Proc. Indian Acad. Sci. (Math. Sci.) Vol. 117, No. 1, February 2007, pp. 13–30.
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Khinchin's inequality, Dunford–Pettis and compact operators on the space $C([0, 1], X)$

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MS received 10 November 2005

Abstract. We prove that if X, Y are Banach spaces, Ω a compact Hausdorff space and $U: C(\Omega, X) \to Y$ is a bounded linear operator, and if U is a Dunford–Pettis operator the range of the representing measure $G(\Sigma) \subseteq DP(X, Y)$ is an uniformly Dunford–Pettis family of operators and $\|G\|$ is continuous at $\emptyset$. As applications of this result we give necessary and/or sufficient conditions that some bounded linear operators on the space $C([0, 1], X)$ with values in c_0 or l_p, $(1 \le p < \infty)$ be Dunford–Pettis and/or compact operators, in which, Khinchin's inequality plays an important role.

Keywords. Banach spaces of continuous functions; tensor products; operator ideals; p-summing operators.

Let Ω be a compact Hausdorff space, X a Banach space, and $C(\Omega, X)$ the Banach space of continuous X-valued functions on Ω under the uniform norm and $C(\Omega)$ when X is the scalar field. It is well-known (see §1 of [1], Theorem 2.2 of [3] (Representation Theorem) or Theorem (Dinculeanu-Singer), p. 182 of [6]) that if Y is a Banach space then any bounded linear operator $U: C(\Omega, X) \to Y$ has a finitely additive vector measure $G: \Sigma \to L(X, Y^{**})$, where Σ is the σ-field of Borel subsets of Ω, such that $y^*U(f) = \int_\Omega f \mathrm{d}G_{y^*}$, $f \in C(\Omega, X)$, $y^* \in Y^*$. The measure G is called the representing measure of U.

Also, for a bounded linear operator $U: C(\Omega, X) \to Y$ we can associate in a natural way two bounded linear operators $U^{\#}: C(\Omega) \to L(X, Y)$ and $U_{\#}: X \to L(C(\Omega), Y)$ defined by $(U^{\#}\varphi)(x) = U(\varphi \otimes x)$ and $(U_{\#}x)(\varphi) = U(\varphi \otimes x)$, where for $\varphi \in C(\Omega)$, $x \in X$ we denote $(\varphi \otimes x)(\omega) = \varphi(\omega)x$.

For a σ-algebra $\Sigma \subseteq \mathcal{P}(S)$, X a Banach space and a vector measure $G: \Sigma \to X$, we denote $\tilde{G}(E) = \sup\{\|G(A)\| | A \in \Sigma, A \subseteq E\}$ the quasivariation of G, by $|G|$ and $\|G\|$ the variation and semivariation of G and we use the fact that $\tilde{G}(E) \le \|G\|(E) \le 4\tilde{G}(E)$ for any $E \in \Sigma$ (see chapter I, Proposition 11, p. 4 of [6]).

We denote by $B(\Sigma, X)$ the space of all totally measurable functions endowed with the supnorm.

Also, for $[0, 1]$ we denote by Σ the σ-field of Borel subsets, $\mu: \Sigma \to [0, 1]$ is the Lebesgue measure and $(r_n)_{n\in\mathbb{N}}$ is the sequence of Rademacher functions.

If $\nu \in rcabv(\Sigma)$, $f: [0, 1] \to \mathbb{K}$ is ν-integrable and $\alpha: \Sigma \to \mathbb{K}$ is defined by $\alpha(E) = \int_E f(t)\mathrm{d}\nu(t)$, then $\int_0^1 |f(t)|\mathrm{d}|\nu|(t) = |\alpha|([0, 1])$ and $\tilde{\alpha}([0, 1]) \le \int_0^1 |f(t)|\mathrm{d}|\nu|(t) \le 4\tilde{\alpha}([0, 1])$.

If $G: \Sigma \to L(X, Y)$ is a vector measure we denote by $\|G\|$ the semivariation of G defined by $\|G\|(E) = \sup\{|G_{y^*}|(E) | \|y^*\| \le 1\}$, $E \in \Sigma$, where $G_{y^*}(E) = \langle G(E)x, y^*\rangle$ and we say that the semivariation $\|G\|$ is continuous at $\emptyset$ if $\|G\|(E_k) \to 0$ for $E_k \searrow \emptyset$,

$(E_k)_{k\in\mathbb{N}} \subset \Sigma$. As is well-known, $\|G\|$ is continuous at $\emptyset$ if and only if there exists $\alpha \geq 0$ a Borel measure on Σ such that $\lim_{\alpha(E)\to 0} \|G(E)\| = 0$. Also, $|G|$ is the variation of G and for $x \in X$ we write $G_x \colon \Sigma \to Y$ defined by $G_x(E) = G(E)(x)$ and if $\lambda \colon \Sigma \to X^*$ is a vector measure for $x \in X$ we write $\lambda x \colon \Sigma \to \mathbb{K}$ defined by $(\lambda x)(E) = \lambda(E)(x)$.

As is well-known, (see chapter 2, p. 32 of [5]) if X is a Banach space, $1 \leq p < \infty$ and $(x_n^*)_{n\in\mathbb{N}} \subseteq X^*$ is such that for any $x \in X$ the series $\sum_{n=1}^{\infty} |x_n^*(x)|^p$ is convergent. Then

$$w_p(x_n^* | n \in \mathbb{N}) = \sup_{\|x\|\leq 1} \left(\sum_{n=1}^{\infty} |x_n^*(x)|^p \right)^{\frac{1}{p}} < \infty$$

and we denote by $w_p(X^*)$ the set of all such sequences.

Observe that if $p = 1$, then $(x_n^*)_{n\in\mathbb{N}} \in w_1(X^*)$ if and only if $\sum_{n=1}^{\infty} x_n^*$ is a weakly Cauchy series in X^*.

We recall that if X and Y are Banach spaces, a bounded linear operator $U \colon X \to Y$ is called Dunford–Pettis operator if and only if for any $x_n \to 0$ *weak* it follows that $U(x_n) \to 0$ in norm. We denote by $DP(X, Y)$ the space of all the Dunford–Pettis operators from X into Y. A Banach space has the Schur property if the identity operator is Dunford–Pettis.

We also need the following characterization of *weak* convergence in a $C(\Omega, X)$ space which will be used later without an explicit reference. If $(f_n)_{n\in\mathbb{N}} \subseteq C(\Omega, X)$, then $f_n \to 0$ *weak* if and only if $\sup_{n\in\mathbb{N},\omega\in\Omega} \|f_n(\omega)\| < \infty$ and $f_n(\omega) \to 0$ *weak* for any $\omega \in \Omega$ (see Theorem 2 of [2]).

For c_0 or l_p with $1 \leq p \leq \infty$ we denote by e_n the standard unit vectors in these spaces.

All notations and notions used and not defined in this paper are either standard or can be found in [5] or [6].

In Theorem 3.1 of [10] it is proved that if $U \colon C(\Omega, X) \to Y$ is a Dunford–Pettis operator, then the representing measure has the property that $G(E) \in DP(X, Y)$ for any $E \in \Sigma$ and $\|G\|$ is continuous at $\emptyset$ and that this condition is necessary and sufficient if and only if X has the Schur property.

We will prove in theorem 4 below, that Dunford–Pettis operators on $C(\Omega, X)$ satisfies a much stronger condition. In order to prove this result we introduce the following notion.

Let X be a Banach space and $(Y_i)_{i\in I}$ a family of Banach spaces.

A family $\{U_i \in L(X, Y_i) | i \in I\}$ is said to be *uniformly Dunford–Pettis family of operators* if and only if for any $x_n \to 0$ *weak* it follows that $\sup_{i\in I} \|U_i(x_n)\| \to 0$.

When there is some risk of confusion we write $\|\cdot\|_i$ for the norm in the Banach space Y_i.

In the sequel we give necessary and sufficient conditions that a sequence of bounded linear operators be an uniformly Dunford–Pettis family.

