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On the problem of isometry of a hypersurface preserving mean curvature

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Abstract. The problem of determining the *Bonnet hypersurfaces in* R^{n+1}, for $n > 1$, is studied here. These hypersurfaces are by definition those that can be isometrically mapped to another hypersurface or to itself (as locus) by at least one nontrivial isometry preserving the mean curvature. The other hypersurface and/or (the locus of) itself is called *Bonnet associate* of the initial hypersurface.

The orthogonal net which is called *A-net* is special and very important for our study and it is described on a hypersurface. It is proved that, non-minimal hypersurface in R^{n+1} with no umbilical points is a Bonnet hypersurface if and only if it has an A-net.

Keywords. Bonnet hypersurface; Bonnet associate; isometry; mean curvature; preserving; Bonnet curve; A-net.

1. Introduction

The isometry problem or more specifically the isometrical deformation problem of surfaces in a 3-dimensional Euclid space form preserving mean curvature has been studied by a number of mathematicians. One of the first mathematicians who has studied this subject is Bonnet [2]. Bonnet, after pointing out that the isometry problem of a surface preserving mean curvature can not be solved in the general case, showed that all surfaces with constant mean curvature can be isometrically mapped to each other and deformable surfaces with non constant mean curvature are isothermic Weingarten surfaces which can be deformable to the revolution surfaces. Cartan [3] considered the problem as an application of differential forms and after rather long calculations, classified Weingarten surfaces which can be deformable to the revolution surfaces and showed that there are finite number surfaces in each class.

Recently, this problem has been reconsidered. Chern [4], using differential forms, obtained a characterization for an isometrical deformation preserving mean curvature. Roussos [11] who has many works on this subject, using Chern's method, in the general case, obtained a characterization for the isometry preserving mean curvature. Some of the important contributions to this subject are by Voss [15], Bobenko and Either [1], Kenmotsu [8], Colares and Kenmotsu [6], Roussos [12], Xiuxiong and Chia-Kuei [16].

The surface which admits an isometry-preserving mean curvature is called Bonnet surface. One of the Bonnet surfaces which can be isometrically mapped to each other is called

Bonnet associate of the other surface. Similar definitions are valid for the Bonnet hypersurfaces.

Kokubu [10] considered deformable Bonnet hypersurfaces and showed that they consist of Bonnet surfaces in R^3 and Bonnet hypersurfaces in R^4.

Later, Soyuçok [13] examined the problem of determining Bonnet surfaces in R^3 and showed that a surface is a Bonnet surface if and only if a surface has a special isothermal parameter system. Moreover, as an application of this, he obtained Cartan's results and explained geometrically why there are a finite number of surfaces in the classes determined by Cartan.

Soyuçok [14], in another work, examined the problem determining Bonnet hypersurfaces in R^4 and obtained quantities of these hypersurfaces. Furthermore, he showed that a 3-dimensional hypersurface is a Bonnet hypersurface if and only if it has a special orthogonal net.

2. Preliminaries

Let us consider M and M' to be non-minimal Bonnet associate hypersurfaces with no umbilical points in R^{n+1}. Therefore, the ranks of their shape operators are the same and less than 3 [10].

Let the local orthonormal frame fields of M and M' consist of the principal vectors $e_1, e_2, \ldots, e_n$ and $e'_1, e'_2, \ldots, e'_n$ where corresponding principal curvatures are $k_1, k_2, (k_1 \neq k_2), k_3(= 0), k_4(= 0), \ldots, k_n(= 0)$ and $k'_1, k'_2, (k'_1 \neq k'_2), k'_3(= 0), k'_4(= 0), \ldots, k'_n(= 0)$ respectively. Since, the null spaces of M and M' are the same at the corresponding points [10],

$$\begin{aligned} &\operatorname{span}\{e_1, e_2\} = \operatorname{span}\{e'_1, e'_2\}, \\ &\operatorname{span}\{e_3, e_4, \ldots, e_n\} = \operatorname{span}\{e'_3, e'_4, \ldots, e'_n\}. \end{aligned}$$

Because M and M' are associate to each other, the first and the second of the curvature lines of M' correspond to the orthogonal curves on M. These curves are called Bonnet curves of M.

