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On equivariant Dirac operators for $SU_q(2)$

PARTHA SARATHI CHAKRABORTY and ARUPKUMAR PAL*

Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai 600 113, India
*Indian Statistical Institute, 7, SJSS Marg, New Delhi 110 016, India
E-mail: parthac@imsc.res.in; arup@isid.ac.in

Dedicated to Prof. Kalyan Sinha on his sixtieth birthday

Abstract. We explain the notion of minimality for an equivariant spectral triple and show that the triple for the quantum $SU(2)$ group constructed by Chakraborty and Pal in [2] is minimal. We also give a decomposition of the spectral triple constructed by Dabrowski *et al* [8] in terms of the minimal triple constructed in [2].

Keywords. Spectral triple; quantum group.

1. Introduction

The interaction between noncommutative geometry and quantum groups, in particular the (noncommutative) geometry of quantum groups, had been one of the less understood and less explored areas of both the theories for a while. In the last few years, however, there has been some progress in this direction. The first important step was taken by the authors in [2] where they found an optimal family of Dirac operators for the quantum $SU(2)$ group acting on $L_2(h)$, the L_2 space of the Haar state h, and equivariant with respect to the (co-)action of the group itself. This family has quite a few remarkable features. They are:

1. Any element of the K-homology group can be realized by a member from this family, which means that all elements of the K-homology group are realizable through some Dirac operator acting on the single Hilbert space $L_2(h)$ in a natural manner.
2. The sign of any equivariant Dirac operator on $L_2(h)$ is a compact perturbation of the sign of a Dirac operator from this family,
3. Given any equivariant Dirac operator $\widetilde{D}$ acting on $L_2(h)$, and any Dirac operator D from this family, there exist two positive reals k_1 and k_2 such that

$$|\widetilde{D}| \leq k_1 + k_2|D|.$$

4. They exhibit features that are unique to the quantum case ($q \neq 1$). It was proved in [2] that for classical $SU(2)$, there does not exist any Dirac operator acting on (one copy of) the L_2 space that is both equivariant as well as 3-summable.

These triples were later analysed by Connes [6] in great detail, where the general theory of Connes–Moscovici was applied to obtain a beautiful local index formula for $SU_q(2)$.

Recently, Dabrowski *et al* [8] have constructed another family of Dirac operators that act on two copies of the L_2 space, has the right summability property, is equivariant in a sense described in [8], and is isospectral to the classical Dirac operator. In this note, we will give a decomposition of this Dirac operator in terms of the Dirac operators constructed in [2].

2. Equivariance and minimality

In this section, we will formulate the notion of an equivariant spectral triple for a compact quantum group and what one means by its minimality, or irreducibility. For basic notions on compact quantum groups, we refer the reader to [12]. To fix the notation, let us recall a few things briefly here. Let $G = (C(G), \Delta)$ be a compact quantum group, where $C(G)$ is the unital C^*-algebra of 'continuous functions on G' and Δ the comultiplication map. The symbols κ and h will denote the antipode map and the Haar state for G. For two functionals ρ and σ on $C(G)$, the convolution product $\rho * \sigma$ is the functional $a \mapsto (\rho \otimes \sigma)\Delta(a)$. For ρ as above and $a \in C(G)$, we will denote by $a * \rho$ the element $(\mathrm{id} \otimes \rho)\Delta(a)$ and by $\rho * a$ the element $(\rho \otimes \mathrm{id})\Delta(a)$. A unitary representation u of G acting on a Hilbert space $\mathcal{H}$ is a unitary element of the multiplier algebra $M(\mathcal{K}(\mathcal{H}) \otimes C(G))$, where $\mathcal{K}(\mathcal{H})$ denotes the space of compact operators on $\mathcal{H}$, that satisfies the condition $(\mathrm{id} \otimes \Delta)u = u_{12}u_{13}$. For a unitary representation u and a continuous linear functional ρ on $C(G)$, we will denote by u_ρ the operator $(\mathrm{id} \otimes \rho)u$ on $\mathcal{H}$. The GNS space associated with the state h will be denoted by $L_2(h)$ and the cyclic vector will be denoted by Ω. While using the comultiplication Δ, we will often use the Sweedler notation (i.e. $\Delta(a) = a_{(1)} \otimes a_{(2)}$).

