Proc. Indian Acad. Sci. (Math. Sci.) Vol. 116, No. 4, November 2006, pp. 489–505.

Stochastic integral representations of quantum martingales on multiple Fock space

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Dedicated to Professor Kalyan B Sinha on the occasion of his 60th birthday

Abstract. In this paper a quantum stochastic integral representation theorem is obtained for unbounded regular martingales with respect to multidimensional quantum noise. This simultaneously extends results of Parthasarathy and Sinha to unbounded martingales and those of the author to multidimensions.

Keywords. Fock space; quantum stochastic process; quantum stochastic integral; quantum martingale.

1. Introduction

The stochastic integral representations of quantum martingales have been studied by many authors (see [2,6,7,10,11,14,15,18,19], etc). In [18], Parthasarathy and Sinha established a stochastic integral representation of a regular bounded quantum martingale on Fock space with respect to the basic martingales, namely the annihilation, creation and conservation processes. A new proof of the Parthasarathy and Sinha representation theorem has been discussed in [15] with the special form of the coefficient of the conservation process. In [9], by using the framework of Gaussian (white noise) analysis (see [5,16]), the author extended the Hudson and Parthasarathy quantum stochastic calculus and generalized the notion of regular martingale in the context of a certain triple of weights [3,12] and then the integral representation theorem for a regular (unbounded) quantum martingale was proved.

In this paper, we extend the results obtained in [9,18] for the representation of a regular martingale to the case of multiple Fock space with an initial Hilbert space. For our purpose, we first extend the quantum stochastic integral studied in [8,19] (see also [17]) to our setting.

The paper is organized as follows. In §2 we construct a rigging of multiple Fock space and briefly recall the basic quantum stochastic processes. In §3 we extend the quantum stochastic integral studied in [19] to a wider class of adapted quantum stochastic processes in our setting. In §4 we prove the main result (Theorem 4.5) for a stochastic integral representation of a (unbounded) regular quantum martingale on multiple Fock space.

We expect that the integral representation of quantum martingales have applications in Markovian cocycles [1,4,13]. Further study is now in progress.

Notations. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be locally convex spaces.
$\mathfrak{X} \otimes \mathfrak{Y}$: the Hilbert space tensor product when $\mathfrak{X}$ and $\mathfrak{Y}$ are Hilbert spaces.
$L(\mathcal{D}, \mathfrak{X})$: the space of all linear operators in $\mathfrak{X}$ with domain $\mathcal{D}$.
$\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$: the space of continuous linear operators from $\mathfrak{X}$ into $\mathfrak{Y}$ equipped with the topology of bounded convergence, see [16].

2. Multiple Fock space and basic processes

Let $H = L^2(\mathbf{R}_+, K) \cong L^2(\mathbf{R}_+) \otimes K$ be the Hilbert space of K-valued square integrable functions on $\mathbf{R}_+$ and B a selfadjoint operator in K with dense domain Dom (B) satisfying inf Spec $(B) \geq 1$, where $\mathbf{R}_+ = [0, \infty)$ and K is a separable Hilbert space called the *multiplicity space*. In fact, we take B of the form $\sum_{i\geq 1} \rho_i |e_i\rangle\langle e_i|$, where $\{e_i\}$ is an orthonormal basis for K and $\{\rho_i\}$ a sequence of real numbers greater than or equal to 1.

For each $p \in \mathbf{R}_+$, put

$$H_p = \text{Dom}\,(I \otimes B^p) \subset H$$

and let H_{-p} be the completion of H with respect to the norm $|I \otimes B^{-p} \cdot|_0$, where $|\cdot|_0$ is the norm on H. Then we have

$$H_\infty = \operatorname*{proj\,lim}_{p\to\infty} H_p \subset H \cong H^* \subset H_\infty^* \cong H_{-\infty} = \operatorname*{ind\,lim}_{p\to\infty} H_{-p},$$

where H_∞^* is the strong dual space of H_∞ with respect to H.

The (Boson) Fock space over H is denoted by $\mathcal{H} = \Gamma(H)$. Then by definition, $\mathcal{H}$ is the space of sequences $\phi = (f_n)_{n=0}^\infty$, where $f_n \in H^{\widehat{\otimes} n}$ (n-fold symmetric tensor power of the Hilbert space H) such that

$$\| \phi \|_0^2 = \sum_{n=0}^\infty n! |f_n|_0^2 < \infty,$$

where $|\cdot|_0$ is the norm on $H^{\widehat{\otimes} n}$ for any $n \in \mathbf{N}$.

