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Positive solutions and eigenvalue intervals for nonlinear systems

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Abstract. This paper deals with the existence of positive solutions for the nonlinear system

$$(q(t)\phi(p(t)u_i'(t)))' + f^i(t, \mathbf{u}) = 0, \quad 0 < t < 1, \quad i = 1, 2, \ldots, n.$$

This system often arises in the study of positive radial solutions of nonlinear elliptic system. Here $\mathbf{u} = (u_1, \ldots, u_n)$ and f^i, $i = 1, 2, \ldots, n$ are continuous and nonnegative functions, $p(t)$, $q(t)$: $[0, 1] \to (0, \infty)$ are continuous functions. Moreover, we characterize the eigenvalue intervals for

$$(q(t)\phi(p(t)u_i'(t)))' + \lambda h_i(t)g^i(\mathbf{u}) = 0, \quad 0 < t < 1, \quad i = 1, 2, \ldots, n.$$

The proof is based on a well-known fixed point theorem in cones.

Keywords. Nonlinear system; p-Laplacian; positive solutions; eigenvalue intervals; fixed point theorem in cones.

1. Introduction

In this paper we study the existence of positive solutions for the nonlinear system

$$\begin{cases} (q(t)\phi(p(t)u_1'(t)))' + f^1(t, \mathbf{u}) = 0, & 0 < t < 1, \\ \cdots \\ (q(t)\phi(p(t)u_n'(t)))' + f^n(t, \mathbf{u}) = 0, & 0 < t < 1, \end{cases} \tag{1.1}$$

with the following boundary condition

$$\mathbf{u}(0) = 0, \quad \mathbf{u}(1) = 0. \tag{1.2}$$

Here $\phi(x) = |x|^{p-2}x$, $p > 1$, $\mathbf{u} = (u_1, \ldots, u_n)$. We always make the following assumptions:

(H_1) f^i: $[0, 1] \times \mathbb{R}_+^n \to (0, \infty)$ is continuous, $i = 1, 2, \ldots, n$.
(H_2) $p(t)$, $q(t)$: $[0, 1] \to (0, \infty)$ are continuous functions and $q(t)$ is nondecreasing on [0,1].

Problems (1.1) and (1.2) often arise from the study of positive radial solutions for the nonlinear elliptic system of the form

$$\begin{cases} \operatorname{div}(|\nabla u_1|^{p-2}\nabla u_1) + k_1(|x|)g^1(\mathbf{u}) = 0, \\ \cdots \\ \operatorname{div}(|\nabla u_n|^{p-2}\nabla u_n) + k_n(|x|)g^n(\mathbf{u}) = 0, \end{cases} \tag{1.3}$$

in the domain $0 < R_1 < |x| < R_2 < \infty,\ x \in \mathbb{R}^N,\ N \geq 2$ with the following boundary condition:

$$u_i = 0 \quad \text{on } |x| = R_1 \quad \text{and} \quad |x| = R_2,\ i = 1, 2, \ldots, n. \tag{1.4}$$

In recent years, positive radial solutions for nonlinear elliptic equation or elliptic system have been studied by many authors and we refer the reader to [3,8,9,15–18]. It was proved in [3,18] that the classical elliptic equation

$$\Delta u + \lambda k(|x|)f(u) = 0,\ \text{in } R_1 < |x| < R_2,\ x \in \mathbb{R}^N,\ N \geq 2$$

has at least one positive radial solution under the assumption that f is either superlinear or sublinear. The one-dimensional p-Laplacian boundary value problem has also attracted considerable attention [2,4,7,14,19,22]. In [11], it was proved that

$$(\phi(u'))' + \lambda h(t)f(u) = 0$$

with boundary condition $u(0) = u(1) = 0$ having at least one positive solution for certain finite intervals of λ if one of f_0 and f_∞ is large enough and the other one is small enough.

In this paper, we also study the following eigenvalue problem

$$\begin{cases} (q(t)\phi(p(t)u_1'(t)))' + \lambda h_1(t)g^1(\mathbf{u}) = 0, \quad 0 < t < 1, \\ \cdots \\ (q(t)\phi(p(t)u_n'(t)))' + \lambda h_n(t)g^n(\mathbf{u}) = 0, \quad 0 < t < 1. \end{cases} \tag{1.5}$$

We prove that (1.5) and (1.2) have at least one positive solution for each λ in an explicit eigenvalue interval. Recently, several eigenvalue characterizations for different kinds of boundary value problems have appeared and we refer the reader to [1,2,5,6,11,12]. In this paper, we will show how our method allows to improve the range of eigenvalue intervals. The new results are easily derived from a general result, stated as Theorem 3.1, which gives sufficient conditions to guarantee the existence of at least one positive solution for systems (1.1) and (1.2).