Let E_1 and E_2 be the unit vectors of the bisectors of Bonnet curves with curvature lines (first and second) of M. We take $\{E_1, E_2, E_3 = e_3, \ldots, E_n = e_n\}$ as the orthonormal frame field for M and we can also choose $\{E_1, E_2, E_3, \ldots, E_n\}$ as the orthonormal frame field for M'.

Let us denote the shape operator and the Riemannian connection of M by L and D respectively. Then

$$L = \frac{1}{2}\begin{pmatrix} k_1(1+\cos\theta) + k_2(1-\cos\theta) & (k_1 - k_2)\sin\theta & 0 & \ldots & 0 \\ (k_1 - k_2)\sin\theta & k_1(1-\cos\theta) + k_2(1+\cos\theta) & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \tag{2.1}$$

can be written to the frame field $\{E_1, E_2, E_3, \ldots, E_n\}$ where θ is the angle between e_1 and e'_1.

Because the angle between e_1' and E_1 is $-\frac{\theta}{2}$, the shape operator of M' with respect to the same frame field can be written as

$$L' = \frac{1}{2}\begin{pmatrix} k_1(1+\cos\theta)+k_2(1-\cos\theta) & -(k_1-k_2)\sin\theta & 0 & \dots & 0 \\ -(k_1-k_2)\sin\theta & k_1(1-\cos\theta)+k_2(1+\cos\theta) & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{2.2}$$

We shall use the following convention on the ranges of indices in this work:

$$1 \le i, j, k, \ldots \le 2,$$
$$3 \le p, q, r, \ldots \le n,$$
$$1 \le A, B, C, \ldots \le n.$$

3. Fundamental equations

3.1 *Codazzi equations*

Let us define the functions k, $\bar{k}$ and t as follows:

$$k = \frac{1}{2}\{k_1(1+\cos\theta) + k_2(1-\cos\theta)\} = H + J\cos\theta,$$
$$\bar{k} = \{k_1(1-\cos\theta) + k_2(1+\cos\theta)\} = H - J\cos\theta,$$
$$t = \frac{1}{2}(k_1 - k_2)\sin\theta = J\sin\theta, \tag{3.1}$$

where H is a mean curvature and

$$H = \frac{k_1 + k_2}{2} \neq 0, \quad J = \frac{k_1 - k_2}{2} \neq 0. \tag{3.2}$$

The Codazzi equation [9] is given by

$$(D^L_{E_i})E_j = (D^L_{E_j})E_i. \tag{3.3}$$

Accordingly, using the shape operator of M and the above definitions we can write the Codazzi equations of M in the form:

$$E_2(k) + (k - \bar{k})w_2^1(E_1) = E_1(t) + 2tw_1^2(E_2),$$
$$E_1(\bar{k}) + (\bar{k} - k)w_1^2(E_2) = E_2(t) + 2tw_2^1(E_1),$$
$$\bar{k}w_2^p(E_1) - kw_1^p(E_2) = t[w_2^p(E_2) - w_1^p(E_1)],$$
$$E_p(k) + kw_p^1(E_1) = t[2w_1^2(E_p) - w_p^2(E_1)],$$
$$E_p(t) + tw_p^1(E_1) = \bar{k}[w_1^2(E_p) - w_p^2(E_1)] - kw_1^2(E_p),$$