Let $\mathcal{A}$ be a unital C^*-algebra, G be a compact quantum group and let τ be an action of G on $\mathcal{A}$, i.e. τ is a unital C^*-homomorphism from $\mathcal{A}$ to $\mathcal{A} \otimes C(G)$ satisfying the condition $(\mathrm{id} \otimes \Delta)\tau = (\tau \otimes \mathrm{id})\tau$. In other words, let $(\mathcal{A}, G, \tau)$ be a C^*-dynamical system. Recall [1] that a covariant representation of $(\mathcal{A}, G, \tau)$ on a Hilbert space $\mathcal{H}$ is a pair (π, u) where π is a unital *-representation of $\mathcal{A}$ on $\mathcal{H}$, u is a unitary representation of G on $\mathcal{H}$ and they obey the condition

$$u(\pi(a) \otimes I)u^* = (\pi \otimes \mathrm{id})\tau(a), \quad a \in \mathcal{A}. \tag{2.1}$$

By an *odd G-equivariant spectral data for $\mathcal{A}$*, we mean a quadruple $(\pi, u, \mathcal{H}, D)$ where

1. (π, u) is a covariant representation of $(\mathcal{A}, G, \tau)$ on the Hilbert space $\mathcal{H}$,
2. π is faithful,
3. $u(D \otimes I)u^* = D \otimes I$,
4. $(\pi, \mathcal{H}, D)$ is an odd spectral triple.

We will often be sloppy and just say $(\pi, \mathcal{H}, D)$ is an odd G-equivariant spectral triple for $\mathcal{A}$, omitting u. We say that an operator D on a Hilbert space $\mathcal{H}$ is an *odd G-equivariant Dirac operator* for $\mathcal{A}$ if there exists a unitary representation u of G on $\mathcal{H}$ such that $(\pi, u, \mathcal{H}, D)$ gives a G-equivariant spectral data for $\mathcal{A}$.

Similarly, an even G-equivariant spectral data for $\mathcal{A}$ consists of an even spectral data $(\pi, u, \mathcal{H}, D, \gamma)$ where $(\pi, u, \mathcal{H}, D)$ obeys conditions 1, 2 and 3 above, and moreover $(\pi, \mathcal{H}, D, \gamma)$ is an even spectral 'triple', and one has $u(\gamma \otimes I)u^* = \gamma \otimes I$. An even G-equivariant Dirac operator is also defined similarly.

We say that an equivariant odd spectral data $(\pi, u, \mathcal{H}, D)$ is *minimal* if the covariant representation (π, u) is irreducible.

Note that if we take $\mathcal{A} = C(G)$, then the groups G and G_{op} have natural actions Δ and Δ_{op} on $\mathcal{A}$. In what follows, we will mainly be concerned about these two systems $(\mathcal{A} = C(G), G, \Delta)$ and $(\mathcal{A} = C(G), G_{\mathrm{op}}, \Delta_{\mathrm{op}})$. A G-equivariant spectral triple for $C(G)$ will be called a *right equivariant spectral triple* for $C(G)$. A *right equivariant Dirac operator* for $C(G)$ will mean a G-equivariant Dirac operator for $C(G)$. Similarly, a G_{op}-equivariant spectral triple for $C(G)$ will be called a *left equivariant spectral triple* for

$C(G)$ and a $G_{\rm op}$-equivariant Dirac operator for $C(G)$ will be called a *left equivariant Dirac operator* for $C(G)$.

We will next study covariant representations of the right G-action on $C(G)$, i.e. representations of the system $(C(G), G, \Delta)$.

Lemma 2.1. *Let (π, u) be a covariant representation of $(C(G), G, \Delta)$. If the Haar state h of G is faithful, then π is faithful.*

Proof. Assume $\pi(a) = 0$. Then $\pi(a^*a) = 0$ and hence $(\pi \otimes \mathrm{id})\Delta(a^*a) = u(\pi(a^*a) \otimes I)u^* = 0$. Applying $(\mathrm{id} \otimes h)$ on both sides, we get $h(a^*a)I = 0$. Since h is faithful, $a = 0$. □

Remark 2.2. The above lemma helps ensure that if we have a compact quantum group with a faithful Haar state, take a covariant representation (π, u) of the system $(C(G), G, \Delta)$ on a Hilbert space $\mathcal{H}$, and look at a Dirac operator D on $\mathcal{H}$, then we really get a spectral triple for the space G rather than that of some subspace (i.e. quotient C^*-algebra of $C(G)$) of it.