Let $\mathcal{I}$ be a separable Hilbert space which is called the *initial Hilbert space* and A a selfadjoint operator in $\mathcal{I}$ with dense domain Dom (A) satisfying inf Spec $(A) \geq 1$. To lighten the notation, the operator $A \otimes \Gamma(eI \otimes B)$ in $\mathcal{I} \otimes \mathcal{H}$ is denoted by $\mathbf{A}$ and

$$\mathbf{A}^{\mathbf{p}} = A^{p_1} \otimes \Gamma(e^{p_2} I \otimes B^{p_3}), \quad \mathbf{p} = (p_1, p_2, p_3) \in \mathbf{R}^3,$$

where $\Gamma(C)$ is the second quantization of the operator C (see [17]). Then by standard arguments we may construct a triplet:

$$\mathcal{G}_\infty \subset \mathcal{G} \subset \mathcal{G}_{-\infty}$$

from $\mathcal{G} = \mathcal{I} \otimes \mathcal{H}$ and $\mathbf{A} = A \otimes \Gamma(eI \otimes B)$. More precisely, for each $\mathbf{p} \in \mathbf{R}_+^3$, put

$$\mathcal{G}_{\mathbf{p}} = \text{Dom}\,(\mathbf{A}^{\mathbf{p}}) \subset \mathcal{G} \equiv \mathcal{I} \otimes \mathcal{H}$$

and then $\mathcal{G}_{\mathbf{p}}$ becomes a Hilbert space with norm $|||\cdot|||_{\mathbf{p}} = |||\,\mathbf{A}^{\mathbf{p}} \cdot|||_0$, where $|||\cdot|||_0$ is the norm on $\mathcal{I}\otimes\mathcal{H}$. Let $\mathcal{G}_{-\mathbf{p}}$ be the completion of $\mathcal{I}\otimes\mathcal{H}$ with respect to the norm $|||\cdot|||_{-\mathbf{p}} = |||\mathbf{A}^{-\mathbf{p}} \cdot|||_0$, and

$$\mathcal{G}_\infty = \operatorname*{proj\,lim}_{p_1,p_2,p_3\to\infty} \mathcal{G}_{(p_1,p_2,p_3)}, \quad \mathcal{G}_{-\infty} = \operatorname*{ind\,lim}_{p_1,p_2,p_3\to\infty} \mathcal{G}_{(-p_1,-p_2,-p_3)}.$$

Note that $\mathcal{G}_{-\infty}$ is topologically isomorphic to the strong dual space $\mathcal{G}_\infty^*$ of $\mathcal{G}_\infty$ with respect to $\mathcal{I} \otimes \mathcal{H}$.

For each $\mathbf{p} = (p_2, p_3) \in \mathbf{R}_+^2$, put

$$\mathcal{H}_{\mathbf{p}} = \mathrm{Dom}\,(\Gamma(e^{p_2} I \otimes B^{p_3}))$$

and let $\mathcal{H}_{-\mathbf{p}}$ be the completion of $\mathcal{H}$ with respect to the norm $\|\cdot\|_{-\mathbf{p}} = \|\Gamma(e^{-p_2} I \otimes B^{-p_3}) \cdot \|_0$, and

$$\mathcal{H}_\infty = \operatorname*{proj\,lim}_{p_2, p_3 \to \infty} \mathcal{H}_{(p_2, p_3)}, \quad \mathcal{H}_{-\infty} = \operatorname*{ind\,lim}_{p_2, p_3 \to \infty} \mathcal{H}_{(-p_2, -p_3)}.$$

For each interval $[a, b] \subset \mathbf{R}_+$, we write $H_{[a,b]} = L^2([a, b], K)$ and then

$$H = H_{s]} \oplus H_{[s,t]} \oplus H_{[t}, \quad 0 < s < t < \infty$$

with abbreviations $H_{s]}$ and $H_{[t}$ when $[a, b] = [0, s]$ and $[a, b] = [t, \infty]$, respectively. Therefore, we have the identification

$$\mathcal{G} = \mathcal{G}_{s]} \otimes \mathcal{H}_{[s,t]} \otimes \mathcal{H}_{[t}, \quad \mathcal{G}_{s]} = \mathcal{I} \otimes \mathcal{H}_{s]},$$

where

$$\mathcal{H}_{s]} = \Gamma(H_{s]}), \quad \mathcal{H}_{[s,t]} = \Gamma(H_{[s,t]}), \quad \mathcal{H}_{[t} = \Gamma(H_{[t}).$$