$$
\begin{aligned}
&E_p(t) + t w_p^2(E_2) = k[w_2^1(E_p) - w_p^1(E_2)] - \bar{k} w_2^1(E_p),\\
&E_p(\bar{k}) + \bar{k} w_p^2(E_2) = t[2w_2^1(E_p) - w_p^1(E_2)],\\
&k w_1^p(E_p) = -t w_2^p(E_p),\\
&k w_1^q(E_p) = -t w_2^q(E_p),\\
&\bar{k} w_2^p(E_p) = -t w_1^p(E_p),\\
&k[w_p^1(E_q) - w_q^1(E_p)] = -t[w_p^2(E_q) - w_q^2(E_p)],\\
&\bar{k}[w_p^2(E_q) - w_q^2(E_p)] = -t[w_p^1(E_q) - w_q^1(E_p)],
\end{aligned}
\tag{3.4}
$$

where w_B^A are the connection forms.

Because of (2.1), (2.2) and (3.1), in order to find the Codazzi equations of M' it is enough to replace t by $-t$ in (3.4). Therefore, from the Codazzi equations of M and M' the following equations are obtained:

$$
\begin{aligned}
&w_1^p(E_p) = w_2^p(E_p) = 0,\\
&w_1^q(E_p) = w_2^q(E_p) = 0,\\
&w_p^2(E_2) = w_p^1(E_1),\\
&w_p^1(E_2) = -w_p^2(E_1) = 2w_2^1(E_p),\\
&H w_p^1(E_2) = 0,\\
&E_p(k) + k T_p = 0,\\
&E_p(\bar{k}) + \bar{k} T_p = 0,\\
&E_p(t) + t T_p = 0,\\
&E_2(k) + (k - \bar{k})h = 0,\\
&E_1(\bar{k}) + (\bar{k} - k)\bar{h} = 0,\\
&E_1(t) + 2t\bar{h} = 0,\\
&E_2(t) + 2th = 0,
\end{aligned}
\tag{3.5}
$$

where the functions T_p, h and $\bar{h}$ are defined as

$$
\begin{aligned}
&w_p^2(E_2) = w_p^1(E_1) = T_p,\\
&h = w_2^1(E_1), \quad \bar{h} = w_1^2(E_2).
\end{aligned}
\tag{3.6}
$$

Since we assume that our hypersurface is a non-minimal one, $H \neq 0$ and from the fifth equation of (3.5) we get

$$w_p^1(E_2) = 0.$$

From the fourth equation of (3.5),

$$w_p^1(E_2) = w_p^2(E_1) = w_2^1(E_p) = 0 \tag{3.7}$$

are obtained.

Therefore, we have the following lemma.

Lemma 1. *A Bonnet hypersurface must satisfy the Codazzi equations*,

$$
\begin{aligned}
&E_p(k) + kT_p = 0,\\
&E_p(\bar{k}) + \bar{k}T_p = 0,\\
&E_p(t) + tT_p = 0,\\
&E_2(k) + (k - \bar{k})h = 0,\\
&E_1(\bar{k}) + (\bar{k} - k)\bar{h} = 0,\\
&E_1(t) + 2t\bar{h} = 0,\\
&E_2(t) + 2th = 0.
\end{aligned} \tag{3.8}
$$

3.2 *Gauss equations*

The Gauss equations of M hypersurface [9] are given by the equations

$$
\begin{aligned}
&D_{E_A}(D_{E_B}^{E_C}) - D_{E_B}(D_{E_A}^{E_C}) - D_{[E_A,E_B]}^{E_C}\\
&\quad = \langle L(E_B), E_C\rangle L(E_A) - \langle L(E_A), E_C\rangle L(E_B).
\end{aligned}
$$

Using (3.7) and (3.8) in the Gauss equations of M, we have the following lemma.