Lemma 2.3. *Let (π, u) be a covariant representation of $(C(G), G, \Delta)$. Then the operator u_h is a projection and for any continuous linear functional ρ on $\mathcal{A}$, one has $u_h u_\rho = u_\rho u_h = \rho(1)u_h$.*

Proof. Using Peter–Weyl decomposition for u, one can assume without loss in generality that u is finite dimensional. Take two vectors w and w' in $\mathcal{H}$. Then

$$
\begin{aligned}
\langle w, u_h w' \rangle &= \langle w \otimes \Omega, u(w' \otimes \Omega) \rangle \\
&= \langle u^*(w \otimes \Omega), w' \otimes \Omega \rangle \\
&= \langle ((\mathrm{id} \otimes \kappa)u)(w \otimes \Omega), w' \otimes \Omega \rangle \\
&= \overline{\langle w' \otimes \Omega, ((\mathrm{id} \otimes \kappa)u)(w \otimes \Omega) \rangle} \\
&= \overline{\langle w', ((\mathrm{id} \otimes h\kappa)u)w \rangle} \\
&= \overline{\langle w', ((\mathrm{id} \otimes h)u)w \rangle} \\
&= \langle u_h w, w' \rangle.
\end{aligned}
$$

Thus u_h is self-adjoint.

Next, for any continuous linear functional ρ,

$$
\begin{aligned}
u_\rho u_h &= (\mathrm{id} \otimes \rho)u(\mathrm{id} \otimes h)u \\
&= (\mathrm{id} \otimes \rho \otimes h)(u_{12}u_{13}) \\
&= (\mathrm{id} \otimes \rho \otimes h)(\mathrm{id} \otimes \Delta)u \\
&= (\mathrm{id} \otimes \rho * h)u \\
&= \rho(1)u_h.
\end{aligned}
$$

Similary one has $u_h u_\rho = \rho(1)u_h$. In particular, $u_h^2 = u_h$, so that u_h is a projection. □

Lemma 2.4. *Let $A \equiv A(G)$ be the *-subalgebra of $C(G)$ generated by matrix entries of all finite dimensional unitary representations of G. Let $(A, \mathcal{U})$ be a dual pair of Hopf*-algebras* (*see* [11]). *Then*

$$u_\rho \pi(a) = \pi(a * \rho_{(1)}) u_{\rho_{(2)}} \quad \text{for all } \rho \in \mathcal{U} \text{ and } a \in A(G). \tag{2.2}$$

Proof. Apply $(\mathrm{id} \otimes \rho)$ on both sides in the equality

$$u(\pi(a) \otimes I) = ((\pi \otimes \mathrm{id})\Delta(a))u$$

and use the fact that $\rho(ab) = \rho_{(1)}(a)\rho_{(2)}(b)$. □

Lemma 2.5. *Let $w \in \mathcal{H}$ be a vector in the range of u_h. Then for any $a \in A(G)$ and $\rho \in \mathcal{U}$, one has $u_\rho \pi(a) w = \pi(a * \rho) w$. In particular, one has $u_h \pi(a) w = h(a) w$.*

Proof. Use Lemma 2.4 to get

$$\begin{aligned} u_\rho \pi(a) w &= \pi(a * \rho_{(1)}) u_{\rho_{(2)}} w \\ &= \pi(a * \rho_{(1)}) u_{\rho_{(2)}} u_h w \\ &= \rho_{(2)}(1) \pi(a * \rho_{(1)}) u_h w \\ &= \pi(\rho_{(2)}(1) a * \rho_{(1)}) w \\ &= \pi(a_{(1)} \rho_{(1)}(a_{(2)}) \rho_{(2)}(1)) w \\ &= \pi(a_{(1)} \rho(a_{(2)})) w \\ &= \pi(a * \rho) w, \end{aligned}$$

for $a \in A(G)$. □

Lemma 2.6. *The linear span of $\{\pi(a) u_h w : a \in A(G), w \in \mathcal{H}\}$ is dense in $\mathcal{H}$. In particular, u_h is nonzero.*

Proof. Using Peter–Weyl decomposition of u and the observation that $h(\kappa(a)) = h(a)$ for all $a \in A$, it follows that $u_h = (u^*)_h$. Now take a vector w' in $\mathcal{H}$ such that $\langle w', \pi(a) u_h w \rangle = 0$ for all $w \in \mathcal{H}$ and $a \in A$. But then $\langle w', \pi(a)(u^*)_h w \rangle = 0$, i.e. $\langle w' \otimes \Omega, (\pi(a) \otimes I) u^* (w \otimes \Omega) \rangle = 0$. The covariance condition (2.1) now gives $\langle u(w' \otimes \Omega), (\pi \otimes \mathrm{id})\Delta(a)(w \otimes \Omega) \rangle = 0$ for all $w \in \mathcal{H}$ and $a \in A$. In particular, one has $\langle u(w' \otimes \Omega), (\pi \otimes \mathrm{id})\Delta(a)(\pi(b) w \otimes \Omega) \rangle = 0$ for all $w \in \mathcal{H}$, and $a, b \in A$. Since $(\pi \otimes \mathrm{id})\Delta(a)(\pi(b) \otimes I) = (\pi \otimes \mathrm{id})(\Delta(a)(b \otimes I))$ and $\{\Delta(a)(b \otimes I) : a, b \in A\}$ is total in $\mathcal{A} \otimes \mathcal{A}$, we get $u(w' \otimes \Omega) = 0$ and consequently $w' = 0$. □

For $w \in \mathcal{H}$, denote by P_w the projection onto the closed linear span of $\{\pi(a) w : a \in A\}$.