Our arguments are based on a well-known fixed point theorem in cones. Many authors [2,5,10] have used this fixed point theorem to discuss the existence of positive solutions for different boundary value problems. In [12,21], the existence, multiplicity and nonexistence of positive solutions for nonlinear systems of ordinary differential equations were considered using fixed point index. The same problems for the quasilinear elliptic system (1.3) were studied in [20]. The novelty of this paper is a choice of a cone different from that used in [12,20,21]. Such a choice of cones (see (2.1)) is well fit with the systems we are considering.

Finally, it is worth remarking here that, we can also deal with the nonlinear system (1.1) with one of the following two sets of boundary conditions

$$\mathbf{u}'(0) = 0, \qquad \mathbf{u}(1) = 0,$$

$$\mathbf{u}(0) = 0, \qquad \mathbf{u}'(1) = 0.$$

However since the arguments are essentially the same (in fact easier), we will restrict our discussion to boundary data (1.2).

The notation used is as follows: $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_+^n = \prod_{i=1}^n \mathbb{R}_+$. For $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$, $\|\mathbf{u}\| = \max_{i=1,2,\ldots,n} |u_i|$.

The remaining part of the paper is organized as follows. In §2, some preliminary results are given and in §3, the main results are proved.

2. Preliminaries

The proof of the main results is based on a well-known fixed point theorem in cones. We recall the statement of this result below, after introducing the definition of a cone.

DEFINITION 2.1

Let X be a Banach space and K be a closed, nonempty subset of X. K is a cone if

(i) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta > 0$,
(ii) $u, -u \in K$ implies $u = 0$.

We also recall that a completely continuous operator means a continuous operator which transforms every bounded set into a relatively compact set. If D is a subset X, we write $D_K = D \cap K$ and $\partial_K D = (\partial D) \cap K$.

Theorem 2.2 [13]. *Let X be a Banach space and K $(\subset X)$ be a cone. Assume that Ω^1, Ω^2 are open subsets of X with $\Omega^1_K \neq \emptyset$, $\overline{\Omega^1}_K \subset \Omega^2_K$. Let*

$$T\colon \overline{\Omega^2}_K \to K$$

be a continuous, completely continuous operator such that either

(i) $\|Tu\| \geq \|u\|, u \in \partial_K \Omega^1$ *and* $\|Tu\| \leq \|u\|, u \in \partial_K \Omega^2$; *or*
(ii) $\|Tu\| \leq \|u\|, u \in \partial_K \Omega^1$ *and* $\|Tu\| \geq \|u\|, u \in \partial_K \Omega^2$.

Then T has a fixed point in $\overline{\Omega^2}_K \backslash \Omega^1_K$.

As usual, we denote by $C[0, 1]$ the space of continuous functions from [0,1] to $\mathbb{R}$. In $C[0, 1]$ we shall consider the norm $|u|_0 = \max_{0 \leq t \leq 1} |u(t)|$. In order to apply Theorem 2.2 below, we take $X = C[0, 1] \times C[0, 1] \times \cdots \times C[0, 1]$ (n times) with the norm $\|\mathbf{u}\| = \max_{i=1,2,\ldots,n} |u_i|_0$ for $\mathbf{u} = (u_1, \ldots, u_n) \in X$. Then X is a Banach space. Define

$$K = \left\{ \mathbf{u} \in X\colon u_i(t) \geq 0 \text{ , } \forall\, t \in [0, 1] \text{ and } \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} u_i(t) \geq \rho |u_i|_0,\ i = 1, 2, \ldots, n \right\}, \tag{2.1}$$

where ρ is given by

$$\rho = \left[\int_0^1 \frac{1}{p(s)} \mathrm{d}s \right]^{-1} \min \left\{ \int_0^{\frac{1}{4}} \frac{1}{p(s)} \mathrm{d}s,\ \int_{\frac{3}{4}}^1 \frac{1}{p(s)} \mathrm{d}s \right\}. \tag{2.2}$$

One can easily verify that K is a cone in X.