Lemma 2. *The Gauss equations of M can be written as follows*:

$$
\begin{aligned}
&E_1(\bar{h}) + E_2(h) + \sum_{p=3}^{n} T_p^2 + h^2 + \bar{h}^2 = t^2 - k\bar{k},\\
&E_p(h) + T_p h = 0,\\
&E_p(\bar{h}) + T_p\bar{h} = 0,\\
&E_p(T_q) - w_q^r(E_p)T_r + T_pT_q = 0,\\
&E_1(T_p) - w_p^q(E_1)T_q = 0,\\
&E_2(T_p) - w_p^q(E_2)T_q = 0,\\
&E_p(w_r^s(E_q)) + w_r^t(E_q)w_t^s(E_p) - E_q(w_r^s(E_p))\\
&\quad - w_r^t(E_p)w_t^s(E_q) - w_p^t(E_q)w_r^s(E_t) + w_q^t(E_p)w_r^s(E_t) = 0.
\end{aligned} \tag{3.9}
$$

3.3 *Structure equations*

Let $\{w^1, w^2, \ldots, w^n\}$ be the dual frame field of $\{E_1, E_2, \ldots, E_n\}$. Thus, by means of (3.6) and (3.7)

$$
\begin{aligned}
&w_2^1 = hw^1 - \bar{h}w^2,\\
&w_p^1 = T_p w^1,\\
&w_p^2 = T_p w^2,\\
&w_p^q = w_p^q(E_A)w^A
\end{aligned}
$$

are obtained. Accordingly, the structure equations $\mathrm{d}w^L = -w^L_J \wedge w^J$ can be written as follows:

$$\begin{aligned}
\mathrm{d}w^1 &= -hw^1 \wedge w^2 - T_p w^1 \wedge w^p,\\
\mathrm{d}w^2 &= \bar{h}w^1 \wedge w^2 - T_p w^2 \wedge w^p,\\
\mathrm{d}w^q &= -w^q_p(E_A)w^A \wedge w^p.
\end{aligned} \tag{3.10}$$

Now, we consider the differential $\mathrm{d}f = f_A w^A$ of f with respect to the dual frame field $\{w^1, w^2, \ldots, w^n\}$. Using (3.7) and (3.10) in $\mathrm{d}^2 f = 0$, we obtain the compatibility equations

$$\begin{aligned}
&f_{12} + hf_1 - f_{21} - \bar{h}f_2 = 0,\\
&f_{1p} + T_p f_1 - f_{p1} + f_q w^q_p(E_1) = 0,\\
&f_{2p} + T_p f_2 - f_{p2} + f_q w^q_p(E_2) = 0,\\
&f_{rp} + T_p f_r - f_{pr} + f_q w^q_p(E_r) = 0.
\end{aligned} \tag{3.11}$$

The sixth and seventh equations of the system (3.7) can be written in the form

$$(\ln|t|)_1 = -2\bar{h}, \quad (\ln|t|)_2 = -2h. \tag{3.12}$$

So, using the first equation of (3.11),

$$h_1 = \bar{h}_2. \tag{3.13}$$

4. New coordinates

Let g_{AB} be the components of the first fundamental tensor. The form w^A can be written as

$$w^A = \sqrt{g_{AA}}\,\mathrm{d}x^A \quad (\text{not sum in } A)$$

in the suitable system of local coordinates $(x^1, x^2, \ldots, x^n)$ [5]. So, from the structure equations (3.10), we get

$$w^q_p(E_1) = 0, \quad w^q_p(E_2) = 0, \tag{4.1}$$

$$w^A_r(E_p) = 0, \tag{4.2}$$

$$g_{pp} = c^p(x^p), \tag{4.3}$$

$$h = \frac{(\sqrt{g_{11}})_{x^2}}{\sqrt{g_{11}g_{22}}} = (\ln\sqrt{g_{11}})_2, \quad \bar{h} = \frac{(\sqrt{g_{22}})_{x^1}}{\sqrt{g_{11}g_{22}}} = (\ln\sqrt{g_{22}})_1, \tag{4.4}$$

$$T_p = \frac{(\sqrt{g_{11}})_{x^p}}{\sqrt{g_{11}g_{pp}}} = \frac{(\sqrt{g_{22}})_{x^p}}{\sqrt{g_{22}g_{pp}}}, \tag{4.5}$$