Lemma 2.7. *Let $w \in u_h \mathcal{H}$. Then $(P_w \otimes I) u (P_w \otimes I) = u(P_w \otimes I)$. If w' is another vector in $u_h \mathcal{H}$ such that $\langle w, w' \rangle = 0$, then the projections P_w and $P_{w'}$ are orthogonal.*

Proof. For the first part, it is enough to show that $P_w u_\rho P_w = u_\rho P_w$ for all $\rho \in \mathcal{U}$. But this is clear because from Lemma 2.5, we have $u_\rho \pi(a) w = \pi(a * \rho) w$.

For the second part, take $a, a' \in A$. Then using Lemma 2.6 one gets

$$\begin{aligned}\langle \pi(a)w, \pi(a')w' \rangle &= \langle w, \pi(a^*a')w' \rangle \\ &= \langle u_h w, \pi(a^*a')w' \rangle \\ &= \langle w, u_h \pi(a^*a')w' \rangle \\ &= \langle w, h(a^*a')w' \rangle \\ &= 0.\end{aligned}$$

Thus P_w and $P_{w'}$ are orthogonal. □

PROPOSITION 2.8

Let $\{w_1, w_2, \ldots\}$ be an orthonormal basis for $u_h\mathcal{H}$. Write P_n for P_{w_n}, and let $\pi_n(\cdot) := P_n\pi(\cdot)P_n$, $u_n := (P_n \otimes I)u(P_n \otimes I)$. Then

1. *For each n, (π_n, u_n) is a covariant representation of the system $(\mathcal{A}, G, \Delta)$ on $P_n\mathcal{H}$,*
2. *$\pi = \oplus\pi_n$, $u = \oplus u_n$,*
3. *(π_n, u_n) is unitarily equivalent to the pair $(\pi_{\mathrm{L}}, u_{\mathrm{R}})$ where π_{L} is the representation of $\mathcal{A}$ on $L_2(G)$ by left multiplications and u_{R} is the right regular representation of G.*

Proof. It follows from Lemmas 2.7 and 2.6 that P_n's are orthogonal, $\sum P_n = I$ and consequently $\pi = \oplus\pi_n$ and $u = \oplus u_n$.

Define V_n: $P_n\mathcal{H} \to L_2(G)$ by

$$V_n\pi(a)w_n = \pi_L(a)\Omega, \quad a \in A.$$

Since $\langle \pi_{\mathrm{L}}(a)\Omega, \pi_{\mathrm{L}}(b)\Omega \rangle = h(a^*b) = \langle \pi(a)w_n, \pi(b)w_n \rangle$, $\{\pi(a)w_n \colon a \in A\}$ is total in $P_n\mathcal{H}$ and $\{\pi_{\mathrm{L}}(a)\Omega \colon a \in A\}$ is total in $L_2(G)$, V_n extends to a unitary from $P_n\mathcal{H}$ onto $L_2(G)$. Next, for $a, b \in A$, one has $V_n\pi(a)\pi(b)w_n = V_n\pi(ab)w_n = \pi_{\mathrm{L}}(ab)\Omega = \pi_{\mathrm{L}}(a)\pi_{\mathrm{L}}(b)\Omega = \pi_{\mathrm{L}}(a)V_n\pi(b)w_n$. So $V_n\pi(a) = \pi_{\mathrm{L}}(a)V_n$ for all $a \in A$ and hence for all $a \in \mathcal{A}$.

Finally, we will show that $(V_n \otimes I)u(V_n^* \otimes I) = u_{\mathrm{R}}$. Write $\tilde{u}_n := (V_n \otimes I)u(V_n^* \otimes I)$. Then for any $\rho \in \mathcal{U}$, one has

$$\begin{aligned}(\mathrm{id} \otimes \rho)\tilde{u}_n\pi_{\mathrm{L}}(a)\Omega &= V_n u_\rho V_n^* \pi_{\mathrm{L}}(a)\Omega \\ &= V_n u_\rho \pi(a) w_n \\ &= V_n u_\rho \pi(a) u_h w_n \\ &= V_n \pi(a * \rho) w_n \\ &= V_n \pi(a * \rho) V_n^* V_n w_n \\ &= \pi_{\mathrm{L}}(a * \rho)\Omega.\end{aligned}$$

By [12], $\tilde{u}_n$ must be the right regular representation u on $L_2(G)$. □

Remark 2.9. The above proposition leads to an alternative proof of the Takesaki–Takai duality for compact quantum groups (Theorem 7.5 of [1]).

