansive mapping; nonexpansive mapping; generalized projection; asymptotic fixed point.

1. Introduction and preliminaries

Let E be a real Banach space, C a nonempty closed convex subset of E and $T: C \to C$ a mapping. Recall that T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C,$$

and T is asymptotically nonexpansive [10] if there exists a sequence $\{k_n\}$ of positive real numbers with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \quad \text{for all } n \geq 1 \text{ and } x, y \in C.$$

A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C: Tx = x\}$.

Some iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first iteration process is now known as Mann's iteration process [14] which is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0 \tag{1.1}$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval [0, 1].

The second iteration process is referred to as Ishikawa's iteration process [11] which is defined recursively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T y_n, \end{cases} \tag{1.2}$$

where the initial guess x_0 is taken in C arbitrarily and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval [0, 1].

In general, not much has been known about the convergence of the iteration processes (1.1) and (1.2) unless the underlying space E has elegant properties which we briefly mention here.

Reich [18] proved that if E is a uniformly convex Banach space with a Fréchet differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.1) converges weakly to a fixed point of T. However we note that Mann's iterations have only weak convergence even in a Hilbert space [9].

Attempts to modify the Mann's iteration method (1.1) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [15] proposed the following modification of the Mann's iteration (1.1) for a single nonexpansive mapping T in a Hilbert space:

$$\begin{cases} x_0 \in C \text{ arbitrarily chosen,} \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C\colon \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C\colon \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.3}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H and proved that sequence $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Recently, Kim and Xu [13] has adapted the iteration (1.1) in a Hilbert space. More precisely, they introduced the following iteration process for asymptotically nonexpansive mappings, with C a closed convex bounded subset of a Hilbert space:

$$\begin{cases} x_0 \in C \text{ arbitrarily chosen,} \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C\colon \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C\colon \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.4}$$

where

$$\theta_n = (1-\alpha_n)(k_n^2 - 1)(\operatorname{diam} C)^2 \to 0 \quad \text{as } n \to \infty.$$

They proved $\{x_n\}$ converges in norm to $P_{F(T)}x_0$ under the conditions that the sequence $\{\alpha_n\}_{n=0}^{\infty}$ in (0, 1) such that $\alpha_n \le \alpha$ for all n and for some $0 < \alpha < 1$.

On the other hand, process (1.2) is indeed more general than process (1.1). But research has been done on the latter due to reasons that the formulation of process (1.1) is simpler than that of (1.2) and that a convergence theorem for process (1.1) may lead to a convergence theorem for process (1.2) provided that $\{\beta_n\}$ satisfies certain appropriate conditions. However, the introduction of the process (1.2) has its own right. Actually, the process (1.1) may fail to converge while process (1.2) can still converge for a Lipschitz pseudo-contractive mapping in a Hilbert space [7].

In [19], Martinez-Yanes and Xu proved the following theorem.

Theorem MYX [19]. *Let C be a closed convex subset of a Hilbert space H and let $T\colon \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in* $[0, 1]$ *such that $\alpha_n \leq 1 - \delta$ for some $\delta \in (0, 1]$ and $\beta_n \to 1$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C by the algorithm*:

$$\begin{cases} x_0 \in C \quad \text{arbitrarily chosen} \\ z_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T z_n, \\ C_n = \{v \in C\colon \|y_n - v\|^2 \leq \|x_n - v\|^2 \\ \qquad +(1 - \alpha_n)(\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, v\rangle)\}, \\ Q_n = \{v \in C\colon \langle x_0 - x_n, x_n - v\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0. \end{cases}$$

Then $\{x_n\}$ converges in norm to $P_{F(T)} x_0$.

The purpose of this paper is to employ Nakajo and Takahashi's idea [15] to modify process (1.2) for relatively asymptotically nonexpansive mappings to have strong convergence theorem in Banach spaces.

Let E be a smooth Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f^* \in E^*\colon \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing.

As we all know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C\colon H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [1] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2 \quad \text{for } x, y \in E, \tag{1.5}$$

where $j(y) \in J(y)$. Observe that, in a Hilbert space H, (1.5) reduces to $\phi(x, y) = \|x - y\|^2$, $x, y \in H$.