PROPOSITION 1

Let X be a Banach space, $(Y_n)_{n\in\mathbb{N}}$ a sequence of Banach spaces and $U_n \in L(X, Y_n)$ for any $n \in \mathbb{N}$. The following assertions are equivalent:

(i) $(U_n)_{n\in\mathbb{N}}$ *is an uniformly Dunford–Pettis family.*
(ii) $U_n \in DP(X, Y_n)$ *for any* $n \in \mathbb{N}$ *and for any sequence* $x_n \to 0$ *weak it follows that* $\|U_n(x_n)\|_n \to 0$ *in norm i.e. the diagonal sequence of the matrix* $(U_n(x_k))_{n,k\in\mathbb{N}}$ *is null convergent.*
(iii) $\sup_{n\in\mathbb{N}} \|U_n\|_n < \infty$ *and the operator* $U \colon X \to l_\infty(Y_n | n \in \mathbb{N})$ *defined by* $U(x) = (U_n(x))_{n\in\mathbb{N}}$ *is Dunford–Pettis.*

Proof.
(i) or (iii)⇒(ii). It is trivial.
(ii)⇒(i). Indeed, it is easy to see that from (ii) it follows that for any sequence $x_n \to 0$ weak and any two subsequences $(k_n)_{n\in\mathbb{N}}$ and $(p_n)_{n\in\mathbb{N}}$ of $\mathbb{N}$ it follows that $\|U_{k_n}(x_{p_n})\|_{k_n} \to 0$. Since U_k is a Dunford–Pettis operator for any $k \in \mathbb{N}$, we have $\lim_{n\to\infty} \|U_k(x_n)\|_k = 0$. Now using the well-known fact that if $(a_{nk})_{n,k\in\mathbb{N}} \subseteq [0,\infty)$ is a double indexed sequence such that for any $k \in \mathbb{N}$ we have $\lim_{n\to\infty} a_{kn} = 0$, then $\lim_{n\to\infty} a_{kn} = 0$ uniformly in $k \in \mathbb{N}$ if and only if for any two subsequences $(k_n)_{n\in\mathbb{N}}$ and $(p_n)_{n\in\mathbb{N}}$ of $\mathbb{N}$ it follows that $\lim_{n\to\infty} a_{k_n p_n} = 0$, we deduce $\sup_{k\in\mathbb{N}} \|U_k(x_n)\|_k \to 0$, i.e. (ii).
(ii)⇒(iii). Let $c_0^{\text{weak}}(X) = \{(x_n)_{n\in\mathbb{N}} \subseteq X | x_n \to 0 \text{ weak}\}$ which is a linear space for the natural operations for addition and scalar multiplication and a Banach space for the norm $\|(x_n)_{n\in\mathbb{N}}\| = \sup_{n\in\mathbb{N}} \|x_n\|$. Then (ii) affirms that the mapping $h\colon c_0^{\text{weak}}(X) \to c_0(Y_n | n \in \mathbb{N})$ defined by $h((x_n)_{n\in\mathbb{N}}) = (U_n(x_n))_{n\in\mathbb{N}}$ takes its values in $c_0(Y_n | n \in \mathbb{N})$ and, by an easy application of the closed graph theorem, h is bounded linear. Then for any $n \in \mathbb{N}$ and $x \in X$ we have $\|U_n(x)\|_n = \|h(0,\ldots,0,x,0,\ldots)\| \le \|h\|\|x\|$ i.e. the family $(U_n)_{n\in\mathbb{N}}$ is pointwise bounded and thus uniformly bounded, (by the uniform boundedness principle) i.e. $\sup_{n\in\mathbb{N}} \|U_n\|_n < \infty$. Then the operator U in (iii) is well-defined, bounded linear and by the equivalence between (i) and (ii), it follows that U is Dunford–Pettis.

As a consequence, from Proposition 1 we give a necessary and sufficient condition that an operator with values in c_0 or l_∞ be Dunford–Pettis, completing a result from Exercise 4, p. 114 of [4]. Probably, this result is well-known, but we do not know a reference.

COROLLARY 2

Let X be a Banach space, $(x_n^*)_{n\in\mathbb{N}} \subseteq X^*$ such that either $(x_n^*)_{n\in\mathbb{N}}$ is bounded or, $x_n^* \to 0$ weak and $U\colon X \to l_\infty$ or c_0 defined by $U(x) = (x_n^*(x))_{n\in\mathbb{N}}$.*

Then the following assertions are equivalent:

(i) *U is a Dunford–Pettis operator.*
(ii) *For any sequence $x_n \to 0$ weak it follows that $x_n^*(x_n) \to 0$.*

In the next proposition the point (a) is an extension of the implication (ii) ⇒ (i) in Theorem 3.1 in [10].

PROPOSITION 3

Let X be a Banach space, $(Y_n)_{n\in\mathbb{N}}$ a sequence of Banach spaces and $T_n, V_n \in L(X, Y_n)$ two sequences with $\sup_{n\in\mathbb{N}} \|T_n\|_n < \infty$ and $\sup_{n\in\mathbb{N}} \|V_n\|_n < \infty$. Let $U\colon C([0,1],X) \to c_0(Y_n | n \in \mathbb{N})$ be the operator defined by

$$U(f) = \left(\int_0^1 (T_n(f(t))\sin 2\pi nt + V_n(f(t))\cos 2\pi nt)\mathrm{d}t\right)_{n\in\mathbb{N}}.$$

Then

(a) *U is Dunford–Pettis ⇔ $(T_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ are uniformly Dunford–Pettis.*
(b) *U is compact ⇔ T_n and V_n are compact for any $n \in \mathbb{N}$ and $\|T_n\| \to 0$ and $\|V_n\| \to 0$.*

Proof. The fact that U takes its values in $c_0(Y_n|n \in \mathbb{N})$ follows from hypothesis, the well-known fact that for any $f \in C[0,1]$ we have $\int_0^1 f(t)\sin 2\pi nt\mathrm{d}t \to 0$ and $\int_0^1 f(t)\cos 2\pi nt\mathrm{d}t \to 0$ and the density of $C[0,1]\otimes X$ in $C([0,1],X)$. For any $n\in\mathbb{N}$, let $S_n\colon C([0,1],X)\to Y_n$ be defined by

$$S_n(f)=\int_0^1 (T_n(f(t))\sin 2\pi nt+V_n(f(t))\cos 2\pi nt)\mathrm{d}t.$$

(a) Suppose U is a Dunford–Pettis operator. Let $k\in\mathbb{N}$ be fixed.

If $x_n\to 0$ weak, then $x_n\sin 2\pi kt\to 0$ weak in $C([0,1],X)$ and thus $U(x_n\sin 2\pi kt)\to 0$ in norm. If $U(x_n\sin 2\pi kt)=\frac{1}{2}(0,\ldots,0,T_k(x_n),0,\ldots)$ then $T_k(x_n)\to 0$ in norm i.e. T_k is Dunford–Pettis. Also, if $x_n\to 0$ weak, then $x_n\sin 2\pi nt\to 0$ weak in $C([0,1],X)$, thus $U(x_n\sin 2\pi nt)\to 0$ in norm and since $U(x_n\sin 2\pi nt)=\frac{1}{2}(0,\ldots,0,T_n(x_n),0,\ldots)$ it follows that $T_n(x_n)\to 0$ in norm. From Proposition 1 it follows that $(T_n)_{n\in\mathbb{N}}$ is an uniformly Dunford–Pettis family. In the same way it can be proved that $(V_n)_{n\in\mathbb{N}}$ is an uniformly Dunford–Pettis family.

Suppose now that $(T_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ are uniformly Dunford–Pettis. Then, by the ideal property of the class of all Dunford–Pettis operators, it follows that S_n is Dunford–Pettis for any $n\in\mathbb{N}$. Let $(f_n)_{n\in\mathbb{N}}\subseteq C([0,1],X)$ be such that $f_n\to 0$ weak. Then $\sup_{n\in\mathbb{N},t\in[0,1]}\|f_n(t)\|<\infty$ and $f_n(t)\to 0$ weak for any $t\in[0,1]$. By Proposition 1, $\|T_n(f_n(t))\sin 2\pi nt\|_n\to 0$ for any $t\in[0,1]$ and obviously $\sup_{n\in\mathbb{N},t\in[0,1]}\|T_n(f_n(t))\sin 2\pi nt\|_n<\infty$. Now by Bartle's convergence theorem (p. 56 of [6]), it follows that $\int_0^1\|T_n(f_n(t))\sin 2\pi nt\|_n\mathrm{d}t\to 0$. Analogously $\int_0^1\|V_n(f_n(t))\cos 2\pi nt\|_n\mathrm{d}t\to 0$. By Proposition 1, U is Dunford–Pettis.

(b) Suppose U is compact. Then (see Exercise 4, p. 144 of [4]) there is $0\le\lambda_n\to 0$ such that $\|S_n(f)\|\le\lambda_n$ for any $f\in C([0,1],X)$ with $\|f\|\le 1$ and any $n\in\mathbb{N}$.

In particular, for any $x\in B_X$ and any $n\in\mathbb{N}$ we have

$$\left\|\int_0^1 (T_n(x)\sin^2 2\pi nt+V_n(x)\sin 2\pi nt\cos 2\pi nt)\mathrm{d}t\right\|\le\lambda_n$$

i.e. $\|T_n\|\le 2\lambda_n$ for any $n\in\mathbb{N}$ and thus $\|T_n\|\to 0$. Similarly $\|V_n\|\to 0$.

Also, by the ideal property of compact operators we obtain that for any $n\in\mathbb{N}$ the operator S_n is compact, in particular the set

$$\left\{\int_0^1 (T_n(x)\sin^2 2\pi nt+V_n(x)\sin 2\pi nt\cos 2\pi nt)\mathrm{d}t\,|\,\|x\|\le 1\right\}\subseteq Y_n$$

is relatively norm compact, $T_n(B_X)$ is relatively norm compact i.e. T_n is compact. Analogously, V_n is compact.

Conversely, by the ideal property of compact operators, it follows that all S_n are compact and also $\|S_n\|\le\|T_n\|+\|V_n\|\to 0$ i.e. U is compact.

The following theorem, which is the main result of our paper, is an extension of Theorem 3.1 in [10].