Let $T\colon K \to X$ be a map with components $(T^1, \dots, T^n)$, where T^i, $i = 1, 2, \dots, n$ is defined by

$$(T^i\mathbf{u})(t) = \begin{cases} \int_0^t \frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_s^{\sigma_i} f^i(\tau, \mathbf{u}(\tau))\mathrm{d}\tau\right)\mathrm{d}s, & 0 \le t \le \sigma_i, \\ \cdots \\ \int_t^1 \frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_{\sigma_i}^{s} f^i(\tau, \mathbf{u}(\tau))\mathrm{d}\tau\right)\mathrm{d}s, & \sigma_i \le t \le 1; \end{cases} \tag{2.3}$$

here $\sigma_i \in (0, 1)$ is a solution of the equation

$$\Theta^i\mathbf{u}(t) = 0, \quad 0 \le t \le 1 \tag{2.4}$$

and the map $\Theta^i\colon K \to C[0, 1]$ is defined by

$$\Theta^i\mathbf{u}(t) = \int_0^t \frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_s^t f^i(\tau, \mathbf{u}(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ - \int_t^1 \frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_t^s f^i(\tau, \mathbf{u}(\tau))\mathrm{d}\tau\right)\mathrm{d}s, \quad 0 < t < 1.$$

Lemma 2.3 [12]. *Assume* (H_1) *and* (H_2) *hold. Then, for any* $\boldsymbol{u} \in K$ *and* $i = 1, 2, \dots, n$, $\Theta^i\boldsymbol{u}(t) = 0$ *has at least one solution in* $(0, 1)$ *and* T *is well-defined.*

Lemma 2.4 [12]. *Assume* (H_2) *holds. Let* $u \in C^1[0, 1]$ *be a nonnegative function and* $q(t)\phi(p(t)u'(t))$ *is nonincreasing on* $[0, 1]$. *Then*

$$u(t) \ge \left[\int_0^1 \frac{1}{p(s)}\mathrm{d}s\right]^{-1} \min\left\{\int_0^t \frac{1}{p(s)}\mathrm{d}s,\ \int_t^1 \frac{1}{p(s)}\mathrm{d}s\right\}|u|_0,\ 0 \le t \le 1.$$

In particular,

$$\min_{\frac{1}{4} \le t \le \frac{3}{4}} u(t) \ge \rho|u|_0.$$

Remark 2.5. If $p(t) = C$ for all $0 \le t \le 1$, then $\rho = \frac{1}{4}$, and this was used in [20].

Lemma 2.6 [12]. *Assume* (H_1) *and* (H_2) *hold. Then* $T(K) \subset K$. *Moreover,* T *is continuous and completely continuous.*

Proof. Lemma 2.4 implies that $T(K) \subset K$. It is easy to see that T is continuous and completely continuous since (H_1) and (H_2) hold. See [21] for a proof. □

3. Main results

In this section we establish the existence of positive solutions for (1.1) and (1.2). Moreover, we characterize the eigenvalues for problems (1.5) and (1.2).

For the given function $a \in C[0,1]$, let

$$\gamma_a(t) = \frac{\rho}{2}\left[\int_{\frac{1}{4}}^{t} \frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_s^t a(\tau)\mathrm{d}\tau\right)\mathrm{d}s + \int_t^{\frac{3}{4}} \frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_t^s a(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right], \quad \frac{1}{4} \le t \le \frac{3}{4}. \tag{3.1}$$

Here ρ is given as in (2.2).

Theorem 3.1. *Assume* (H_1) *and* (H_2) *hold. Furthermore, it is assumed that for all* $i = 1, 2, \ldots, n$, *the following hypotheses hold*:

(D_1) *There exist a constant* $\alpha > 0$ *and a continuous function* $\psi_i\colon \left[\frac{1}{4}, \frac{3}{4}\right] \to (0, \infty)$ *such that*

$$f^i(t, \mathbf{u}) \ge (\rho\alpha)^{p-1}\psi_i(t), \qquad \frac{1}{4} \le t \le \frac{3}{4}$$

for all $0 \le u_j \le \alpha$ $(j \in \{1, 2, \ldots, n\} \setminus \{i\})$ *and* $\rho\alpha \le u_i \le \alpha$, *and*

$$\inf_{\frac{1}{4} \le t \le \frac{3}{4}} \gamma_{\psi_i}(t) \ge 1.$$

(D_2) *There exist a constant* $\beta > 0$, $\beta \ne \alpha$ *and a continuous function* $\varphi_i\colon [0, 1] \to (0, \infty)$ *such that*

$$f^i(t, \boldsymbol{u}) \le \beta^{p-1}\varphi_i(t) \quad \text{for } 0 \le t \le 1 \text{ and } 0 < u_j \le \beta, \quad j \in \{1, \ldots, n\}$$

and

$$\int_0^1 \frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_0^1 \varphi_i(\tau)\mathrm{d}\tau\right)\mathrm{d}s \le 1.$$

Then problems (1.1) *and* (1.2) *have at least one positive solution* $\mathbf{u}$ *satisfying*

$$\min\{\alpha, \beta\} \le \|\mathbf{u}\| \le \max\{\alpha, \beta\}.$$

Proof. We assume that $\alpha < \beta$. The case $\alpha > \beta$ is analogous.

Define the sets

$$\Omega^1 = \{x \in X\colon \|x\| < \alpha\} \quad \text{and} \quad \Omega^2 = \{x \in X\colon \|x\| < \beta\}.$$

We claim that

(i) $\|T\mathbf{u}\| \ge \|\mathbf{u}\|$ for $\mathbf{u} \in \partial_K\Omega^1$;

and

(ii) $\|T\mathbf{u}\| \le \|\mathbf{u}\|$ for $\mathbf{u} \in \partial_K\Omega^2$.

First we shall prove (i). Note, from the definition of $T\mathbf{u}$, that $T^i\mathbf{u}(\sigma_i)$ is the maximum of $T^i\mathbf{u}$ on $[0, 1]$. Since $\|\mathbf{u}\| = \alpha$, there exists $i \in \{1, \ldots, n\}$ such that $\rho\alpha \le u_i(t) \le \alpha$, $\frac{1}{4} \le t \le \frac{3}{4}$ and $0 \le u_j \le \alpha$ $(j \in \{1, 2, \ldots, n\} \setminus \{i\})$. We consider three cases.

Case 1. $\sigma_i \in \left[\frac{1}{4}, \frac{3}{4}\right]$. Then

$$\begin{aligned}\sup_{0\le t\le 1}(T^i\mathbf{u})(t) &\ge \frac{1}{2}\left[\int_{\frac{1}{4}}^{\sigma_i}\frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_s^{\sigma_i} f^i(\tau,\mathbf{u}(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right.\\ &\quad\left.+\int_{\sigma_i}^{\frac{3}{4}}\frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_{\sigma_i}^{s} f^i(\tau,\mathbf{u}(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right]\\ &\ge \frac{\rho\alpha}{2}\left[\int_{\frac{1}{4}}^{\sigma_i}\frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_s^{\sigma_i}\psi_i(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right.\\ &\quad\left.+\int_{\sigma_i}^{\frac{3}{4}}\frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_{\sigma_i}^{s}\psi_i(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right]\\ &= \alpha\gamma_{\psi_i}(\sigma_i) \ge \alpha\min_{\frac{1}{4}\le t\le\frac{3}{4}}\gamma_{\psi_i}(t)\ge\alpha.\end{aligned}$$

Case 2. $\sigma_i > \frac{3}{4}$. Then

$$\begin{aligned}\sup_{0\le t\le 1}(T^i\mathbf{u})(t) &\ge \int_{\frac{1}{4}}^{\frac{3}{4}}\frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_s^{\frac{3}{4}} f^i(\tau,\mathbf{u}(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\ &\ge \rho\alpha\int_{\frac{1}{4}}^{\frac{3}{4}}\frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_s^{\frac{3}{4}}\psi_i(\tau)\mathrm{d}\tau\right)\mathrm{d}s\\ &= 2\alpha\gamma_{\psi_i}\left(\frac{3}{4}\right)\ge\alpha\min_{\frac{1}{4}\le t\le\frac{3}{4}}\gamma_{\psi_i}(t)\ge\alpha.\end{aligned}$$