The generalized projection $\Pi_C\colon E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x). \tag{1.6}$$

The existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example, [3]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2 \quad \text{for all } x, y \in E. \tag{1.7}$$

Remark. If E is a strictly convex and smooth Banach space, then for $x, y \in E, \phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From (1.7), we have $\|x\| = \|y\|$. This implies $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definitions of J, we have $Jx = Jy$. Since J is one-to-one, we have $x = y$; see [8,19] for more details.

Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point of p in C is said to be an asymptotically fixed point of T [17] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(Tx_n - x_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A mapping T from C into itself is called relatively nonexpansive [1–3] if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. A mapping T from C into itself is called relatively asymptotically nonexpansive if $\hat{F}(T) = F(T)$ and $\phi(p, T^n x) \leq k_n^2 \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$, $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E\colon \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in E$. It is well-known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. A Banach space is said to have the Kadec–Klee property if a sequence $\{x_n\} \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if E is uniformly convex then E has the Kadec–Klee property; see [10,19] for more details.

We need the following Lemmas for the proof of our main results.

Lemma 1.1 [12]. *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Lemma 1.2 [1–3]. *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \geq 0 \quad \textit{for } y \in C.$$

Lemma 1.3 [1–3]. *Let E be a reflexive, strictly convex and smooth Banach space. Let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_c x) + \phi(\Pi_c x, x) \leq \phi(y, x) \quad \textit{for all } y \in C.$$

Lemma 1.4. *Let E be a uniformly convex and uniformly smooth Banach space. Let C be a closed convex subset of E and let T be a relatively asymptotically nonexpansive mapping from C into itself. If T is continuous, then $F(T)$ is closed and convex.*

Proof. We first show that $F(T)$ is closed. Since T is continuous, we can obtain the closedness of $F(T)$ easily. Next, we show that $F(T)$ is convex for $x, y \in F(T)$ and $t \in (0, 1)$. Put $p = tx + (1 - t)y$. It is sufficient to show $Tp = p$. In fact, we have

$$
\begin{aligned}
\phi(p, T^n p) &= \|p\|^2 - 2\langle p, JT^n p\rangle + \|T^n p\|^2 \\
&= \|p\|^2 - 2\langle tx + (1-t)y, JT^n p\rangle + \|T^n p\|^2 \\
&= \|p\|^2 - 2t\langle x, JT^n p\rangle - 2(1-t)\langle y, JT^n p\rangle + \|T^n p\|^2 \\
&= \|p\|^2 + t\phi(x, T^n p) + (1-t)\phi(y, T^n p) - t\|x\|^2 \\
&\quad - (1-t)\|y\|^2 \\
&\le \|p\|^2 + k_n t\phi(x, p) + k_n(1-t)\phi(y, p) - t\|x\|^2 \\
&\quad - (1-t)\|y\|^2 \\
&= (k_n - 1)(t\|x\|^2 + (1-t)\|y\|^2 - \|p\|^2).
\end{aligned}
$$

Take the limit as $n \to \infty$ yields

$$\lim_{n\to\infty} \phi(p, T^n p) = 0.$$

Now we apply Lemma 1.1 to see that $T^n p \to p$ strongly. By continuity of T we obtain $p \in F(T)$. This completes the lemma 1.4. □

2. Main results

Theorem 2.1. *Let E be a uniformly convex and uniformly smooth Banach space. Let C be a nonempty bounded closed convex subset of E. Let $T: C \to C$ be a relatively asymptotically nonexpansive mapping with sequence $\{k_n\}$ such that $k_n \to 1$ as $n \to \infty$ and $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in* $[0, 1]$ *such that* $\limsup_{n\to\infty} \alpha_n < 1$ *and $\beta_n \to 1$. Define a sequence $\{x_n\}$ in C by the following algorithm*:

$$
\begin{cases}
x_0 \in C \quad \text{arbitrarily chosen}, \\
z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JT^n x_n), \\
y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n z_n), \\
C_n = \{v \in C: \phi(v, y_n) \le \phi(v, x_n) \\
\qquad +(1-\alpha_n)(k_n^2\|z_n\|^2 - \|x_n\|^2 + (k_n^2 - 1)M - 2\langle v, k_n^2 Jz_n - Jx_n\rangle)\}, \\
Q_n = \{v \in C: \langle Jx_0 - Jx_n, x_n - v\rangle \ge 0\}, \\
x_{n+1} = \Pi_{C_n \cap Q_n} x_0,
\end{cases}
\tag{2.1}
$$

where J is the duality mapping on E and M is an appropriate constant such that $M > \|v\|^2$ for each $v \in C$. If T is uniformly continuous, then $\{x_n\}$ converges to some $q = P_{F(T)}x_0$.