Theorem 4. *Let X, Y be Banach spaces, Ω a compact Hausdorff space and $U\colon C(\Omega,X)\to Y$ a bounded linear operator with G its representing measure.*

If U is a Dunford–Pettis operator, then the range of the representing measure $G(\Sigma) \subseteq DP(X, Y)$ is an uniformly Dunford–Pettis family of operators and $\|G\|$ is continuous at $\emptyset$, or equivalently, for any $x_n \to 0$ weak it follows that $\widetilde{G_{x_n}}(\Omega) \to 0$ and $\|G\|$ is continuous at $\emptyset$.

Proof. If U is a Dunford–Pettis operator, then clearly for any $x \in X$ we have $U_\#(x) \in DP(C(\Omega), Y)$. Let $x_n \to 0$ weak. For $n \in \mathbb{N}$, let $\varphi_n \in C(\Omega)$ with $\|\varphi_n\| \leq 1$ such that

$$\|U_\#(x_n)\| - \frac{1}{n} < \|U_\#(x_n)(\varphi_n)\| = \|U(\varphi_n \otimes x_n)\|.$$

For any $\omega \in \Omega$ and any $x^* \in X^*$ we have

$$|x^*(\varphi_n(\omega)x_n)| \leq |x^*(x_n)| \to 0 \text{ and } \sup_{n\in\mathbb{N}} \|\varphi_n \otimes x_n\| \leq \sup_{n\in\mathbb{N}} \|x_n\| < \infty,$$

hence $\varphi_n \otimes x_n \to 0$ weak. Since U is Dunford–Pettis we have $\|U(\varphi_n \otimes x_n)\| \to 0$ and thus $\|U_\#(x_n)\| \to 0$, which means that $U_\#\colon X \to DP(C(\Omega), Y)$ is a Dunford–Pettis operator. Because for any $x \in X$ the operator $U_\#(x)\colon C[0, 1] \to Y$ has the representing measure $G_x\colon \Sigma \to Y$ and $\widetilde{G_x}(\Omega) \leq \|U_\#(x)\| = \|G_x\|(\Omega) \leq 4\widetilde{G_x}(\Omega)$ we get that for any $x_n \to 0$ weak it follows that $\sup_{E\in\Sigma} \|G(E)(x_n)\| = \widetilde{G_{x_n}}(\Omega) \to 0$.

The fact that $\|G\|$ is continuous at $\emptyset$ is proved in Theorem 3.1 of [10].

We observe that the above proof is an obvious modification of the proof of Proposition 7 in [8].

Now we analyze the case of operators with values in c_0.

Theorem 5.

(i) *Let X be a Banach space, $(\lambda_n)_{n\in\mathbb{N}} \subseteq rcabv(\Sigma, X^*)$ such that $\lambda_n(E) \to 0$ weak* for any $E \in \Sigma$ and let $U\colon C([0, 1], X) \to c_0$ be the operator defined by*

$$U(f) = \left(\int_0^1 f(t)\mathrm{d}\lambda_n(t)\right)_{n\in\mathbb{N}}.$$

If U is a Dunford–Pettis operator, then for any $x_n \to 0$ weak it follows that $\sup_{k\in\mathbb{N}} |\lambda_k x_n|([0, 1]) \to 0$ and $(\lambda_n)_{n\in\mathbb{N}}$ is uniformly countably additive.

(ii) *Let X be a Banach space, $(\varphi_n)_{n\in\mathbb{N}} \subseteq B(\Sigma, X^*)$ with $\sup_{n\in\mathbb{N}, t\in[0,1]} \|\varphi_n(t)\| = M < \infty$, $(\nu_n)_{n\in\mathbb{N}} \subseteq rcabv(\Sigma)$ an uniformly countably additive pointwise bounded family such that for any $E \in \Sigma$ and any $x \in X$ we have $\int_E \varphi_n(t)(x)\mathrm{d}\nu_n(t) \to 0$. Let $U\colon C([0, 1], X) \to c_0$ be the operator defined by*

$$U(f) = \left(\int_0^1 \varphi_n(t)(f(t))\mathrm{d}\nu_n(t)\right)_{n\in\mathbb{N}}.$$

(a) *If U is a Dunford–Pettis operator, then for any $x_n \to 0$ weak it follows that $\int_0^1 |\varphi_n(t)(x_n)|\mathrm{d}|\nu_n|(t) \to 0$.*

(b) *If for any $x_n \to 0$ weak it follows that $\varphi_n(t)(x_n) \to 0$ for any $t \in [0, 1]$, then U is a Dunford–Pettis operator.*

(c) *U is a compact operator if and only if $\int_0^1 \|\varphi_n(t)\|\mathrm{d}|\nu_n|(t) \to 0$.*

Proof.

(i) Indeed, by hypothesis and the Nikodym boundedness theorem it follows that $\sup_{n\in\mathbb{N}} \|\lambda_n\|([0,1]) < \infty$. From $\lambda_n(E) \to 0$ weak* for any $E \in \Sigma$ it follows that for any simple function $f\colon [0,1] \to X$ we have $\int_0^1 f(t)\mathrm{d}\lambda_n(t) \to 0$. From the well-known inequality $\| \int_0^1 f(t)\mathrm{d}\lambda_n(t)\| \le \|f\|\|\lambda_n\|([0,1])$, $f \in B(\Sigma, X)$, we deduce that U takes its values in c_0 and that it is bounded linear.

The representing measure of U is $G(E) = (\lambda_k(E))_{k\in\mathbb{N}}\colon X \to c_0$ and for any $x \in X$ we have $\widetilde{G_x}([0,1]) = \sup_{E\in\Sigma} \sup_{k\in\mathbb{N}} |\lambda_k(E)(x)| = \sup_{k\in\mathbb{N}} \widetilde{\lambda_k x}([0,1])$. Using that for any $x \in X$ we have $\frac{1}{4}|\lambda_k x|([0,1]) \le \widetilde{\lambda_k x}([0,1]) \le |\lambda_k x|([0,1])$. Applying Theorem 4 we obtain what needs to be proved. (Observe that in this case, $\|G\|$ is continuous at $\emptyset$ and is equivalent to the fact that the family $(\lambda_n)_{n\in\mathbb{N}}$ is uniformly countably additive (see chapter I, Theorem 4, p. 11 of [6]).

(ii) Let $(\lambda_n)_{n\in\mathbb{N}} \subseteq rcabv(\Sigma, X^*)$ be defined by $\lambda_n(E)(x) = \int_E \varphi_n(t)(x)\mathrm{d}\nu_n(t)$ and observe that $\lambda_n(E) = \text{Bochner} - \int_E \varphi_n(t)\mathrm{d}\nu_n(t)$.

Then, by hypothesis, we have that $\lambda_n(E) \to 0$ weak* for any $E \in \Sigma$. Also, from $\|\lambda_n(E)\| \le M|\nu_n|(E)$ and the fact that $(\nu_n)_{n\in\mathbb{N}}$ is uniformly countably additive it follows that $(\lambda_n)_{n\in\mathbb{N}}$ is uniformly countably additive. Then $U(f) = \left(\int_0^1 f(t)\mathrm{d}\lambda_n(t)\right)_{n\in\mathbb{N}}$.

(a) For any $x \in X$, $k \in \mathbb{N}$ we have that $\lambda_k x\colon \Sigma \to \mathbb{K}$ is defined by $(\lambda_k x)(E) = \int_E \varphi_k(t)(x)\mathrm{d}\nu_k(t)$ and $|\lambda_k x|([0,1]) = \int_0^1 |\varphi_k(t)(x)|\mathrm{d}|\nu_k|(t)$. Now by (i) if $x_n \to 0$ weak, then $\sup_{k\in\mathbb{N}} \int_0^1 |\varphi_k(t)(x_n)|\mathrm{d}|\nu_k|(t) \to 0$.

(b) Let $(f_n)_{n\in\mathbb{N}} \subseteq C([0,1], X)$ be such that $f_n \to 0$ weak. Then $\sup_{n\in\mathbb{N},\, t\in[0,1]} \|f_n(t)\| = L < \infty$ and $f_n(t) \to 0$ weak for any $t \in [0,1]$.

By hypothesis, $\varphi_n(t)(f_n(t)) \to 0$ for any $t \in [0,1]$ and $|\varphi_n(t)(f_n(t))| \le ML$ for any $t \in [0,1]$. Then, by Bartle's convergence theorem, it follows that $\int_0^1 |\varphi_n(t)(f_n(t))|\mathrm{d}|\nu_n|(t) \to 0$ and, by Corollary 2, U is a Dunford–Pettis operator.

(c) We observe that $|\lambda_n|([0,1]) = \int_0^1 \|\varphi_n(t)\|\mathrm{d}|\nu_n|(t)$ and the proof will be finished since (see Exercise 4, p. 114 of [4]) U is a compact operator if and only if $|\lambda_n|([0,1]) \to 0$.

Remark 6.

(a) If $(\varphi_n)_{n\in\mathbb{N}} \subseteq B(\Sigma, X^*)$ is such that $\sup_{n\in\mathbb{N},\, t\in[0,1]} \|\varphi_n(t)\| < \infty$, $(\nu_n)_{n\in\mathbb{N}} \subseteq rcabv(\Sigma)$ is an uniformly countably additive pointwise bounded family and in addition, $\varphi_n(t) \to 0$ weak* for any $t \in [0,1]$, then it follows that for any $E \in \Sigma$ and any $x \in X$ we have $\int_E \varphi_n(t)(x)\mathrm{d}\nu_n(t) \to 0$.