Moreover, for any $\mathbf{p} = (p_1, p_2, p_3) \in \mathbf{R}_+^3 \cup \mathbf{R}_-^3$ ($\mathbf{R}_- = (-\infty, 0]$) and $0 < s < t < \infty$, we have

$$\mathcal{G}_{\mathbf{p}} = \mathcal{G}_{\mathbf{p};s]} \otimes \mathcal{H}_{\mathbf{p}';[s,t]} \otimes \mathcal{H}_{\mathbf{p}';[t},$$

where $\mathbf{p}' = (p_2, p_3)$ and

$$\mathcal{G}_{\mathbf{p};s]} = \mathcal{G}_{\mathbf{p}} \cap \mathcal{G}_{s]}, \quad \mathcal{H}_{\mathbf{p}';[s,t]} = \mathcal{H}_{\mathbf{p}'} \cap \mathcal{H}_{[s,t]}, \quad \mathcal{H}_{\mathbf{p}';[t} = \mathcal{H}_{\mathbf{p}'} \cap \mathcal{H}_{[t}$$

(closures when $\mathbf{p} \in \mathbf{R}_-^3$).

For each $g, h \in H_\infty$ and $T \in \mathcal{L}(H_\infty, H_\infty)$, the *annihilation*, *creation* and *conservation operators* are defined on $\mathcal{H}_\infty$ as follows:

$$a(g)\phi = (ng \widehat{\otimes}^1 f_n)_{n=1}^\infty,$$

$$a^*(h)\phi = (S_{1+n}(h \otimes f_n))_{n=0}^\infty,$$

$$\lambda(T)\phi = ((n+1)S_{1+n}(T \otimes I^{\otimes n}) f_{n+1})_{n=0}^\infty,$$

respectively, for any $\phi = (f_n)_{n=0}^\infty \in \mathcal{H}_\infty$, where $g \widehat{\otimes}^1 f_n$ is the left 1-contraction of g and f_n [16], and S_{l+m} stands for the symmetrizing operator. Then we can easily show that $a(g)$, $a^*(h)$ and $\lambda(T)$ are continuous linear operators acting on $\mathcal{H}_\infty$. The operators $a(g)$ and $a^*(g)$ are adjoint to each other and $\lambda(T^*) = (\lambda(T))^*$.

The three basic (quantum stochastic) processes called *annihilation*, *creation* and *conservation processes* are defined by

$$A_i(t) = I \otimes a(\mathbf{1}_{[0,t]} \otimes e_i),$$

$$A_i^*(t) = I \otimes a^*(\mathbf{1}_{[0,t]} \otimes e_i),$$

$$\Lambda_{ij}(t) = I \otimes \lambda(\mathbf{1}_{[0,t]} \otimes P_{ij}),$$

respectively, where I is the identity operator on $\mathcal{I}$ and $\mathbf{1}_{[0,t]}$ the indicator function. In the definition of $A_i(t)$ and $A_i^*(t)$ the indicator function $\mathbf{1}_{[0,t]}$ is a vector in $L^2(\mathbf{R}_+)$ while it is considered as a multiplication operator in $L^2(\mathbf{R}_+)$ in the definition of $\Lambda_{ij}(t)$.

For each $f \in H$, a vector of the form:

$$\phi_f = \left(1, f, \frac{f^{\otimes 2}}{2!}, \dots, \frac{f^{\otimes n}}{n!}, \dots\right)$$

is called an *exponential vector* or a *coherent vector*. Note that ϕ_f belongs to $\mathcal{H}_\infty$ (resp. $\mathcal{H}_{-\infty}$) if and only if f belongs to H_∞ (resp. $H_{-\infty}$). The exponential vectors $\{\phi_f\ ;\ f \in H_\infty\}$ span a dense subspace of $\mathcal{H}_\infty$, hence of $\mathcal{H}_\mathbf{p}$ for all $\mathbf{p} \in \mathbf{R}_+^2$ and of $\mathcal{H}_{-\infty}$. We denote $\mathcal{E}(D)$ the linear subspace generated by $\{\phi_f\ ;\ f \in D\}$ for $D \subset H$. Then for any $f, g \in H_\infty$ and $t \in \mathbf{R}_+$ we have