Proof. We first show that C_n and Q_n are closed and convex for each $n \ge 0$. From the definition of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for each $n \ge 0$. We prove that C_n is convex. For $v_1, v_2 \in C_n$ and $t \in (0, 1)$, put $v = tv_1 + (1-t)v_2$. It is sufficient to show that $v \in C_n$. Since

$$
\begin{aligned}
\phi(v, y_n) \le \phi(v, x_n) &+ (1-\alpha_n)(k_n^2\|z_n\|^2 - \|x_n\|^2 \\
&+ (k_n^2 - 1)M - 2\langle v, k_n^2 Jz_n - Jx_n\rangle)
\end{aligned}
$$

is equivalent to

$$
\begin{aligned}
&2\langle v, Jx_n\rangle + 2(1-\alpha_n)\langle v, k_n^2 Jz_n - Jx_n\rangle - 2\langle v, Jy_n\rangle \\
&\le (2-\alpha_n)\|x_n\|^2 + (1-\alpha_n)(k_n^2\|z_n\|^2 + (k_n^2-1)M) - \|y_n\|^2,
\end{aligned}
$$

we have

$$
\begin{aligned}
&2\langle v, Jx_n\rangle + 2(1-\alpha_n)\langle v, k_n^2 Jz_n - Jx_n\rangle - 2\langle v, Jy_n\rangle \\
&\quad = 2\langle tv_1 + (1-t)v_2, Jx_n\rangle + 2(1-\alpha_n)\langle tv_1 + (1-t)v_2, k_n^2 Jz_n - Jx_n\rangle \\
&\qquad - 2\langle tv_1 + (1-t)v_2, Jy_n\rangle \\
&\quad = 2t\langle v_1, Jx_n\rangle + 2(1-t)\langle v_2, Jx_n\rangle + 2(1-\alpha_n)t\langle v_1, k_n^2 Jz_n - Jx_n\rangle \\
&\qquad + 2(1-\alpha_n)(1-t)\langle v_2, k_n^2 Jz_n - Jx_n\rangle \\
&\qquad - 2t\langle v_1, Jy_n\rangle - 2(1-t)\langle v_2, Jy_n\rangle \\
&\quad \le (2-\alpha_n)\|x_n\|^2 + (1-\alpha_n)(k_n^2\|z_n\|^2 + (k_n^2-1)M) - \|y_n\|^2.
\end{aligned}
$$

This implies $v \in C_n$. So C_n is convex. Next, we show that $F(T) \subset C_n$ for all n. Indeed, we have, for all $p \in F(T)$,

$$
\begin{aligned}
\phi(p, y_n) &= \phi(p, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n z_n)) \\
&= \|p\|^2 - 2\langle p, \alpha_n Jx_n + (1-\alpha_n)JT^n z_n\rangle \\
&\quad + \|\alpha_n Jx_n + (1-\alpha_n)JT^n z_n)\|^2 \\
&\le \|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2(1-\alpha_n)\langle p, JT^n z_n\rangle \\
&\quad + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|T^n z_n\|^2 \\
&\le \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, T^n z_n) \\
&\le \alpha_n\phi(p, x_n) + k_n^2(1-\alpha_n)\phi(p, z_n) \\
&= \phi(p, x_n) + (1-\alpha_n)[k_n^2\phi(p, z_n) - \phi(p, x_n)] \\
&\le \phi(p, x_n) + (1-\alpha_n)(k_n^2\|z_n\|^2 - \|x_n\|^2 \\
&\quad + (k_n^2-1)\|p\|^2 - 2\langle p, k_n^2 Jz_n - Jx_n\rangle) \\
&\le \phi(p, x_n) + (1-\alpha_n)(k_n^2\|z_n\|^2 - \|x_n\|^2 \\
&\quad + (k_n^2-1)M - 2\langle p, k_n^2 Jz_n - Jx_n\rangle).
\end{aligned}
$$