(b) Under the hypothesis of theorem 5 (ii), for any $k \in \mathbb{N}$, the operator $T_k\colon X \to L_1(|\nu_k|)$ defined by $T_k(x) = \varphi_k x$ is Dunford–Pettis and the condition: for any $x_n \to 0$ weak it follows that $\int_0^1 |\varphi_n(t)(x_n)|\mathrm{d}|\nu_n|(t) \to 0$ is equivalent to the fact that the family $(T_k)_{k\in\mathbb{N}}$ is an uniformly Dunford–Pettis family. (For this reason we formulate (a) in theorem 5(ii)).

Proof.

(a) Indeed, in our hypotheses, we can apply again Bartle's convergence theorem, to deduce that for any $E \in \Sigma$ we have $\int_E \varphi_n(t)(x)\mathrm{d}\nu_n(t) \to 0$.

(b) Let $k \in \mathbb{N}$ be fixed. If $x_n \to 0$ weak, then $\varphi_k(t)(x_n) \to 0$ for any $t \in [0, 1]$ and $\sup_{n\in\mathbb{N},\, t\in[0,1]} |\varphi_k(t)(x_n)| < \infty$. From the Lebesgue dominated convergence theorem, it follows that $\int_0^1 |\varphi_k(t)(x_n)|\mathrm{d}|\nu_k|(t) \to 0$. The last part of the statement follows by Proposition 1.

In the following corollary we indicate a way to construct examples of Dunford–Pettis operators from a Dunford–Pettis one. In view of Theorem 9 of [2] and Theorem 3.1 of [10], this result is, perhaps, natural; see also [9] for other examples in the scalar case.

COROLLARY 7

(a) *Let X be a Banach space, $(x_n^*)_{n\in\mathbb{N}} \subseteq X^*$ a bounded sequence, $T\colon X \to l_\infty$ defined by $T(x) = (x_n^*(x))_{n\in\mathbb{N}}$ and $(\nu_n)_{n\in\mathbb{N}} \subseteq rcabv(\Sigma)$ such that $\nu_n(E) \to 0$ for any $E \in \Sigma$ and $\liminf_{n\to\infty} |\nu_n|([0, 1]) > 0$, or $x_n^* \to 0$ weak*, $T\colon X \to c_0$ defined by $T(x) = (x_n^*(x))_{n\in\mathbb{N}}$ and $(\nu_n)_{n\in\mathbb{N}} \subseteq rcabv(\Sigma)$ uniformly countably additive pointwise bounded such that $\liminf_{n\to\infty} |\nu_n|([0, 1]) > 0$.*
Let $U\colon C([0, 1], X) \to c_0$ be the operator defined by

$$U(f) = \left(\int_0^1 x_n^* f(t)\mathrm{d}\nu_n(t)\right)_{n\in\mathbb{N}}.$$

Then

(i) *U is Dunford–Pettis $\Leftrightarrow$ T is Dunford–Pettis.*
(ii) *U is compact $\Leftrightarrow$ T is compact.*

(b) *Let $a = (a_n)_{n\in\mathbb{N}} \in l_\infty$ and let $U\colon C([0, 1], L_1[0, 1]) \to c_0$ be the operator defined by*

$$U(f) = \left(a_n \int_0^1 \left(\int_0^1 f(t)(s)s^n ds\right) r_n(t)\mathrm{d}t\right)_{n\in\mathbb{N}}.$$

Then U is Dunford–Pettis, while U is compact $\Leftrightarrow$ $a \in c_0$.

Proof.

(a) Define $\varphi_n(t) = x_n^*$ and observe that, in our hypotheses, in both cases for any $E \in \Sigma$ and $x \in X$ we have $\int_E \varphi_n(t)(x)\mathrm{d}\nu_n(t) \to 0$. Thus, the hypotheses from Theorem 5(ii) are satisfied and $U(f) = \left(\int_0^1 \varphi_n(t)(f(t))\mathrm{d}\nu_n(t)\right)_{n\in\mathbb{N}}$.

(i) Suppose U is a Dunford–Pettis operator. In both cases, from Theorem 5(ii)(a) if $x_n \to 0$ weak, it follows that $\int_0^1 |\varphi_n(t)(x_n)|\mathrm{d}|\nu_n|(t) \to 0$, or $|x_n^*(x_n)||\nu_n|([0, 1]) \to 0$. From here, since $\liminf_{n\to\infty} |\nu_n|([0, 1]) > 0$, we deduce $|x_n^*(x_n)| \to 0$ i.e., by Corollary 2, T is Dunford–Pettis.
The converse follows from Corollary 2 and Theorem 5(ii)(b).

(ii) By Theorem 5(ii)(c), U is compact if and only if $\|x_n^*\||\nu_n|([0, 1]) = \int_0^1 \|\varphi_n(t)\|\mathrm{d}|\nu_n|(t) \to 0$ or equivalently, by hypothesis, $\|x_n^*\| \to 0$ i.e. T is compact.

We remark that in (i) and (ii) the converses are true without the hypothesis $\liminf_{n\to\infty} |\nu_n|([0, 1]) > 0$.

(b) Since any positive bounded linear operator from $L_1[0,1]$ into c_0 is Dunford–Pettis (see Corollary 2.3 of [7]) the operator $V\colon L_1[0,1] \to c_0$ defined by $V(f) = \left(\int_0^1 f(s)s^n \mathrm{d}s\right)_{n\in\mathbb{N}}$ is Dunford–Pettis, and thus $T\colon L_1[0,1] \to c_0$ defined by $T(f) = \left(a_n \int_0^1 f(s)s^n \mathrm{d}s\right)_{n\in\mathbb{N}}$ is Dunford–Pettis. By (a)(i), U is Dunford–Pettis. By (a)(ii), U is compact $\Leftrightarrow$ T is compact $\Leftrightarrow$ $|a_n| \to 0$.

Examples of measures as in Corollary 7(a) can be obtained in the following ways:

1. Let $(\alpha_n)_{n\in\mathbb{N}} \in l_1$ be a non-null element and define $\nu_n(E) = \alpha_1 \int_E r_n(t)\mathrm{d}t + \cdots + \alpha_n \int_E r_1(t)\mathrm{d}t$. Then $\nu_n(E) \to 0$ for any $E \in \Sigma$ and $\liminf_{n\to\infty} |\nu_n|([0,1]) > 0$;
2. If $h\colon [0,1] \to \mathbb{R}$ is a differentiable function with $h'(0) \neq 0$, then $\nu_n(E) = n \int_E \left(h\left(\frac{t}{n}\right) - h(0)\right)\mathrm{d}t \to h'(0) \int_E t\mathrm{d}t = \nu(E)$ uniformly for $E \in \Sigma$ and $\lim_{n\to\infty} |\nu_n|([0,1]) = \frac{|h'(0)|}{2} > 0$.

1. Indeed, in our hypothesis, by a well-known classical result (see Chapter IX, Exercise 17 of [11]) it follows that $\nu_n(E) \to 0$ for any $E \in \Sigma$ and, by Khinchin's inequality (see p. 10 of [5]) for any $n \in \mathbb{N}$ we have $|\nu_n|([0,1]) \geq \frac{1}{\sqrt{2}}(|\alpha_1|^2 + \cdots + |\alpha_n|^2)^{\frac{1}{2}}$ and $\liminf_{n\to\infty} |\nu_n|([0,1]) \geq \frac{1}{\sqrt{2}}\|(\alpha_n)_{n\in\mathbb{N}}\|_2 > 0$.
2. Let $\varepsilon > 0$. Then there is $\delta_\varepsilon > 0$ such that for any $0 \leq t < \delta_\varepsilon$ we have $(h'(0) - \varepsilon)t \leq h(t) - h(0) \leq (h'(0) + \varepsilon)t$. There is also $n_\varepsilon \in \mathbb{N}$ such that $\frac{1}{n_\varepsilon} < \delta_\varepsilon$. Take $n \geq n_\varepsilon$. Then for any $t \in [0,1]$ we have $\frac{t}{n} < \delta_\varepsilon$ from where $(h'(0) - \varepsilon)t \leq n\left[h\left(\frac{t}{n}\right) - h(0)\right] \leq (h'(0) + \varepsilon)t$. For any $E \in \Sigma$ we obtain $(h'(0) - \varepsilon) \int_E t \leq n \int_E \left[h\left(\frac{t}{n}\right) - h(0)\right]\mathrm{d}t \leq (h'(0) + \varepsilon) \int_E t$, or $\left|n \int_E \left[h\left(\frac{t}{n}\right) - h(0)\right]\mathrm{d}t - h'(0) \int_E t\mathrm{d}t\right| \leq \varepsilon \int_E t\mathrm{d}t \leq \frac{\varepsilon}{2}$. Also

$$|\nu_n|([0,1]) = n \int_0^1 \left|h\left(\frac{t}{n}\right) - h(0)\right| \mathrm{d}t = n^2 \int_0^{\frac{1}{n}} |h(t) - h(0)|\mathrm{d}t \to \frac{|h'(0)|}{2}.$$

The next example is different from what was used in Theorem 9 of [2] and Theorem 3.1 of [10] and in the scalar case appear in Example 11 of [9].