$$\langle\langle A_i(t)u \otimes \phi_f,\ v \otimes \phi_g \rangle\rangle = \langle u,\ v\rangle \left(\int_0^t f_i(s)\mathrm{d}s\right) e^{\langle f,\, g\rangle},$$

$$\langle\langle A_i^*(t)u \otimes \phi_f,\ v \otimes \phi_g \rangle\rangle = \langle u,\ v\rangle \left(\int_0^t g_i(s)\mathrm{d}s\right) e^{\langle f,\, g\rangle},$$

$$\langle\langle \Lambda_{ij}(t)u \otimes \phi_f,\ v \otimes \phi_g \rangle\rangle = \langle u,\ v\rangle \left(\int_0^t f_j(s)g_i(s)\mathrm{d}s\right) e^{\langle f,\, g\rangle},$$

where $\langle\langle\cdot,\ \cdot\rangle\rangle$ is the $\mathbf{C}$-bilinear form on $\mathcal{G}_{-\infty} \times \mathcal{G}_\infty$ and $h_i(s) = \langle h(s),\ e_i\rangle$ for $h \in H$.

The quantum Ito's formula established in [8] is summarized by the following table:

	$\mathrm{d}t$	$\mathrm{d}A_k$	$\mathrm{d}A_k^*$	$\mathrm{d}\Lambda_{kl}$
$\mathrm{d}t$	0	0	0	0
$\mathrm{d}A_i$	0	0	$\delta_{ik}\mathrm{d}t$	$\delta_{ik}\mathrm{d}A_l$
$\mathrm{d}A_i^*$	0	0	0	0
$\mathrm{d}\Lambda_{ij}$	0	0	$\delta_{jk}\mathrm{d}A_i^*$	$\delta_{jk}\mathrm{d}\Lambda_{il}$

. (2.1)

3. Quantum stochastic integral

Let $\mathcal{D}_0$ and M be dense linear subspaces of $\mathcal{I}_\infty$ and H_∞, respectively, such that $\mathbf{1}_{[0,t]}f \in M$ for any $t \in \mathbf{R}_+$ and $f \in M$, and let

$$M_{t]} = \{\mathbf{1}_{[0,t]}f\ ;\ f \in M\}, \quad M_{[t} = \{\mathbf{1}_{[t,\infty)}f\ ;\ f \in M\}.$$

We put $\widetilde{\mathcal{E}} = \mathcal{D}_0 \otimes_{\mathrm{al}} \mathcal{E}(M) \subset \mathcal{G}_\infty$, where $\otimes_{\mathrm{al}}$ is the algebraic tensor product, and put

$$\widetilde{\mathcal{E}}_{t]} = \mathcal{D}_0 \otimes_{\mathrm{al}} \mathcal{E}(M_{t]}), \quad \mathcal{E}_{[t} = \mathcal{E}(M_{[t}) \quad \text{and then} \quad \widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}_{t]} \otimes_{\mathrm{al}} \mathcal{E}_{[t}.$$

A family of operators $\Xi = \{\Xi(t)\}_{t\ge 0} \subset L(\widetilde{\mathcal{E}}, \mathcal{G}_{-\infty})$ is called a $\mathcal{G}_\mathbf{p}$-*quantum stochastic process* if there exists $\mathbf{p} \in \mathbf{R}_+^3 \cup \mathbf{R}_-^3$ (independent of $t \ge 0$) such that $\Xi(t) \in L(\widetilde{\mathcal{E}}, \mathcal{G}_\mathbf{p})$ for each $t \ge 0$ and for each $\psi \in \widetilde{\mathcal{E}}$ the map $\mathbf{R}_+ \ni t \mapsto \Xi(t)\psi \in \mathcal{G}_\mathbf{p}$ is strongly measurable. We may then think of $\Xi(t)$ as a densely defined operator on the Hilbert space $\mathcal{G}_\mathbf{p}$; and call Ξ *adapted* if $\Xi(t) = \Xi(t]) \otimes_{\mathrm{alg}} I([t)$ for some $\Xi(t]) \in L(\widetilde{\mathcal{E}}_{t]}, \mathcal{G}_{\mathbf{p};t]})$, where $I([t)$ is the identity operator on $\mathcal{G}_{\mathbf{p};[t}$.