So $p \in C_n$ for all n. Next we show that

$$
F(T) \subset Q_n, \quad \text{for all } n \ge 0. \tag{2.2}
$$

We prove this by induction. For $n = 0$, we have $F(T) \subset C = Q_0$. Assume that $F(T) \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, by Lemma 1.2 we have

$$
\langle Jx_0 - Jx_{n+1}, x_{n+1} - z\rangle \ge 0, \quad \forall z \in C_n \cap Q_n.
$$

As $F(T) \subset C_n \cap Q_n$ by the induction assumptions, the last inequality holds, in particular, for all $z \in F(T)$. This together with the definition of Q_{n+1} implies that $F(T) \subset Q_{n+1}$. Hence (2.2) holds for all $n \geq 0$. This implies that $\{x_n\}$ is well defined. Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in Q_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) \quad \text{for all } n \geq 0.$$

Therefore $\{\phi(x_n, x_0)\}$ is nondecreasing. Since C is bounded, $\phi(x_n, x_0)$ is bounded. Moreover from (1.7), we have that $\{x_n\}$ is bounded. So, we obtain that the limit of $\{\phi(x_n, x_0)\}$ exists. From Lemma 1.3, we have

$$\begin{aligned}\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_0) \leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0)\end{aligned}$$

for all $n \geq 0$. This implies that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{2.3}$$

By using Lemma 1.1, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.4}$$

Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n$, from the definition of C_n, we also have

$$\begin{aligned}\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) &+ (1-\alpha_n)(k_n^2\|z_n\|^2 - \|x_n\|^2 \\ &+ (k_n^2 - 1)M - 2\langle x_{n+1}, k_n^2 J z_n - J x_n\rangle).\end{aligned} \tag{2.5}$$

However, since $\lim_{n\to\infty} \beta_n = 1$ and $\{x_n\}$ is bounded, we obtain

$$\begin{aligned}\phi(x_{n+1}, z_n) &= \phi(x_{n+1}, J^{-1}(\beta_n J x_n + (1-\beta_n) J T^n x_n)) \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, \beta_n J x_n + (1-\beta_n) J T^n x_n\rangle \\ &\quad + \|\beta_n J x_n + (1-\beta_n) J T^n x_n\|^2 \\ &\leq \|x_{n+1}\|^2 - 2\beta_n \langle x_{n+1}, J x_n\rangle - 2(1-\beta_n)\langle x_{n+1}, J T^n x_n\rangle \\ &\quad + \beta_n \|x_n\|^2 + (1-\beta_n)\|T^n x_n\|^2 \\ &= \beta_n \phi(x_{n+1}, x_n) + (1-\beta_n)\phi(x_{n+1}, T^n x_n).\end{aligned}$$

Therefore, we obtain

$$\phi(x_{n+1}, z_n) \to 0, \tag{2.6}$$

which yields

$$\|x_{n+1}\|^2 + \|z_n\|^2 - 2\langle x_{n+1}, J z_n\rangle \to 0. \tag{2.7}$$

On the other hand, we have

$$\begin{aligned}&k_n^2\|z_n\|^2 - \|x_n\|^2 - 2\langle x_{n+1}, k_n^2 J z_n - J x_n\rangle \\ &\quad = k_n^2\|z_n\|^2 - \|x_n\|^2 - 2k_n^2\langle x_{n+1}, J z_n\rangle + 2\langle x_{n+1}, J x_n\rangle \\ &\quad = (k_n^2\|z_n\|^2 + k_n^2\|x_{n+1}\|^2 - 2k_n^2\langle x_{n+1}, J z_n\rangle) \\ &\qquad + 2\langle x_{n+1}, J x_n\rangle - k_n^2\|x_{n+1}\|^2 - \|x_n\|^2.\end{aligned} \tag{2.8}$$