COROLLARY 8

Let X be a Banach space, $\sum_{n=1}^\infty x_n^*$ a weakly Cauchy series in X^* and let $U\colon C([0,1],X) \to c_0$ be the operator defined by

$$U(f) = \left(\int_0^1 (x_n^*(f(t))r_1(t) + \cdots + x_1^*(f(t))r_n(t))\mathrm{d}t\right)_{n\in\mathbb{N}}.$$

Then U is a Dunford–Pettis operator and U is a compact operator if and only if $U = 0$.

Proof. Let $T\colon X \to l_1$ be the operator defined by $T(x) = (x_n^*(x))_{n\in\mathbb{N}}$ and $V\colon C([0,1],X) \to C([0,1],l_1)$ the operator defined by $V(f) = T \circ f$. Define also $S\colon C([0,1],l_1) \to c_0$ by

$$S(f) = \left(\int_0^1 (\langle f(t), e_n\rangle r_1(t) + \cdots + \langle f(t), e_1\rangle r_n(t))\mathrm{d}t\right)_{n\in\mathbb{N}}$$

and observe that $U = SV$.

(S takes its values in c_0 (Chapter IX, Exercise 17 of [11]).) Because l_1 has the Schur property, from Theorem 3.1 of [10], S is a Dunford–Pettis operator and hence U is also a Dunford–Pettis operator.

Define $\varphi_n(t) = x_n^* r_1(t) + x_{n-1}^* r_2(t) + \cdots + x_1^* r_n(t)$ and observe that (Chapter IX, Exercise 17 of [11]) $\int_E \varphi_n(t)\mathrm{d}t \to 0$ weak* for any $E \in \Sigma$, $\|\varphi_n(t)\| \le w_1(x_n^* | n \in \mathbb{N})$ for any $t \in [0,1]$ and $U(f) = \left(\int_0^1 \varphi_n(t)(f(t))\mathrm{d}t\right)_{n\in\mathbb{N}}$.

By Theorem 5(ii)(c) U is compact if and only if $\int_0^1 \|\varphi_n(t)\|\mathrm{d}t \to 0$. However, for any $n \in \mathbb{N}$, again by Khinchin's inequality we have

$$\frac{1}{\sqrt{2}} \sup_{\|x\|\le 1} (|x_1^*(x)|^2 + \cdots + |x_n^*(x)|^2)^{\frac{1}{2}} \le \int_0^1 \|\varphi_n(t)\|\mathrm{d}t.$$

If U is compact, then $\sup_{\|x\|\le 1}\left(\sum_{n=1}^{\infty} |x_n^*(x)|^2\right)^{\frac{1}{2}} = 0$, which implies $x_n^* = 0$ for any $n \in \mathbb{N}$ i.e. $U = 0$.

We now state a remark which is certainly well-known, but, unfortunately, we do not know a reference.

Remark 9.

(a) The space $w_1(L_1[0,1]^*)$ is isometrically isomorph with $L_\infty([0,1], l_1)$, more precisely, if $g_n \in L_\infty[0,1] = L_1[0,1]^*$, then $\sum_{n=1}^{\infty} g_n$ is a weakly Cauchy series in $L_\infty[0,1]$ $\Leftrightarrow$ the function $g = (g_n)_{n\in\mathbb{N}} \in L_\infty([0,1], l_1)$.
(b) A weakly Cauchy series $\sum_{n=1}^{\infty} g_n$ in $L_\infty[0,1]$ is unconditionally norm convergent $\Leftrightarrow$ the function $g = (g_n)_{n\in\mathbb{N}} \in L_\infty([0,1], l_1)$ has an essentially relatively compact range in l_1.

Proof.

(a) Indeed, $\sum_{n=1}^{\infty} g_n$ is a weakly Cauchy series in $L_\infty[0,1] \Leftrightarrow$ the operator $T\colon L_1[0,1] \to l_1$ defined by $T(f) = \left(\int_0^1 g_n(t)f(t)\mathrm{d}t\right)_{n\in\mathbb{N}}$ is bounded linear. Since l_1 has the Radon–Nikodym property (see Theorem, p. 63 of [6]), this is equivalent to the fact that T is representable i.e. there is $h = (h_n)_{n\in\mathbb{N}} \in L_\infty([0,1], l_1)$ such that $T(f) = \text{Bochner} - \int_0^1 f(t)h(t)\mathrm{d}t$. Then for any $n \in \mathbb{N}$ we have that $\int_E g_n(t)\mathrm{d}t = \int_E h_n(t)\mathrm{d}t$ for any $E \in \Sigma$. Thus $g_n = h_n$ μ-a.e. and the statement follows.
(b) By Theorem 1.9, p. 9 of [5], the unconditionality norm convergence of series $\sum_{n=1}^{\infty} g_n$ is equivalent to the fact that the operator $T\colon L_1[0,1] \to l_1$ as in (a) is compact. By the representation of compact operators on $L_1(\mu)$ (see p. 68 of [6]), this is equivalent to the fact that g has an essentially relatively compact range in l_1.

In the sequel we analyze the same kind of operators as in Corollary 7, but with values in l_p, where $1 \le p < \infty$. We begin with a lemma which is, probably, a well-known result but, we do not know a reference.

Lemma 10. *Let* $1 \le p < \infty$, $(\alpha_n)_{n\in\mathbb{N}} \in l_p$ *and let* $G\colon \Sigma \to l_p$ *be defined by*

$$G(E) = \left(\alpha_n \int_E r_n(t)\mathrm{d}t\right)_{n\in\mathbb{N}}.$$

(i) *If* $1 \le p < 2$, *then* $\frac{1}{\sqrt{2}}\|(\alpha_n)_{n\in\mathbb{N}}\|_r \le \|G\|([0,1]) \le \|(\alpha_n)_{n\in\mathbb{N}}\|_r$, *where* $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$.
(ii) *If* $2 \le p < \infty$, *then* $\frac{1}{\sqrt{2}} \sup_{n\in\mathbb{N}} |\alpha_n| \le \|G\|([0,1]) \le \sup_{n\in\mathbb{N}} |\alpha_n|$.

Proof. Let $h\colon [0,1] \to l_p$ be defined by $h(t) = (\alpha_n r_n(t))_{n\in\mathbb{N}}$ and observe that $G(E) =$ Bochner $- \int_E h(t)\mathrm{d}t$. Then

$$\|G\|([0,1]) = \sup_{\|y^*\|\le 1} |G_{y^*}|([0,1]) = \sup_{\|y^*\|\le 1} \int_0^1 |y^* h(t)|\mathrm{d}t,$$

because $G_{y^*}(E) = \int_E y^* h(t)\mathrm{d}t$ and $|G_{y^*}|([0,1]) = \int_0^1 |y^* h(t)|\mathrm{d}t$.

However, for any $y^* = (\xi_n)_{n\in\mathbb{N}} \in l_p^* = l_q$ (q is the conjugate of p), we have $y^* h(t) = \sum_{n=1}^{\infty} \xi_n \alpha_n r_n(t)$ and, by Khinchin's inequality

$$\frac{1}{\sqrt{2}}\left(\sum_{n=1}^{\infty} |\xi_n \alpha_n|^2\right)^{\frac{1}{2}} \le \int_0^1 |y^* h(t)|\mathrm{d}t \le \left(\sum_{n=1}^{\infty} |\xi_n \alpha_n|^2\right)^{\frac{1}{2}},$$

i.e. $\frac{1}{\sqrt{2}}\|M\| \le \|G\|([0,1]) \le \|M\|$, where $M\colon l_q \to l_2$ is the multiplication operator $M((\xi_n)_{n\in\mathbb{N}}) = (\alpha_n \xi_n)_{n\in\mathbb{N}}$. Now, as is well-known

(i) if $2 < q$ i.e. $1 \le p < 2$, then $\|M\| = \|(\alpha_n)_{n\in\mathbb{N}}\|_r = \left(\sum_{n=1}^{\infty} |\alpha_n|^r\right)^{\frac{1}{r}}$, where $\frac{1}{2} = \frac{1}{q} + \frac{1}{r}$ i.e. $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$.
(ii) if $q \le 2$ i.e. $2 \le p < \infty$, then $\|M\| = \sup_{n\in\mathbb{N}} |\alpha_n|$.

In case of operators on $C([0,1], X)$ with values in l_p, Khinchin's inequality gives a distinction for $1 < p < 2$ and $2 \le p < \infty$.

PROPOSITION 11

Let $1 < p < \infty$, p^* *be the conjugate of* p, X *a Banach space*, $(x_n^*)_{n\in\mathbb{N}} \in w_p(X^*)$ *and* $T\colon X \to l_p$ *defined by* $T(x) = (x_n^*(x))_{n\in\mathbb{N}}$.