where

$$f_{x^A} = \frac{\partial f}{\partial x^A}. \tag{4.6}$$

Because of (4.4) in (3.12), equation (3.13) is equivalent to

$$\left(\ln \frac{g_{11}}{g_{22}}\right)_{x^1x^2} = 0.$$

Therefore, we have

$$\frac{g_{11}}{a(x^1)} = \frac{g_{22}}{b(x^2)}, \tag{4.7}$$

where $a(x^1)$ and $b(x^2)$ are arbitrary functions. Let us introduce the new coordinates by means of the following scaling transformation:

$$\bar{x}^1 = \int \sqrt{a(x^1)}\mathrm{d}x^1, \quad \bar{x}^2 = \int \sqrt{b(x^2)}\mathrm{d}x^2, \quad \bar{x}^p = \int \sqrt{c^p(x^p)}\mathrm{d}x^p.$$

Then, from (4.7) and (4.3) we get

$$g_{11} = g_{22}, \quad g_{pp} = 1. \tag{4.8}$$

Here we have again denoted the new coordinates by $x^1, x^2, \ldots, x^n$. Accordingly (4.5) reduces to

$$T_p = \frac{(\sqrt{g_{11}})_{x^p}}{\sqrt{g_{11}}} = \frac{(\sqrt{g_{22}})_{x^p}}{\sqrt{g_{22}}}. \tag{4.9}$$

Because of (4.1) the fifth and sixth equations of the Gauss equations (3.9) take the forms

$$E_1(T_p) = 0 \quad \text{and} \quad E_2(T_p) = 0.$$

So, T_p does not depend on x^1 and x^2.

Moreover, using (4.2), the fourth equation of (3.8) can be written as

$$E_p(T_q) + T_pT_q = 0 \tag{4.10}$$

and so

$$E_p(T_p) + (T_p)^2 = 0. \tag{4.11}$$

Now, let us consider the following two cases:

Case 1. Some T_p's are not zero.
Case 2. All T_p's are zero.

Case 1. Assume that some T_p's are not zero. Let the number of non zero T_p's be $m-2$. Renewing the indices p of non zero T_p's we can choose them as $3, 4, \ldots, r$. Hence we can take the non zero T_p's to be $T_3, T_4, \ldots, T_r$ $(3 \leq r \leq n)$.

Solving the equations (4.10) and (4.11), we find

$$T_p = \frac{C^p}{C^3x^3 + \cdots + C^px^p + \cdots + C^rx^r}, \tag{4.12}$$

where $C^3, \ldots, C^r$ are non zero constants of integration.

Accordingly, from (4.9) we have

$$g_{11} = g_{22} = (C^3x^3 + C^4x^4 + \cdots + C^rx^r)^2\xi(x^1, x^2), \tag{4.13}$$

where $\xi(x^1, x^2)$ is an arbitrary function.

By means of (4.4) and (4.13), the second and the third Gauss equations of (3.9) are automatically satisfied

Using (4.8) and (4.9), let us solve the third equation of (3.7) and then let us solve eqs. (3.12), (4.4) and (4.13). Thus

$$t = \frac{C}{(C^3x^3 + \cdots + C^rx^r)\xi(x^1, x^2)}, \tag{4.14}$$

where C is a constant.

On the other hand, using (4.8) and (4.12), solutions of the first and the second equations of the system (3.7) are obtained as

$$k = \frac{\Psi(x^1, x^2)}{(C^3x^3 + \cdots + C^rx^r)}, \quad \bar{k} = \frac{\bar{\Psi}(x^1, x^2)}{(C^3x^3 + \cdots + C^rx^r)}. \tag{4.15}$$

Now, let us find the mean curvature H. Since $H = \frac{k+\bar{k}}{2}$, according to (3.1) we have

$$H = \frac{\mathfrak{H}(x^1, x^2)}{(C^3x^3 + \cdots + C^rx^r)}. \tag{4.16}$$

Moreover, from (3.1), (4.15) and (4.16), we have

$$\theta = \theta(x^1, x^2) \tag{4.17}$$

and

$$J = \frac{\mathfrak{J}(x^1, x^2)}{(C^3x^3 + \cdots + C^rx^r)}. \tag{4.18}$$

All of Codazzi's equations (3.8) are satisfied except the fourth and the fifth equation and all of Gauss' equations (3.8) are satisfied except the first equation. Now, we have to consider these three equations.