Now, we consider

$$
\begin{aligned}
&2\langle x_{n+1}, Jx_n\rangle - k_n^2\|x_{n+1}\|^2 - \|x_n\|^2 \\
&\quad = \langle x_{n+1}, Jx_n\rangle + \langle x_{n+1}, Jx_n\rangle - k_n^2\|x_{n+1}\|^2 - \|x_n\|^2 \\
&\quad = \langle x_n + x_{n+1} - x_n, Jx_n\rangle + \langle x_{n+1}, Jx_{n+1} + Jx_n - Jx_{n+1}\rangle \\
&\qquad - k_n^2\|x_{n+1}\|^2 - \|x_n\|^2 \\
&\quad = \langle x_{n+1} - x_n, Jx_n\rangle + \langle x_{n+1}, Jx_n - Jx_{n+1}\rangle - (k_n^2 - 1)\|x_{n+1}\|^2.
\end{aligned}
$$

It follows from (2.4) that

$$
2\langle x_{n+1}, Jx_n\rangle - k_n^2\|x_{n+1}\|^2 - \|x_n\|^2 \to 0. \tag{2.9}
$$

It follows from (2.7) and (2.9) that

$$
k_n^2\|z_n\|^2 - \|x_n\|^2 - 2\langle x_{n+1}, k_nJz_n - Jx_n\rangle \to 0. \tag{2.10}
$$

Combining (2.3), (2.5) and (2.10), we have

$$
\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0.
$$

Using Lemma 1.1, we obtain

$$
\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{2.11}
$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$
\lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = 0. \tag{2.12}
$$

Notice that

$$
\begin{aligned}
\|Jx_{n+1} - Jy_n\| &= \|Jx_{n+1} - (\alpha_nJx_n + (1-\alpha_n)JT^nz_n)\| \\
&= \|\alpha_n(Jx_{n+1} - Jx_n) + (1-\alpha_n)(Jx_{n+1} - JT^nz_n)\| \\
&= \|(1-\alpha_n)(Jx_{n+1} - JT^nz_n) - \alpha_n(Jx_n - Jx_{n+1})\| \\
&\geq (1-\alpha_n)\|Jx_{n+1} - JT^nz_n\| - \alpha_n\|Jx_n - Jx_{n+1}\|.
\end{aligned}
$$

We have

$$
\|Jx_{n+1} - JT^nz_n\| \leq \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Jy_n\| + \alpha_n\|Jx_n - Jx_{n+1}\|).
$$

From (2.12) and $\limsup_{n\to\infty}\alpha_n < 1$, we obtain

$$
\lim_{n\to\infty} \|Jx_{n+1} - JT^nz_n\| = 0.
$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain

$$
\lim_{n\to\infty} \|x_{n+1} - T^nz_n\| = 0 \tag{2.13}
$$

and hence

$$\|x_n - T^n x_n\| \le \|x_{n+1} - x_n\| + \|x_{n+1} - T^n z_n\|.$$

It follows from (2.4) and (2.13) that $\lim_{n\to\infty} \|T^n x_n - x_n\| = 0$. Putting $L = \sup\{k_n : n \ge 1\} < \infty$, we obtain

$$\|Tx_n - x_n\| \le \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\|.$$

Since T is uniformly continuous, we have

$$\|Tx_n - x_n\| \to 0, \quad \text{as } n \to \infty.$$

Finally, we prove that $x_n \to q = \Pi_{F(T)}x_0$. Assume that $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ such that $\{x_{n_i}\} \rightharpoonup q \in C$. Then $q \in \hat{F}(T) = F(T)$. Next we show that $q = \Pi_{F(T)}x_0$ and convergence is strong. Putting $q' = \Pi_{F(T)}x_0$ from $x_{n+1} = \Pi_{C_n \cap Q_n}x_0$ and $q' \in F(T) \subset C_n \cap Q_n$. We have $\phi(x_{n+1}, x_0) \le \phi(q', x_0)$. On the other hand, from weakly lower semicontinuity of the norm, we obtain

$$\begin{aligned}
\phi(q, x_0) &= \|q\|^2 - 2\langle q, Jx_0\rangle + \|x_0\|^2 \\
&\le \liminf_{i\to\infty} (\|x_{n_i}\|^2 - \langle x_{n_i}, Jx_0\rangle + \|x_0\|^2) \\
&\le \liminf_{i\to\infty} \phi(x_{n_i}, x_0) \le \limsup_{i\to\infty} \phi(x_{n_i}, x_0) \\
&\le \phi(q', x_0).
\end{aligned}$$