Let $U\colon C([0,1], X) \to l_p$ *be the operator defined by*

$$U(f) = \left(\int_0^1 x_n^* f(t) r_n(t)\mathrm{d}t\right)_{n\in\mathbb{N}}.$$

(a) *If* T *is a Dunford–Pettis* (*resp. compact*) *operator*, *then* U *is a Dunford–Pettis* (*resp. compact*) *operator.*
(b) *If* X *is reflexive and* T *is Dunford–Pettis*, *or* $X = c_0$, *then* T *is compact and thus* U *is compact.*
(c) *If* U *is a compact operator*, *then* $x_n^* \to 0$ *in norm.*
(d) *Suppose* $1 < p < 2$. *Then*

  (i) U *is a Dunford–Pettis operator* $\Leftrightarrow$ T *is a Dunford–Pettis operator.*
  (ii) U *is a compact operator* $\Leftrightarrow$ T *is a compact operator.*

(e) *Suppose* $2 \leq p < \infty$.

(i) *If U is a Dunford–Pettis operator, then* $T\colon X \to c_0$ *is a Dunford–Pettis operator.*
(ii) *If* $T\colon X \to c_0$ *is a compact operator i.e.* $x_n^* \to 0$ *in norm and* X^* *has type* a, *where* $1 < p^* \leq a \leq 2$, *then U is a compact operator.*

Proof. The fact that U is well-defined, bounded linear is clear. Also, the representing measure of U is $G(E)(x) = \left(x_n^*(x)\int_E r_n(t)\mathrm{d}t\right)_{n\in\mathbb{N}}$.
(a) We observe that for any $f \in C([0,1], X)$ we have

$$\|U(f)\|^p = \sum_{n=1}^{\infty}\left|\int_0^1 x_n^* f(t) r_n(t)\mathrm{d}t\right|^p \leq \int_0^1 \sum_{n=1}^{\infty} |x_n^* f(t)|^p \mathrm{d}t$$

$$= \int_0^1 \|T(f(t))\|^p \mathrm{d}t$$

i.e.

$$(*) \qquad \|U(f)\| \leq \left(\int_0^1 \|T(f(t))\|^p \mathrm{d}t\right)^{1/p}.$$

From this inequality it is easy to prove that if T is a Dunford–Pettis operator, then U is a Dunford–Pettis operator.

Indeed, let $(f_n)_{n\in\mathbb{N}} \subseteq C([0,1], X)$ be such that $f_n \to 0$ weak. Then $\sup_{n\in\mathbb{N},\, t\in[0,1]} \|f_n(t)\| = L < \infty$ and $f_n(t) \to 0$ weak for any $t \in [0,1]$. Since T is a Dunford–Pettis operator, it follows that $\|T(f_k(t))\|^p \to 0$ for any $t \in [0,1]$ and $\|T(f_k(t))\| \leq \|T\|\|f_k(t)\| \leq L\|T\|$.

From the Lebesgue dominated convergence theorem we get $\int_0^1 \|T(f_k(t))\|^p \mathrm{d}t \to 0$ and by $(*)$, it follows that $\|U(f_k)\| \to 0$.

Suppose $T\colon X \to l_p$ is compact and let $\varepsilon > 0$. From Exercise 6, p. 6 of [4], there is $n_\varepsilon \in \mathbb{N}$ such that $\sup_{\|x\|\leq 1} \sum_{k=n_\varepsilon}^{\infty} |x_k^*(x)|^p < \varepsilon$.

Take $f \in C([0,1], X)$ with $\|f\| \leq 1$. Since for any $n \in \mathbb{N}$ we have $\sum_{k=n}^{\infty}\left|\int_0^1 x_n^* f(t) r_n(t)\mathrm{d}t\right|^p \leq \int_0^1 \sum_{k=n}^{\infty} |x_k^*(f(t))|^p \mathrm{d}t$, we deduce $\sum_{k=n_\varepsilon}^{\infty}\left|\int_0^1 x_n^* f(t) r_n(t)\mathrm{d}t\right|^p \leq \varepsilon$, hence, again by Exercise 6, p. 6 of [4], U is compact.
(b) First, it follows from the well-known fact that a Dunford–Pettis operator whose domain is reflexive is compact and second, by the same reasoning and the fact that l_1 has the Schur property, the dual is compact, hence compact by Schauder's theorem.
(c) We have that $U^*\colon l_{p^*} \to rcabv(\Sigma, X^*) = C([0,1], X)^*$ is defined by $U^*(\xi) = G_\xi$, where $G_\xi(E)(x) = \sum_{n=1}^{\infty} \xi_n x_n^*(x)\int_E r_n(t)\mathrm{d}t$.

Since $\xi = (\xi_n)_{n\in\mathbb{N}} \in l_{p^*}$ and $(x_n^*)_{n\in\mathbb{N}} \in w_p(X^*)$ it follows that the series $\sum_{n=1}^{\infty} \xi_n x_n^* r_n(t)$ is unconditionally norm convergent for any $t \in [0,1]$ and let $h_\xi\colon [0,1] \to X^*$ be defined by $h_\xi(t) = \sum_{n=1}^{\infty} \xi_n x_n^* r_n(t)$. Then the function h_ξ is Bochner integrable (by an easy application of the Lebesgue dominated convergence theorem for the Bochner integral), $G_\xi(E) = \text{Bochner} - \int_E h_\xi(t)\mathrm{d}t$ and thus $\|U^*(\xi)\| = |G_\xi|([0,1]) = \int_0^1 \|h_\xi(t)\|\mathrm{d}t$.

If U is a compact operator, then by the Schauder's theorem U^* is compact, in particular $U^*(e_n) \to 0$ and the statement follows since $\|U^*(e_n)\| = \int_0^1 \|h_{e_n}(t)\|\mathrm{d}t = \int_0^1 \|x_n^* r_n(t)\|\mathrm{d}t = \|x_n^*\|$.

(d) Define r such that $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$ and $T: X \to l_r$ by $T(x) = (x_n^*(x))_{n\in\mathbb{N}}$.
We prove the following equivalences:

(i) U is a Dunford–Pettis operator $\Leftrightarrow T: X \to l_r$ is a Dunford–Pettis operator $\Leftrightarrow T: X \to l_p$ is a Dunford–Pettis operator.

We observe that for any $x \in X$ the measure $G_x: \Sigma \to l_p$ is defined by $G_x(E) = \left(x_n^*(x) \int_E r_n(t)\mathrm{d}t\right)_{n\in\mathbb{N}}$ and by lemma 10(i)

$$\frac{1}{\sqrt{2}}\|T(x)\|_r \leq \|G_x\|([0,1]) \leq \|T(x)\|_r.$$

If U is a Dunford–Pettis operator, then by Theorem 4, for any $x_n \to 0$ weak it follows that $\|G_{x_n}\|([0,1]) \to 0$ and by the above $\|T(x_n)\|_r \to 0$ i.e. $T: X \to l_r$ is Dunford–Pettis.

We now prove that for any $x \in X$ we have the inequalities

$$(**) \qquad \|T(x)\|_r \leq \|T(x)\|_p \leq w_2(x_n^*|n \in \mathbb{N})\|x\|\|T(x)\|_r$$

and then from these inequalities it is easy to prove the second equivalence.

Indeed, first inequality follows from the inclusion $l_p \subset l_r$ and for the second we use the Hölder inequality $\|T(x)\|_p \leq \|T(x)\|_2\|T(x)\|_r$.

In (a) we prove that if $T: X \to l_p$ is a Dunford–Pettis operator, then U is a Dunford–Pettis operator.

(ii) U is a compact operator $\Leftrightarrow T: X \to l_r$ is a compact operator $\Leftrightarrow T: X \to l_p$ is a compact operator.

Suppose U is compact and $\varepsilon > 0$. Then (see Theorem 6 of [1] and Exercise 6, p. 6 of [4]) there is $n_\varepsilon \in \mathbb{N}$ such that for any $f \in B(\Sigma, X)$ with $\|f\| \leq 1$ we have

$$\left(\sum_{k=n_\varepsilon}^{\infty}\left|\int_0^1 x_k^* f(t) r_k(t)\mathrm{d}t\right|^p\right)^{1/p} < \varepsilon.$$

By Hölder's inequality we obtain that for any $f \in B(\Sigma, X)$ with $\|f\| \leq 1$ and any $\xi = (\xi_n)_{n\in\mathbb{N}} \in l_{p^*}$ we have

$$\left|\sum_{k=n_\varepsilon}^{\infty} \xi_k \int_0^1 x_k^* f(t) r_k(t)\mathrm{d}t\right| \leq \varepsilon\left(\sum_{k=n_\varepsilon}^{\infty} |\xi_k|^{p^*}\right)^{1/p^*}.$$

In particular, for any $E \in \Sigma$ and $\|x\| \leq 1$ and any $\xi \in l_{p^*}$ we have

$$\left|\int_E \left(\sum_{k=n_\varepsilon}^{\infty} \xi_k x_k^*(x) r_k(t)\right)\mathrm{d}t\right| \leq \varepsilon\left(\sum_{k=n_\varepsilon}^{\infty} |\xi_k|^{p^*}\right)^{1/p^*}.$$

Then for any $\|x\| \leq 1$ and any $\xi \in l_{p^*}$ we deduce

$$\frac{1}{4}\int_0^1 \left|\sum_{k=n_\varepsilon}^{\infty} \xi_k x_k^*(x) r_k(t)\right| \mathrm{d}t \leq \sup_{E\in\Sigma}\left|\int_E \left(\sum_{k=n_\varepsilon}^{\infty} x_k^*(x) r_k(t)\right)\mathrm{d}t\right|$$
$$\leq \varepsilon\left(\sum_{k=n_\varepsilon}^{\infty} |\xi_k|^{p^*}\right)^{1/p^*}$$

and, by Khinchin's inequality we get

$$\frac{1}{4\sqrt{2}}\left(\sum_{k=n_\varepsilon}^{\infty}|\xi_k x_k^*(x)|^2\right)^{1/2} \le \varepsilon\left(\sum_{k=n_\varepsilon}^{\infty}|\xi_k|^{p^*}\right)^{1/p^*}.$$

Taking $\xi_k \in \mathbb{K}$ such that $|\xi_k| = |x_k^*(x)|^{(r-2)/2}$ and using $\frac{1}{p^*} = \frac{1}{2} - \frac{1}{r}$ we obtain $\frac{1}{4\sqrt{2}}\left(\sum_{k=n_\varepsilon}^{\infty}|x_k^*(x)|^r\right)^{1/r} \le \varepsilon$ i.e., by Exercise 6, p. 6 of [4], $T\colon X \to l_r$ is compact. Now, from the inequality $(**)$ in (i) it can be proved, using Exercise 6, p. 6 of [4], that if $T\colon X \to l_r$ is a compact operator, then $T\colon X \to l_p$ is a compact operator and by (a), U is compact.