By using (3.1), (4.4), (4.6), (4.14), (4.16), (4.17) and (4.18), the fourth and the fifth equations of Codazzi's equations (3.8) can be written as follows:

$$\frac{\mathfrak{H}_{x^2}}{\mathfrak{J}} = \frac{\theta_{x^2}}{\sin\theta}, \quad \frac{\mathfrak{H}_{x^1}}{\mathfrak{J}} = -\frac{\theta_{x^1}}{\sin\theta}. \tag{4.19}$$

The first equation of Gauss's equations (3.8) is reduced to the following equation:

$$\begin{aligned} &|\mathfrak{J}|\sin\theta\{(\ln|\mathfrak{J}|\sin\theta)_{x^1x^1} + (\ln|\mathfrak{J}|\sin\theta)_{x^2x^2}\} - 2\{(C^3)^2 + \cdots + (C^r)^2\} \\ &\quad = 2(\mathfrak{H}^2 - \mathfrak{J}^2). \end{aligned} \tag{4.20}$$

5. The fundamental theorem

Let us denote the components of the second fundamental tensor with b_{AB}. They are given by [7]

$$b_{AB} = \left\langle L\left(\frac{\partial}{\partial x^A}\right), \frac{\partial}{\partial x^B}\right\rangle.$$

Thus, we get

$$b_{11} = g_{11}k, \quad b_{12} = \sqrt{g_{11}g_{22}}t, \quad b_{22} = g_{22}\bar{k}, \quad b_{iq} = b_{pq} = 0.$$

By using (4.13), (4.14) and (4.15), non zero components can be written as

$$b_{11} = (C^3x^3 + \cdots + C^rx^r)\frac{(\mathfrak{H} + \mathfrak{J}\sin\theta)}{|\mathfrak{J}|\sin\theta},$$

$$b_{22} = (C^3x^3 + \cdots + C^rx^r)\frac{(\mathfrak{H} - \mathfrak{J}\sin\theta)}{|\mathfrak{J}|\sin\theta},$$

$$b_{12} = C(C^3x^3 + \cdots + C^rx^r).$$

We can make a suitable scale transformation $C = \in = sgn\mathfrak{J}$.

Thus, the components of the first and the second fundamental tensors of Bonnet hypersurface M in an orthogonal coordinate system $x^1, x^2, \ldots, x^n$ are obtained as

$$g_{11} = g_{22} = (C^3x^3 + \cdots + C^rx^r)^2\frac{1}{|\mathfrak{J}|\sin\theta}, \quad g_{pp} = 1$$

$$b_{11} = (C^3x^3 + \cdots + C^rx^r)\frac{(\mathfrak{H} + \mathfrak{J}\sin\theta)}{|\mathfrak{J}|\sin\theta},$$

$$b_{22} = (C^3x^3 + \cdots + C^rx^r)\frac{(\mathfrak{H} - \mathfrak{J}\sin\theta)}{|\mathfrak{J}|\sin\theta},$$

$$b_{12} = \in (C^3x^3 + \cdots + C^rx^r), \quad \in = \pm 1$$

$$b_{iq} = b_{pq} = 0, \quad \mathfrak{H} = \mathfrak{H}(x^1, x^2), \ldots, \mathfrak{J} = \mathfrak{J}(x^1, x^2). \tag{5.1}$$