Theorem 2.10. *The covariant representation (π, u) is irreducible if and only if the operator u_h is a rank one projection.*

Proof. Immediate corollary of Proposition 2.8. □

Remark 2.11. In particular, it follows from the above theorem that the covariant representation $(\pi_{\mathrm{L}}, u_{\mathrm{R}})$ on $L_2(G)$ is irreducible. Thus the equivariant Dirac operator constructed in [2] is *minimal*.

3. The decomposition

Canonical triples for $SU_q(2)$

Let q be a real number in the interval (0, 1). Let $\mathcal{A}$ denote the C^*-algebra of continuous functions on $SU_q(2)$, which is the universal C^*-algebra generated by two elements α and β subject to the relations

$$\alpha^*\alpha + \beta^*\beta = I = \alpha\alpha^* + q^2\beta\beta^*, \quad \alpha\beta - q\beta\alpha = 0 = \alpha\beta^* - q\beta^*\alpha,$$
$$\beta^*\beta = \beta\beta^*$$

as in [2]. Let $\pi\colon \mathcal{A} \to \mathcal{L}(L_2(h))$ be the representation given by left multiplication by elements in $\mathcal{A}$. Let u denote the right regular representation of $SU_q(2)$. Recall [12] that u is the unique representation acting on $L_2(h)$ that obeys the condition

$$((\mathrm{id} \otimes \rho)u)\pi(a)\Omega = \pi((\mathrm{id} \otimes \rho)\Delta(a))\Omega \tag{3.3}$$

for all $a \in \mathcal{A}$ and for all continuous linear functionals ρ on $\mathcal{A}$. In [2], the authors studied right equivariant Dirac operators, those Dirac operators that commute with the right regular representation, i.e. D acting on $L_2(h)$ for which

$$(D \otimes I)u = u(D \otimes I).$$

In particular, an optimal family of equivariant Dirac operators were found. A generic member of this family is of the form

$$e_{ij}^{(n)} \mapsto \begin{cases} (an+b)e_{ij}^{(n)}, & \text{if } -n \leq i < n-k, \\ (cn+d)e_{ij}^{(n)}, & \text{if } i = n-k, n-k+1, \ldots, n, \end{cases}$$

where k is a fixed nonnegative integer and a, b, c, d are reals with $ac < 0$. If one looks at left equivariant Dirac operators, the same arguments would then lead to the following theorem.

Theorem 3.1. *Let v be the left regular representation of $SU_q(2)$. Let k be a nonnegative integer and let a, b, c, d be real numbers with $ac < 0$. Then the operator $D \equiv D(k, a, b, c, d)$ on $L_2(h)$ given by*

$$e_{ij}^{(n)} \mapsto \begin{cases} (an+b)e_{ij}^{(n)}, & \text{if } -n \leq j < n-k, \\ (cn+d)e_{ij}^{(n)}, & \text{if } j = n-k, n-k+1, \ldots, n, \end{cases}$$

gives a spectral triple $(\pi, L_2(h), D)$ having nontrivial Chern character and obeys

$$(D \otimes I)v = v(D \otimes I). \tag{3.4}$$

Conversely, given any spectral triple $(\pi, L_2(h), \tilde{D})$ with nontrivial Chern character such that $(\tilde{D} \otimes I)v = v(\tilde{D} \otimes I)$, there exist a nonnegative integer k and reals a, b, c, d with $ac < 0$ such that

1. sign $\tilde{D}$ *is a compact perturbation of the sign of* $D \equiv D(k, a, b, c, d)$*, and*
2. *there exist constants* k_1 *and* k_2 *such that*

$$|\tilde{D}| \leq k_1 + k_2|D|.$$

Proof. The key point is to note that the characterizing property of the left regular representation v is

$$((\mathrm{id} \otimes \rho)v^*)\pi(a)\Omega = \pi((\rho \otimes \mathrm{id})\Delta(a))\Omega. \tag{3.5}$$

Thus on the right-hand side, one now has left convolution of a by ρ instead of right convolution by ρ. Therefore any self-adjoint operator on $L_2(h)$ with discrete spectrum that obeys $(D \otimes I)v = v(D \otimes I)$ will be of the form

$$e_{ij}^{(n)} \mapsto \lambda(n, j)e_{ij}^{(n)}.$$

Hence if one now proceeds exactly along the same lines as in [2], one gets all the desired conclusions. □