Case 3. $\sigma_i < \frac{1}{4}$. Then

$$\begin{aligned}\sup_{0\le t\le 1}(T^i\mathbf{u})(t) &\ge \int_{\frac{1}{4}}^{\frac{3}{4}}\frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_{\frac{1}{4}}^{s} f^i(\tau,\mathbf{u}(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\ &\ge \rho\alpha\int_{\frac{1}{4}}^{\frac{3}{4}}\frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_{\frac{1}{4}}^{s}\psi_i(\tau)\mathrm{d}\tau\right)\mathrm{d}s\\ &= 2\alpha\gamma_{\psi_i}\left(\frac{1}{4}\right)\ge\alpha\min_{\frac{1}{4}\le t\le\frac{3}{4}}\gamma_{\psi_i}(t)\ge\alpha.\end{aligned}$$

Hence $\sup_{0\le t\le 1}(T^i\mathbf{u})(t)\ge\alpha$. This implies that (i) holds.

Next we prove (ii). In fact, for any $\mathbf{u}\in\partial_K\Omega^2$, we have $|u_i|_0\le\beta$ for each $i\in\{1,\ldots,n\}$. Fix $i\in\{1,\ldots,n\}$. Then

$$(T^i\mathbf{u})(t)\le\int_0^1\frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_0^1 f^i(\tau,\mathbf{u}(\tau))\mathrm{d}\tau\right)\mathrm{d}s$$

$$\leq \beta \int_0^1 \frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_0^1 \varphi_i(\tau)\mathrm{d}\tau\right)\mathrm{d}s$$

$$\leq \beta.$$

Therefore, $|T_i\mathbf{u}|_0 \leq \|\mathbf{u}\|$ for each $i = 1, 2, \dots, n$. This implies that (ii) holds.

Now Theorem 2.2 guarantees that T has a fixed point $\mathbf{u} \in \overline{\Omega^2}_K \setminus \Omega^1_K$. Thus $\alpha \leq \|\mathbf{u}\| \leq \beta$. Clearly, $\mathbf{u}$ is a positive solution of (1.1) and (1.2). □

Next we employ Theorem 3.1 to establish the existence of positive solutions of the following problem:

$$\begin{cases} (q(t)\phi(p(t)u_1'(t)))' + h_1(t)g^1(\mathbf{u}) = 0, & 0 < t < 1, \\ \dots \\ (q(t)\phi(p(t)u_n'(t)))' + h_n(t)g^n(\mathbf{u}) = 0, & 0 < t < 1. \end{cases} \tag{3.2}$$

We assume the following conditions:

(H_3) $g^i: \mathbb{R}^n_+ \to \mathbb{R}_+$ is continuous with $g^i(\mathbf{u}) > 0$ for $\|\mathbf{u}\| > 0$, $i = 1, 2, \dots, n$.
(H_4) $h_i(t)$: $[0, 1] \to \mathbb{R}_+$ is continuous and $h_i(t) \not\equiv 0$ on any subinterval of [0,1], $i = 1, 2, \dots, n$.

In order to state the result, we introduce the notation

$$g_0^i = \lim_{\|\mathbf{u}\|\to 0}\frac{g^i(\mathbf{u})}{\phi(\|\mathbf{u}\|)}, \quad g_\infty^i = \lim_{\|\mathbf{u}\|\to \infty}\frac{g^i(\mathbf{u})}{\phi(\|\mathbf{u}\|)}, \ \mathbf{u} \in \mathbb{R}^n_+, \ i = 1, 2, \dots, n.$$

$$A_i = \int_0^1 \frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_0^1 h_i(\tau)\mathrm{d}\tau\right)\mathrm{d}s, \quad B_i = \min_{\frac{1}{4}\leq t\leq \frac{3}{4}} \gamma_{h_i}(t)$$

$$A = \max\{A_i, i = 1, 2, \dots, n\}, \quad B = \min\{B_i, i = 1, 2, \dots, n\}.$$

Theorem 3.2. *Suppose that conditions* (H_2) *and* (H_4) *hold. Then problems* (3.2) *and* (1.2) *have at least one positive solution* $\mathbf{u}$ *with* $\mathbf{u}(t) \not\equiv 0$ *for* $t \in (0, 1)$ *if one of the following conditions holds.*

(h_1) $0 \leq g_0^i < \left(\frac{1}{A_i}\right)^{p-1}$ *and* $\left(\frac{1}{B_i}\right)^{p-1} < g_\infty^i \leq \infty, \ i = 1, 2, \dots, n;$
(h_2) $0 \leq g_\infty^i < \left(\frac{1}{A_i}\right)^{p-1}$ *and* $\left(\frac{1}{B_i}\right)^{p-1} < g_0^i \leq \infty, \ i = 1, 2, \dots, n.$

Proof. To see this, we will apply Theorem 3.1 with $f^i(t, \mathbf{u}) = h_i(t)g^i(\mathbf{u}), \ i = 1, 2, \dots, n$. We assume that ($h_1$) holds. The case when ($h_2$) holds is similar.