(e)

(i) Is the same as in (d)(i) and use Lemma 10(ii). We omit the proof. Another way to prove this fact is to use Theorem 5(ii)(a).

(ii) We prove that in our hypothesis it follows that U^* is compact, hence by Schauder's theorem U will be compact. With the same notations as in (c), $U^*\colon l_{p^*} \to rcabv(\Sigma, X^*)$ is defined by $U^*(\xi) = G_\xi$ and

$$\|U^*(\xi)\| = |G_\xi|([0,1]) = \int_0^1 \|h_\xi(t)\|\mathrm{d}t.$$

Since X^* has type a, then by definition of type a (see chapter 11, p. 217 of [5]) for any $\xi = (\xi_n)_{n\in\mathbb{N}} \in l_{p^*}$, we have

$$\|U^*(\xi)\| = \int_0^1 \|h_\xi(t)\|\mathrm{d}t \le T_a(X^*)\left(\sum_{n=1}^{\infty}|\xi_n|^a\|x_n^*\|^a\right)^{1/a}.$$

From $p^* \le a$ we obtain that

$$\|U^*(\xi)\| \le T_a(X^*)\left(\sum_{n=1}^{\infty}|\xi_n|^{p^*}\|x_n^*\|^{p^*}\right)^{1/p^*}.$$

If we define the finite rank operators $V_n\colon l_{p^*} \to rcabv(\Sigma, X^*)$ by $V_n(\xi) = U^*(\xi_1,\ldots,\xi_n,0,\ldots)$, then by the above inequality, for any $\xi = (\xi_n)_{n\in\mathbb{N}} \in l_{p^*}$ with $\|\xi\| \le 1$ we obtain

$$\begin{aligned}\|U^*(\xi) - V_n(\xi)\| &\le T_a(X^*)\left(\sum_{k=n+1}^{\infty}|\xi_k|^{p^*}\|x_k^*\|^{p^*}\right)^{1/p^*}\\ &\le T_a(X^*)\sup_{k\ge n+1}\|x_k^*\|\end{aligned}$$

and hence

$$\|U^* - V_n\| \le T_a(X^*)\sup_{k\ge n+1}\|x_k^*\| \to 0.$$

Thus U^* is compact.

With the help of Proposition 11 we can give some concrete examples.

Example 12.

(a) The operator $U\colon C([0,1], L_2[0,1]) \to l_2$ defined by

$$U(f) = \left(\int_0^1 \left(\int_0^1 f(t)(s)s^n \mathrm{d}s \right) r_n(t)\mathrm{d}t \right)_{n\in\mathbb{N}}$$

is compact.

(b) Let $1 < p < \infty$ and $U\colon C([0,1], l_p) \to l_p$ be the operator defined by

$$U(f) = \left(\frac{\int_0^1 (\langle f(t), e_1\rangle + \cdots + \langle f(t), e_n\rangle) r_n(t)\mathrm{d}t}{n} \right)_{n\in\mathbb{N}}.$$

(i) If $1 < p < 2$, then U is not Dunford–Pettis.
(ii) If $2 \le p < \infty$, then U is compact.

(c) Let $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in l_\infty$, $1 < p < \infty$ and $U\colon C([0,1], l_1) \to l_p$ be the operator defined by

$$U(f) = \left(\alpha_n \int_0^1 \langle f(t), e_n\rangle r_n(t)\mathrm{d}t \right)_{n\in\mathbb{N}}.$$

Then U is a Dunford–Pettis operator, and U is compact $\Leftrightarrow \alpha \in c_0$.

(d) Let $1 < p < 2$. Define $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$ and take $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in l_r$. Then the operator $U\colon C([0,1], C[0,1]) \to l_p$ defined by

$$U(f) = \left(\alpha_n \int_0^1 \left(\int_0^1 f(t)(s) r_n(s)\mathrm{d}s \right) r_n(t)\mathrm{d}t \right)_{n\in\mathbb{N}}$$

is compact.

(e) Let $2 \le p < \infty$, $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in l_\infty$ and let $U\colon C([0,1], C[0,1]) \to l_p$ be the operator defined by

$$U(f) = \left(\alpha_n \int_0^1 \left(\int_0^1 f(t)(s) r_n(s)\mathrm{d}s \right) r_n(t)\mathrm{d}t \right)_{n\in\mathbb{N}}.$$

Then U is a Dunford–Pettis operator, and U is compact $\Leftrightarrow \alpha \in c_0$.

(f) Let $2 \le p < \infty$, $1 < p^* \le 2$ be the conjugate of p, $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in l_\infty$ and $U\colon C([0,1], L_{p^*}[0,1]) \to l_p$ the operator defined by

$$U(f) = \left(\alpha_n \int_0^1 \left(\int_0^1 f(t)(s) r_n(s)\mathrm{d}s \right) r_n(t)\mathrm{d}t \right)_{n\in\mathbb{N}}.$$

Then U is a Dunford–Pettis operator $\Leftrightarrow U$ is compact $\Leftrightarrow \alpha \in c_0$.

(g) Let $1 < p < 2$, $g = (g_n)_{n\in\mathbb{N}} \in L_\infty([0,1], l_p)$ and let $U\colon C([0,1], L_1[0,1]) \to l_p$ be the operator defined by

$$U(f) = \left(\int_0^1 \left(\int_0^1 f(t)(s)g_n(s)\mathrm{d}s\right) r_n(t)\mathrm{d}t\right)_{n\in\mathbb{N}}.$$

Then U is a Dunford–Pettis operator, and U is compact $\Leftrightarrow$ g has an essentially relatively compact range.

Proof.

(a) Take $x_n^* \in L_2[0,1]$ defined by $x_n^*(f) = \int_0^1 f(s)s^n\mathrm{d}s$. By Hilbert's theorem, the operator $H\colon L_2[0,1] \to l_2$ defined by $H(f) = \left(\int_0^1 f(s)s^n\mathrm{d}t\right)_{n\in\mathbb{N}}$ is bounded linear. Now $H\colon L_2[0,1] \to c_0$ is compact, because $\|x_n^*\| = \frac{1}{\sqrt{2n}} \to 0$, $L_2[0,1]$ is a Hilbert space, hence of type 2, and by Proposition 11(e)(ii) we obtain the statement.

(b) Let $x_n^* \in l_p^*$ be defined by $x_n^*((x_n)_{n\in\mathbb{N}}) = \frac{x_1+\cdots+x_n}{n}$. Then the Hardy operator $H\colon l_p \to l_p$ defined by $H(x) = (x_n^*(x))_{n\in\mathbb{N}}$ is bounded linear.

(i) It follows from Proposition 11(d)(i) and (b) and the well-known fact that the Hardy operator is not compact.

(ii) In this case $l_p^* = l_{p^*}$ has type p^*, where $1 < p^* \le 2$, $\|x_n^*\| = \frac{1}{n^{\frac{1}{p}}} \to 0$ and we apply Proposition 11(e)(ii).

(c) The fact that U is a Dunford–Pettis operator follows from Theorem 3.1 in [10], because l_1 has the Schur property. Define $x_n^* \in l_1^*$ by $x_n^*(x) = \alpha_n x_n$ and observe that, since $\alpha \in l_\infty$, then the multiplication operator $M\colon l_1 \to l_p$ defined by $M(x) = (x_n^*(x))_{n\in\mathbb{N}}$ is well-defined bounded linear. If U is compact, then by Proposition 11(c), $|\alpha_n| = \|x_n^*\| \to 0$.

Conversely, if $\alpha \in c_0$, then M is compact and hence by Proposition 11(a), U is compact.

(d) and (e). In our hypothesis, for any $1 < p < \infty$ the operator $T\colon C[0,1] \to l_p$ defined by $T(f) = \left(\alpha_n \int_0^1 f(s)r_n(s)\mathrm{d}s\right)_{n\in\mathbb{N}}$ is bounded linear (for $1 < p < 2$ by the Hölder–Bessel inequality and $\alpha \in l_r$, while for $2 \le p < \infty$ by $\|\|_p \le \|\|_2$ and the Bessel inequality).