Observe at this point that the whole analysis carried out in [6] will go through for this Dirac operator as well. Let us now take two such Dirac operators D_1 and D_2 on $L_2(h)$ given by

$$D_1 e_{ij}^{(n)} = \begin{cases} -2n e_{ij}^{(n)}, & \text{if } j \neq n \\ (2n+1)e_{ij}^{(n)}, & \text{if } j = n \end{cases},$$

$$D_2 e_{ij}^{(n)} = \begin{cases} (-2n-1) e_{ij}^{(n)}, & \text{if } j \neq n \\ (2n+1)e_{ij}^{(n)}, & \text{if } j = n \end{cases}. \tag{3.6}$$

Now look at the triple

$$(\pi \oplus \pi, L_2(h) \oplus L_2(h), D_1 \oplus |D_2|).$$

It is easy to see that this is a spectral triple. Nontriviality of its Chern character is a direct consequence of that of D_1. We will show in the next paragraph that in a certain sense, the spectral triple constructed in [8] is equivalent to this above triple.

The decomposition

Let us briefly recall the Dirac operator constructed in [8]. The carrier Hilbert space $\mathscr{H}$ is a direct sum of two copies of $L_2(h)$ that decomposes as

$$\mathscr{H} = W_0^{\uparrow} \oplus \left(\bigoplus_{n \in \frac{1}{2}\mathbb{Z}_+} (W_n^{\uparrow} \oplus W_n^{\downarrow}) \right),$$

where

$$W_n^{\uparrow} = \operatorname{span}\Big\{u_{ij}^n \colon i = -n, -n+1, \ldots, n,$$
$$j = -n - \frac{1}{2}, -n + \frac{1}{2}, \ldots, n + \frac{1}{2}\Big\},$$

$$W_n^{\downarrow} = \operatorname{span}\Big\{d_{ij}^n \colon i = -n, -n+1, \ldots, n,$$
$$j = -n + \frac{1}{2}, -n + \frac{3}{2}, \ldots, n - \frac{1}{2}\Big\}.$$

(u_{ij}^n and d_{ij}^n correspond to the basis elements $|nij\uparrow\rangle$ and $|nij\downarrow\rangle$ respectively in the notation of [8].) Now write

$$v_{ij}^n = \begin{pmatrix} u_{ij}^n \\ d_{ij}^n \end{pmatrix}$$

with the convention that $d_{ij}^n = 0$ for $j = \pm\big(n + \frac{1}{2}\big)$. Then the representation π' of $\mathcal{A}$ on $\mathscr{H}$ is given by

$$\pi'(\alpha^*)\, v_{ij}^n = a_{nij}^+ v_{i+\frac{1}{2},j+\frac{1}{2}}^{n+\frac{1}{2}} + a_{nij}^- v_{i+\frac{1}{2},j+\frac{1}{2}}^{n-\frac{1}{2}},$$

$$\pi'(-\beta)\, v_{ij}^n = b_{nij}^+ v_{i+\frac{1}{2},j-\frac{1}{2}}^{n+\frac{1}{2}} + b_{nij}^- v_{i+\frac{1}{2},j-\frac{1}{2}}^{n-\frac{1}{2}},$$

$$\pi'(\alpha)\, v_{ij}^n = \tilde{a}_{nij}^+ v_{i-\frac{1}{2},j-\frac{1}{2}}^{n+\frac{1}{2}} + \tilde{a}_{nij}^- v_{i-\frac{1}{2},j-\frac{1}{2}}^{n-\frac{1}{2}},$$

$$\pi'(-\beta^*)\, v_{ij}^n = \tilde{b}_{nij}^+ v_{i-\frac{1}{2},j+\frac{1}{2}}^{n+\frac{1}{2}} + \tilde{b}_{nij}^- v_{i-\frac{1}{2},j+\frac{1}{2}}^{n-\frac{1}{2}},$$

where $a_{nij}^{\pm}$ and $b_{nij}^{\pm}$ are the following 2×2 matrices:

$$a_{nij}^+ = q^{\left(i+j-\frac{1}{2}\right)/2}[n+i+1]^{\frac{1}{2}} \begin{pmatrix} q^{-n-\frac{1}{2}} \frac{[n+j+\frac{3}{2}]^{1/2}}{[2n+2]} & 0 \\ q^{\frac{1}{2}} \frac{[n-j+\frac{1}{2}]^{1/2}}{[2n+1]\,[2n+2]} & q^{-n} \frac{[n+j+\frac{1}{2}]^{1/2}}{[2n+1]} \end{pmatrix},$$