Because of (2.1) and (2.2), the components of the first and the second fundamental tensors of associate Bonnet hypersurface M' can be written in the form:

$$g'_{11} = g'_{22} = (C^3x^3 + \cdots + C^rx^r)^2\frac{1}{|\mathfrak{J}|\sin\theta}, \quad g'_{pp} = 1$$

$$b'_{11} = (C^3x^3 + \cdots + C^rx^r)\frac{(\mathfrak{H} + \mathfrak{J}\sin\theta)}{|\mathfrak{J}|\sin\theta},$$

$$b'_{22} = (C^3x^3 + \cdots + C^rx^r)\frac{(\mathfrak{H} - \mathfrak{J}\sin\theta)}{|\mathfrak{J}|\sin\theta},$$

$$b'_{12} = - \in (C^3x^3 + \cdots + C^rx^r), \quad \in = \pm 1$$

$$b'_{iq} = b'_{pq} = 0. \tag{5.2}$$

It is seen that all the fundamental quantities of M and M' are the same, except b_{12} and b'_{12}. There is the relation $b_{12} = -b'_{12}$ between b_{12} and b'_{12}.

Case 2. Let us consider the case where all $T_p = 0$. Then, it is easily seen that in order to obtain the quantities in this case, it is sufficient to replace $C^3x^3 + \cdots + C^r x^r$ by 1 and $(C^3)^2 + \cdots + (C^r)^2$ by 0 in the quantities of Case 1.

Thus, in this case, the fundamental quantities of M and M' in an orthogonal coordinate system $x^1, x^2, \ldots, x^n$ are respectively given by

$$
\begin{aligned}
g_{11} &= g_{22} = \frac{1}{|\mathfrak{J}| \sin\theta}, \quad g_{pp} = 1\\
b_{11} &= \frac{(\mathfrak{H} + \mathfrak{J}\sin\theta)}{|\mathfrak{J}| \sin\theta},\\
b_{22} &= \frac{(\mathfrak{H} - \mathfrak{J}\sin\theta)}{|\mathfrak{J}| \sin\theta},\\
b_{12} &= \in, \quad \in = \pm 1\\
b_{iq} &= b_{pq} = 0, \quad \mathfrak{H} = \mathfrak{H}(x^1, x^2), \ldots, \quad \mathfrak{J} = \mathfrak{J}(x^1, x^2)
\end{aligned}
\tag{5.3}
$$

and

$$
\begin{aligned}
g'_{11} &= g'_{22} = \frac{1}{|\mathfrak{J}| \sin\theta}, \quad g'_{pp} = 1\\
b'_{11} &= \frac{(\mathfrak{H} + \mathfrak{J}\sin\theta)}{|\mathfrak{J}| \sin\theta},\\
b'_{22} &= \frac{(\mathfrak{H} - \mathfrak{J}\sin\theta)}{|\mathfrak{J}| \sin\theta},\\
b'_{12} &= - \in, \quad \in = \pm 1\\
b'_{iq} &= b'_{pq} = 0.
\end{aligned}
\tag{5.4}
$$

DEFINITION 1

If the fundamental quantities in an orthogonal coordinate system $x^1, x^2, \ldots, x^n$ are in the form

$$g_{11} = g_{22}, \quad g_{pp} = 1, \quad b_{12} = \in (C^3x^3 + \cdots + C^r x^r), \quad b_{iq} = b_{pq} = 0$$

or

$$g_{11} = g_{22}, \quad g_{pp} = 1, \quad b_{12} = \in, \quad b_{iq} = b_{pq} = 0,$$

where C^p are constants. The net which consists of this orthogonal coordinate system is called A-net.

Therefore, we have the following theorem.

Theorem 1 (*The Fundamental Theorem*)**.** *A non-minimal hypersurface in R^{n+1} with no umbilical points is a Bonnet hypersurface if and only if it has an A-net. Moreover, the fundamental quantities of the Bonnet hypersurface and its Bonnet associate are given by* (4.21) *and* (4.22) *respectively or* (4.23) *and* (4.24) *respectively.*

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