It follows from the definition of $\Pi_{F(T)}x_0$, that $q = \Pi_{F(T)}x_0$ and hence

$$\lim_{i\to\infty} \phi(x_{n_i}, x_0) = \phi(q', x_0) = \phi(q, x_0).$$

So we have $\lim_{i\to\infty} \|x_{n_i}\| = \|q\|$. Using the Kadec–Klee property of E, we obtain that $\{x_{n_i}\}$ converges strongly to $q = P_{F(T)}x_0$. Since $\{x_{n_i}\}$ is an arbitrarily weakly convergent sequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$. This completes the proof. □

3. Applications

Theorem 3.1. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $T: C \to C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}$ such that $k_n \to 1$ as $n \to \infty$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in* $[0, 1]$ *such that* $\limsup_{n\to\infty} \alpha_n < 1$ *and $\beta_n \to 1$. Define a sequence $\{x_n\}$ in C by the following algorithm*:

$$\begin{cases}
x_0 \in C \quad \text{arbitrarily chosen}, \\
z_n = \beta_n x_n + (1-\beta_n)T^n x_n, \\
y_n = \alpha_n x_n + (1-\alpha_n)T^n z_n, \\
C_n = \{v \in C : \|y_n - v\|^2 \le \|x_n - v\|^2 \\
\qquad + (1-\alpha_n)(k_n^2\|z_n\|^2 - \|x_n\|^2 + (k_n^2 - 1)M - 2\langle v, k_n^2 z_n - x_n\rangle)\}, \\
Q_n = \{v \in C : \langle x_0 - x_n, x_n - v\rangle \ge 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x_0,
\end{cases}$$

where M is an appropriate constant such that $M > \|v\|^2$ for each $v \in C$. Then $\{x_n\}$ converges to some $q = P_{F(T)}x_0$.

Proof. Note that T has a fixed point in C [10]. The key is to show that if T is asymptotically nonexpansive, then T is also relatively asymptotically nonexpansive. Take $p \in \hat{F}(T)$. There exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} \|Tx_n - x_n\|$. Since T is a asymptotically nonexpansive, it is well-known that T is demiclosed, which yields $p \in F(T)$. On the other hand, we have $F(T) \subset \hat{F}(T)$. In Hilbert spaces we know (1.5) reduces to $\phi(x, y) = \|x-y\|^2$, $x, y \in H$. That is, $\phi(T^n x, T^n y) \le k_n^2 \phi(x, y)$ is equivalent to $\|T^n x - T^n y\| \le k_n\|x-y\|$. Therefore, T is also relatively asymptotically nonexpansive. By using Theorem 2.1, it is easy to obtain the desired conclusion. This completes the proof. □

Theorem 3.2 [19]. *Let C be a closed convex subset of a Hilbert space H and let $T: C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$ such that $\alpha_n \le 1 - \delta$ for some $\delta \in (0, 1]$ and $\beta_n \to 1$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C by the algorithm:*

$$\begin{cases} x_0 \in C \quad \textit{arbitrarily chosen,} \\ z_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tz_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \le \|x_n - v\|^2 \\ \qquad +(1-\alpha_n)(\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, v\rangle)\}, \\ Q_n = \{v \in C : \langle x_0 - x_n, x_n - v\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases}$$

Then $\{x_n\}$ converges in norm to $P_{F(T)}x_0$.

Proof. It is well-known that the nonexpansive map is an asymptotically nonexpansive map when $k_n = 1$. By using Theorem 3.1, it is easy to obtain the desired conclusion. This completes the proof. □

Theorem 3.3 [15]. *Let C be a closed convex subset of a Hilbert space H and let $T: C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $(0, 1)$ such that $\alpha_n \le 1 - \delta$ for some $\delta \in (0, 1]$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C by the algorithm:*

$$\begin{cases} x_0 \in C \quad \textit{arbitrarily,} \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{v \in C : \|y_n - v\| \le \|x_n - v\|, \\ Q_n = \{v \in C : \langle x_0 - x_n, x_n - v\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases}$$

Then $\{x_n\}$ converges in norm to $P_{F(T)}x_0$.

Proof. By taking $\beta_n = 1$ in Theorem 3.2, we can obtain the desired conclusion. □

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