$$a_{nij}^- = q^{\left(i+j-\frac{1}{2}\right)/2}[n-i]^{\frac{1}{2}} \begin{pmatrix} q^{n+1} \frac{[n-j+\frac{1}{2}]^{1/2}}{[2n+1]} & -q^{\frac{1}{2}} \frac{[n+j+\frac{1}{2}]^{1/2}}{[2n]\,[2n+1]} \\ 0 & q^{n+\frac{1}{2}} \frac{[n-j-\frac{1}{2}]^{1/2}}{[2n]} \end{pmatrix},$$

$$b_{nij}^+ = q^{\left(i+j-\frac{1}{2}\right)/2}[n+i+1]^{\frac{1}{2}} \begin{pmatrix} \frac{[n-j+\frac{3}{2}]^{1/2}}{[2n+2]} & 0 \\ -q^{-n-1} \frac{[n+j+\frac{1}{2}]^{1/2}}{[2n+1]\,[2n+2]} & q^{-\frac{1}{2}} \frac{[n-j+\frac{1}{2}]^{1/2}}{[2n+1]} \end{pmatrix},$$

$$b_{nij}^- = q^{\left(i+j-\frac{1}{2}\right)/2}[n-i]^{\frac{1}{2}} \begin{pmatrix} -q^{-\frac{1}{2}} \frac{[n+j+\frac{1}{2}]^{1/2}}{[2n+1]} & -q^{n} \frac{[n-j+\frac{1}{2}]^{1/2}}{[2n]\,[2n+1]} \\ 0 & -\frac{[n+j-\frac{1}{2}]^{1/2}}{[2n]} \end{pmatrix},$$

($[m]$ being the q-number $\frac{q^m-q^{-m}}{q-q^{-1}}$) and $\tilde{a}^{\pm}_{nij}$ and $\tilde{b}^{\pm}_{nij}$ are the hermitian conjugates of the above ones:

$$\tilde{a}^{\pm}_{nij} = (a^{\mp}_{n\pm\frac{1}{2},i-\frac{1}{2},j-\frac{1}{2}})^*, \qquad \tilde{b}^{\pm}_{nij} = (b^{\mp}_{n\pm\frac{1}{2},i-\frac{1}{2},j+\frac{1}{2}})^*.$$

The operator D is given by

$$Du^n_{ij} = (2n+1)u^n_{ij}, \qquad Dd^n_{ij} = -2nd^n_{ij}.$$

The triple $(\pi', \mathscr{H}, D)$ is precisely the triple constructed in [8].

Theorem 3.2. *Let $\mathcal{K}_q$ be the two-sided ideal of $\mathcal{L}(\mathscr{H})$ generated by the operator*

$$d^n_{ij} \mapsto q^n d^n_{ij}, \quad u^n_{ij} \mapsto q^n u^n_{ij},$$

*and let $\mathcal{A}_f$ denote the *-subalgebra of $\mathcal{A}$ generated by α and β. Then there is a unitary $U: L_2(h) \oplus L_2(h) \to \mathscr{H}$ such that*

$$U(D_1 \oplus |D_2|)U^* = D, \tag{3.7}$$

$$U(\pi(a) \oplus \pi(a))U^* - \pi'(a) \in \mathcal{K}_q \quad \textit{for all } a \in \mathcal{A}_f. \tag{3.8}$$

Proof. Define $U: L_2(h) \oplus L_2(h) \to \mathscr{H}$ as follows:

$$U(e^{(n)}_{ij} \oplus 0) = d^n_{i,j+\frac{1}{2}}, \quad i = -n, -n+1, \ldots, n,$$
$$j = -n, -n+1, \ldots, n-1,$$
$$U(e^{(n)}_{in} \oplus 0) = u^n_{i,n+\frac{1}{2}}, \quad i = -n, -n+1, \ldots, n,$$
$$U(0 \oplus e^{(n)}_{ij}) = u^n_{i,j-\frac{1}{2}}, \quad i = -n, -n+1, \ldots, n, \ j = -n, -n+1, \ldots, n.$$