From the first part of (h_1), there exists $\beta > 0$ such that $g^i(\mathbf{u}) \leq \left(\frac{1}{A_i}\right)^{p-1}\beta$ for $\|\mathbf{u}\| \leq \beta$. Choose $\varphi_i(t) = \left(\frac{1}{A_i}\right)^{p-1} h_i(t)$ for $i = 1, 2, \dots, n$. Fix $i \in \{1, \dots, n\}$. Then

$$f^i(t, \mathbf{u}) = h_i(t)g^i(\mathbf{u}) \leq \left(\frac{1}{A_i}\right)^{p-1}\beta h_i(t) = \beta\varphi_i(t)$$

$$\text{if } 0 \leq t \leq 1 \text{ and } 0 < u_j \leq \beta \text{ for } j \in \{1, \dots, n\}$$

and

$$\int_0^1 \frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_0^1 \varphi_i(\tau)\mathrm{d}\tau\right)\mathrm{d}s$$

$$= \frac{1}{A_i}\int_0^1 \frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_0^1 h_i(\tau)\mathrm{d}\tau\right)\mathrm{d}s = 1.$$

Thus hypothesis (D_2) holds.

From the second part of (h_1), there exists $\alpha > 0$ such that $\alpha > \max\{\rho^{-1}\beta, \beta\}$ and $g^i(\mathbf{u}) \geq \left(\frac{1}{B_i}\right)^{p-1}(\rho\alpha)^{p-1}$ for $\|\mathbf{u}\| \geq \rho\alpha,\ i = 1, 2, \ldots, n$.

Thus $g^i(\mathbf{u}) \geq \left(\frac{1}{B_i}\right)^{p-1}(\rho\alpha)^{p-1}$ for $u_i \geq \rho\alpha,\ i = 1, 2, \ldots, n$. Choose $\psi_i(t) = \left(\frac{1}{B_i}\right)^{p-1}h_i(t)$ for $i = 1, 2, \ldots, n$. Then

$$f^i(t, \mathbf{u}) = h_i(t)g^i(\mathbf{u}) \geq \left(\frac{1}{B_i}\right)^{p-1}(\rho\alpha)^{p-1}h_i(t) = (\rho\alpha)^{p-1}\psi_i(t),$$

$$\frac{1}{4} \leq t \leq \frac{3}{4},\ u_i \geq \rho\alpha,$$

(so in particular for $\rho\alpha \leq u_i \leq \alpha$) and for $\frac{1}{4} \leq t \leq \frac{3}{4}$,

$$\begin{aligned}\gamma_{\psi_i}(t) &= \frac{\rho}{2}\left[\int_{\frac{1}{4}}^t \frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_s^t \psi_i(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right.\\ &\quad \left.+ \int_t^{\frac{3}{4}} \frac{1}{p(s)}\phi^{-1}\left(\frac{1}{q(s)}\int_t^s \psi_i(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right]\\ &= \frac{1}{B_i}\gamma_{h_i}(t).\end{aligned}$$

Thus

$$\inf_{\frac{1}{4}\leq t\leq\frac{3}{4}} \gamma_{\psi_i}(t) = \frac{1}{B_i}\inf_{\frac{1}{4}\leq t\leq\frac{3}{4}} \gamma_{h_i}(t) = 1.$$

This implies that hypothesis (D_1) holds. The result now follows from Theorem 3.1. □

Next we consider the nonlinear eigenvalue problems (1.5) and (1.2). By applying Theorem 3.2, we easily get the following result.