(e) Since $\alpha \in l_r$ for any $f \in C[0,1]$ we have the inequality

$$l_r\left(\alpha_k \int_0^1 f(s)r_k(s)\mathrm{d}s | k \ge n\right) \le l_r(\alpha_k | k \ge n)\|f\|.$$

From Exercise 6, p. 6 of [4], $T\colon C[0,1] \to l_r$ is compact. Hence by the equivalences in the proof of Proposition 11(d)(ii) $\left(\frac{1}{p} = \frac{1}{2} + \frac{1}{r}\right)$, U is compact.

(f) Since $C([0,1], C[0,1])$ is isometric and isomorph with $C([0,1]^2)$ and l_p is reflexive, U is weakly compact and hence is Dunford–Pettis (see Corollary 6, p. 154 of [6]).

If U is compact, then by Proposition 11(c), $|\alpha_n| = \|x_n^*\| \to 0$.

For the converse, observe that for any $f \in C[0,1]$ we have the following chain of inequalities:

$$l_p\left(\alpha_k \int_0^1 f(s)r_k(s)\mathrm{d}s | k \ge n\right) \le (\sup_{k\ge n} |\alpha_n|) l_p\left(\int_0^1 f(s)r_k(s)\mathrm{d}s | k \ge n\right)$$

$$\leq (\sup_{k\geq n} |\alpha_n|) l_2 \left(\int_0^1 f(s) r_k(s) \mathrm{d}s | k \geq n \right)$$

$$\leq (\sup_{k\geq n} |\alpha_n|) \left(\int_0^1 |f(s)|^2 \mathrm{d}s \right)^{1/2}$$

$$\leq \|f\| (\sup_{k\geq n} |\alpha_n|).$$

Since $\alpha \in c_0$ and from Exercise 6, p. 6 of [4], it follows that $T\colon C[0,1] \to l_p$ is compact and by Proposition 11(a), U will be compact.

(g) By the Hausdorff–Young inequality, the operator $T\colon L_{p^*}[0,1] \to l_p$ defined by $T(f) = \left(\alpha_n \int_0^1 f(s) r_n(s) \mathrm{d}s\right)_{n\in\mathbb{N}}$ is bounded linear.

Suppose U is a Dunford–Pettis operator. Then by Proposition 11(e)(i) $T\colon L_{p^*}[0,1] \to c_0$ is a Dunford–Pettis operator and thus is compact, since $L_{p^*}[0,1]$ is reflexive, hence $|\alpha_n| = \|x_n^*\| \to 0$. The converse follows from Proposition 11(e)(ii), since $(L_{p^*}[0,1])^* = L_p[0,1]$ has type $\min(p,2) = 2$ (see Corollary 11.7, p. 219 of [5]).

(h) In our hypothesis the operator $T\colon L_1[0,1] \to l_p$ defined by $T(f) = \left(\int_0^1 f(s) g_n(s) \mathrm{d}s\right)_{n\in\mathbb{N}}$ is bounded linear and weakly compact, and hence is a Dunford–Pettis operator (see Lemma 4, p. 62 of [6]) and Dunford–Pettis theorem (p. 76). By Proposition 11(d)(i), U is a Dunford–Pettis operator. For the compactness we use Proposition 11(d)(ii) and the representation of compact operators on $L_1(\mu)$, (see p. 68 of [6]).

Now we analyze the case l_1. Since l_1 has the Schur property (see Chapter 1 of [5]), we study only the compactness.

PROPOSITION 13

Let X be a Banach space, $(x_n^*)_{n\in\mathbb{N}} \in w_1(X^*)$, $T\colon X \to l_1$ defined by $T(x) = (x_n^*(x))_{n\in\mathbb{N}}$ and let $U\colon C([0,1], X) \to l_1$ be the operator defined by

$$U(f) = \left(\int_0^1 x_n^* f(t) r_n(t) \mathrm{d}t \right)_{n\in\mathbb{N}}.$$

(a) *If the series $\sum_{n=1}^{\infty} x_n^*$ is unconditionally norm convergent i.e. $T\colon X \to l_1$ is a compact operator, then U is a compact operator.*

(b) *If U is a compact operator, then $T\colon X \to l_2$ is a compact operator.*

Proof.

(a) It is analogous with we gave in Proposition 11(a).

(b) If U is compact, then, see Theorem 6 of [1] and Exercise 6, p. 6 of [4]. For any $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that for any $n \geq n_\varepsilon$ it follows that for any $f \in B(\Sigma, X)$ with $\|f\| \leq 1$ we have

$$\sum_{k=n_\varepsilon}^{n} \left| \int_0^1 x_k^* f(t) r_k(t) \mathrm{d}t \right| < \varepsilon.$$

Let $n \geq n_\varepsilon$. In particular, by the above inequality, for any $E \in \Sigma$ and $\|x\| \leq 1$ we have

$$\left| \int_E \left(\sum_{k=n_\varepsilon}^{n} x_k^*(x) r_k(t) \right) \mathrm{d}t \right| < \varepsilon.$$

Then for any $\|x\| \leq 1$ we obtain

$$\frac{1}{4} \int_0^1 \left| \sum_{k=n_\varepsilon}^{n} x_k^*(x) r_k(t) \right| \mathrm{d}t \leq \sup_{E \in \Sigma} \left| \int_E \left(\sum_{k=n_\varepsilon}^{n} x_k^*(x) r_k(t) \right) \mathrm{d}t \right| \leq \varepsilon$$

and, by Khinchin's inequality, $\frac{1}{4\sqrt{2}} \left(\sum_{k=n_\varepsilon}^{n} |x_k^*(x)|^2 \right)^{1/2} \leq \varepsilon$, which by Exercise 6, p. 6 of [4], means that $T\colon X \to l_2$ is compact.

Now a concrete example.

Example 14.

(a) Let $a = (a_n)_{n \in \mathbb{N}} \in l_\infty$ and let $U\colon C([0,1], l_1) \to l_1$ be the operator defined by

$$U(f) = \left(a_n \int_0^1 \langle f(t), e_n \rangle r_n(t) \mathrm{d}t \right)_{n \in \mathbb{N}}.$$

Then U is a compact operator $\Leftrightarrow a \in c_0$.

(b) Let $g = (g_n)_{n \in \mathbb{N}} \in L_\infty([0,1], l_1)$ and $U\colon C([0,1], L_1[0,1]) \to l_1$ be the operator defined by

$$U(f) = \left(\int_0^1 \left(\int_0^1 f(t)(s) g_n(s) \mathrm{d}s \right) r_n(t) \mathrm{d}t \right)_{n \in \mathbb{N}}.$$

(i) If the function g has an essentially relatively compact range in l_1, then the operator U is compact.

(ii) If U is a compact operator, then the function $g \in L_\infty([0,1], l_1)$ has an essentially relatively compact range in l_2.

(c) Let $a = (a_n)_{n \in \mathbb{N}} \in l_\infty$, $(E_n)_{n \in \mathbb{N}} \subseteq \Sigma$ be a sequence of pair-wise disjoint and non-negligible Lebesgue sets. The operator $U\colon C([0,1], L_1[0,1]) \to l_1$ defined by

$$U(f) = \left(\int_0^1 a_n \left(\int_{E_n} f(t)(s) \mathrm{d}s \right) r_n(t) \mathrm{d}t \right)_{n \in \mathbb{N}}$$

is a compact operator $\Leftrightarrow a \in c_0$.

Proof.

(a) If U is compact, then by Proposition 13(b) the multiplication operator $M\colon l_1 \to l_2$ defined by $M(x_n)_{n \in \mathbb{N}} = (a_n x_n)_{n \in \mathbb{N}}$ is compact, which implies that $a \in c_0$. Conversely, if $a \in c_0$, then the operator $M\colon l_1 \to l_1$ defined by $M(x_n)_{n \in \mathbb{N}} = (a_n x_n)_{n \in \mathbb{N}}$ is compact and thus, by Proposition 13(a), U is compact.

(b) (i) It follows from Remark 9(b) and Proposition 13(a). (ii) If U is a compact operator, then by Proposition 13(b), the operator $T\colon L_1[0,1] \to l_2$ defined by $T(f) = \int_0^1 f(t)g(t)$ is compact and using the representation of compact operators on $L_1(\mu)$ in p. 68 of [6], the statement follows.

(c) By hypothesis $g = (a_n \chi_{E_n})_{n\in\mathbb{N}} \in L_\infty([0,1], l_1)$. If U is compact, then by (b)(ii), g has an essentially relatively compact range in l_2 i.e. there is $A \in \Sigma$ with $\mu(A) = 0$ such that for any $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that $\sup_{t\notin A} \sum_{k=n_\varepsilon}^{\infty} |a_k \chi_{E_k}(t)|^2 < \varepsilon^2$. Let $n \geq n_\varepsilon$. Since E_n is a non-negligible Lebesgue, there is $t \in E_n - A$ and thus $|a_n|^2 < \varepsilon^2$ i.e. $a_n \to 0$.

The converse follows from inequality $\sum_{k=n}^{\infty} |a_n \chi_{E_k}(t)| \leq \sup_{k\geq n} |a_k|$ and (b)(i).

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