It is immediate that $U(D_1 \oplus |D_2|)U^* = D$. Therefore all that we need to prove now is that $U(\pi(a) \oplus \pi(a))U^* - \pi'(a) \in \mathcal{K}_q$ for all $a \in \mathcal{A}_f$. For this, let us introduce the representation $\hat{\pi}: \mathcal{A} \to \mathcal{L}(L_2(h))$ given by

$$\hat{\pi}(\alpha) = \hat{\alpha}, \qquad \hat{\pi}(\beta) = \hat{\beta},$$

where $\hat{\alpha}$ and $\hat{\beta}$ are the following operators on $L_2(h)$ (see Lemma 2.2 of [3]):

$$\hat{\alpha}: e^{(n)}_{ij} \mapsto q^{2n+i+j+1} e^{\left(n+\frac{1}{2}\right)}_{i-\frac{1}{2},j-\frac{1}{2}} + (1-q^{2n+2i})^{\frac{1}{2}}(1-q^{2n+2j})^{\frac{1}{2}} e^{\left(n-\frac{1}{2}\right)}_{i-\frac{1}{2},j-\frac{1}{2}}, \tag{3.9}$$

$$\hat{\beta}: e^{(n)}_{ij} \mapsto -q^{n+j}(1-q^{2n+2i+2})^{\frac{1}{2}} e^{\left(n+\frac{1}{2}\right)}_{i+\frac{1}{2},j-\frac{1}{2}} + q^{n+i}(1-q^{2n+2j})^{\frac{1}{2}} e^{\left(n-\frac{1}{2}\right)}_{i+\frac{1}{2},j-\frac{1}{2}}. \tag{3.10}$$

It is easy to see that

$$\pi(a) \oplus \pi(a) - \hat{\pi}(a) \oplus \hat{\pi}(a) \in U^* \mathcal{K}_q U$$

for $a = \alpha^*$ and $a = \beta$. Therefore it is enough to verify that

$$U(\hat{\pi}(a) \oplus \hat{\pi}(a))U^* - \pi'(a) \in \mathcal{K}_q$$

for $a = \alpha^*$ and for $a = \beta$.

Next observe that

$$a^+_{nij} = (1 - q^{2n+2i+2})^{\frac{1}{2}} \begin{pmatrix} (1 - q^{2n+2j+3})^{\frac{1}{2}} & 0 \\ 0 & (1 - q^{2n+2j+1})^{\frac{1}{2}} \end{pmatrix} + O(q^{2n}),$$

$$a^-_{nij} = q^{2n+i+j+\frac{1}{2}}(1 - q^{2n-2i})^{\frac{1}{2}} \begin{pmatrix} q(1 - q^{2n-2j+1})^{\frac{1}{2}} & 0 \\ 0 & (1 - q^{2n-2j-1})^{\frac{1}{2}} \end{pmatrix}$$
$$+ O(q^{2n}),$$

$$b^+_{nij} = q^{n+j-\frac{1}{2}}(1 - q^{2n+2i+2})^{\frac{1}{2}} \begin{pmatrix} q(1 - q^{2n-2j+3})^{\frac{1}{2}} & 0 \\ 0 & (1 - q^{2n-2j+1})^{\frac{1}{2}} \end{pmatrix}$$
$$+ O(q^{2n}),$$
$$= q^{n+j-\frac{1}{2}}(1 - q^{2n+2i+2})^{\frac{1}{2}} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} + O(q^{2n}),$$

$$b^-_{nij} = -q^{n+i}(1 - q^{2n-2i})^{\frac{1}{2}} \begin{pmatrix} (1 - q^{2n+2j+1})^{\frac{1}{2}} & 0 \\ 0 & (1 - q^{2n+2j-1})^{\frac{1}{2}} \end{pmatrix}$$
$$+ O(q^{2n})$$
$$= -q^{n+i} \begin{pmatrix} (1 - q^{2n+2j+1})^{\frac{1}{2}} & 0 \\ 0 & (1 - q^{2n+2j-1})^{\frac{1}{2}} \end{pmatrix} + O(q^{2n}).$$

The required result now follows from this easily. □

Remark 3.3. The above decomposition in particular tells us that the spectral triples $(\pi \oplus \pi, L_2(h) \oplus L_2(h), D_1 \oplus |D_2|)$ and $(\pi', \mathscr{H}, D)$ are essentially unitarily equivalent at the Fredholm module level. Therefore by Proposition 8.3.14 of [9], they give rise to the same element in K-homology.

Remark 3.4. In the spectral triple in [8], the Hilbert space can be decomposed as a direct sum of two isomorphic copies in such a manner that in each half Dirac operator has constant sign. So positive and negative signs come with equal frequency. However this symmetry is only superficial, as the decomposition above illustrates. This asymmetry might be a reflection of the inherent asymmetry in the *growth graph* associated with quantum $SU(2)$ (see [4]). For classical $SU(2)$ the graph is symmetric whereas in the quantum case it is not.

It should also be pointed out here that, at least as far as classical odd dimensional spaces are concerned, this kind of sign symmetry is always superficial. They are always inherent in the even cases, but not in the odd ones.

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