Theorem 3.3. *Suppose that conditions* (H_2) *and* (H_4) *hold. Then problems* (1.5) *and* (1.2) *have at least one positive solution for each*

$$\lambda \in \left(\frac{1}{B^{p-1}\min\limits_{i=1,2,\ldots,n}\{g^i_\infty\}}, \frac{1}{A^{p-1}\max\limits_{i=1,2,\ldots,n}\{g^i_0\}}\right) \tag{3.3}$$

if $1/(B^{p-1}\min_{i=1,2,\ldots,n}\{g_\infty^i\}) < 1/(A^{p-1}\max_{i=1,2,\ldots,n}\{g_0^i\})$. *The same result remains valid for each*

$$\lambda \in \left(\frac{1}{B^{p-1}\min\limits_{i=1,2,\ldots,n}\{g_0^i\}}, \frac{1}{A^{p-1}\max\limits_{i=1,2,\ldots,n}\{g_\infty^i\}}\right) \tag{3.4}$$

if $1/(B^{p-1}\min_{i=1,2,\ldots,n}\{g_0^i\}) < 1/(A^{p-1}\max_{i=1,2,\ldots,n}\{g_\infty^i\})$. *Here we write* $1/g_\alpha^i = 0$ *if* $g_\alpha^i = \infty$ *and* $1/g_\alpha^i = \infty$ *if* $g_\alpha^i = 0$, *where* $\alpha = 0, \infty$.

Proof. We consider the case (3.3). The case (3.4) is similar. If λ satisfies (3.3), then

$$\lambda g_0^i \le \lambda \max_{i=1,2,\ldots,n}\{g_0^i\} < \frac{1}{A^{p-1}} \le \left(\frac{1}{A_i}\right)^{p-1}, \qquad i = 1, 2, \ldots, n,$$

and

$$\lambda g_\infty^i \ge \lambda \min_{i=1,2,\ldots,n}\{g_\infty^i\} > \frac{1}{B^{p-1}} \ge \left(\frac{1}{B_i}\right)^{p-1}, \qquad i = 1, 2, \ldots, n.$$

So Theorem 3.2 applies directly. □

COROLLARY 3.4

Problems (1.5) *and* (1.2) *have at least one positive solution for each* $\lambda \in (0, \infty)$ *if one of the following two conditions holds*:

(i) $g_\infty^i = \infty$ *and* $g_0^i = 0$, $i = 1, 2, \ldots, n$;
(ii) $g_0^i = \infty$ *and* $g_\infty^i = 0$, $i = 1, 2, \ldots, n$.

Finally, it is worth remarking here that, we can apply Theorems 3.2 and 3.3 to study the existence of positive radial solutions for the nonlinear elliptic systems (1.3) and (1.4).

In fact, a radial solution of (1.3) and (1.4) can be considered as a solution of the system

$$\begin{cases} (r^{N-1}\phi(u_1'(r)))' + r^{N-1}k_1(r)g^1(\mathbf{u}) = 0, \\ \ldots \\ (r^{N-1}\phi(u_n'(r)))' + r^{N-1}k_n(r)g^n(\mathbf{u}) = 0, \end{cases} \tag{3.5}$$

$0 < R_1 < r < R_2 < \infty$, with the following boundary condition

$$\mathbf{u}(R_1) = 0, \quad \mathbf{u}(R_2) = 0. \tag{3.6}$$

Applying the change of variables, $r = (R_2 - R_1)t + R_1$, we can transform (3.5) and (3.6) into the form

$$\begin{cases} (q(t)\phi(\xi u_1'(t)))' + h_1(t)g^1(\mathbf{u}) = 0, \quad 0 < t < 1, \\ \ldots \\ (q(t)\phi(\xi u_n'(t)))' + h_n(t)g^n(\mathbf{u}) = 0, \quad 0 < t < 1, \end{cases} \tag{3.7}$$

with boundary condition (1.2), where

$$q(t) = ((R_2 - R_1)t + R_1)^{N-1}, \ \zeta = \frac{1}{R_2 - R_1}$$

and

$$h_i(t) = (R_2 - R_1)((R_2 - R_1)t + R_1)^{N-1} k_i((R_2 - R_1)t + R_1), \ i = 1, 2, \ldots, n,$$

which correspond to problems (1.1) and (1.2) with $p(t) = \xi$ and $f^i(t, \mathbf{u}) = h_i(t) g^i(\mathbf{u}), \ i = 1, 2, \ldots, n.$

Thus, we only need to consider problems (3.7) and (1.2) and we can obtain results similar to those in Theorems 3.2 and 3.3. Here we omit the details.

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