For certain sets of $\{E^{(k)}\}_{k=1,2,3,4}$ of families of adapted process, stochastic integrals of the type

$$\int_0^t \left\{ \sum_{i,j} E_{ij}^{(1)} \mathrm{d}\Lambda_{ij} + \sum_i E_i^{(2)} \mathrm{d}A_i + \sum_i E_i^{(3)} \mathrm{d}A_i^+ + E^{(4)} \mathrm{d}s \right\}$$

can be defined as in [9]. We first define the integrals for a finite family of simple adapted processes $\{E^{(k)}\}_{k=1,2,3,4}$ and then the definition can be extended to a certain class of countable families $\{E^{(k)}\}_{k=1,2,3,4}$ with a norm estimate (see (3.3)) induced by the quantum Itô formula. For detailed calculations, we refer to [8] and [19].

For each $\mathbf{p} = (p_1, p_2, p_3) \in \mathbf{R}_+^3 \cup \mathbf{R}_-^3$ we denote $\mathcal{A}_2(\widetilde{\mathcal{E}}, \mathcal{G}_{\mathbf{p}})$ the class of all (ordered) quadruples of families of adapted processes

$$\mathbf{E} \equiv \{E_{ij}^{(1)}(t),\ E_i^{(2)}(t),\ E_i^{(3)}(t),\ E^{(4)}(t);\ 1 \le i, j < \infty,\ t \ge 0\}$$

satisfying

$$\int_0^t \left\{ \sum_i \rho_i^{2p_3} \left|\!\left|\!\left| \sum_j f_j(s) E_{ij}^{(1)}(s) u \otimes \phi_f \right|\!\right|\!\right|_{\mathbf{p}}^2 + \sum_{k=2}^{3} \sum_i \rho_i^{2p_3} ||| E_i^{(k)}(s) u \otimes \phi_f |||_{\mathbf{p}}^2 + ||| E^{(4)}(s) u \otimes \phi_f |||_{\mathbf{p}}^2 \right\} \mathrm{d}s < \infty \tag{3.1}$$

for all $t > 0$, $u \in \mathcal{D}_0$ and $f \in M$.

Theorem 3.1. *Let* $\mathbf{p} = (p_1, p_2, p_3) \in \mathbf{R}_+^3 \cup \mathbf{R}_-^3$ *and* $\mathbf{E} \in \mathcal{A}_2(\widetilde{\mathcal{E}}, \mathcal{G}_{\mathbf{p}})$. *Then the stochastic integral*

$$\Xi(t) = \int_0^t \sum_{i,j} E_{ij}^{(1)}(s) \mathrm{d}\Lambda_{ij}(s) + \int_0^t \sum_i E_i^{(2)}(s) \mathrm{d}A_i(s) + \int_0^t \sum_i E_i^{(3)}(s) \mathrm{d}A_i^+(s) + \int_0^t E^{(4)}(s) \mathrm{d}s$$

is well-defined as an adapted process in $L(\widetilde{\mathcal{E}}, \mathcal{G}_{\mathbf{p}})$. *Moreover, for any* $u, v \in \mathcal{D}_0$ *and* $f, g \in M$ *we have*

$$\begin{aligned} &\langle\!\langle \Xi(t) u \otimes \phi_f,\ v \otimes \phi_g \rangle\!\rangle \\ &= \int_0^t \left\langle\!\!\left\langle \left\{ \sum_{i,j} g_i(s) f_j(s) E_{ij}^{(1)}(s) + \sum_i f_i(s) E_i^{(2)}(s) \right\} u \otimes \phi_f,\ v \otimes \phi_g \right\rangle\!\!\right\rangle \mathrm{d}s \\ &\quad + \int_0^t \left\langle\!\!\left\langle \left\{ \sum_i g_i(s) E_i^{(3)}(s) + E^{(4)}(s) \right\} u \otimes \phi_f,\ v \otimes \phi_g \right\rangle\!\!\right\rangle \mathrm{d}s \end{aligned} \tag{3.2}$$

and

$$||| \Xi(t) u \otimes \phi_f |||_{\mathbf{p}}^2 \le \exp\left\{ t + 3e^{2p_2} \int_0^t |\, f(u) \,|_{p_3}^2 \, \mathrm{d}u \right\} \left(\int_0^t G(s) \mathrm{d}s \right) < \infty, \tag{3.3}$$

where, for each $t \in \mathbf{R}_+$,

$$G(t) = 3e^{2p_2} \sum_i \rho_i^{2p_3} \left|\!\left|\!\left| \sum_j f_j(t) E_{ij}^{(1)}(t) u \otimes \phi_f \right|\!\right|\!\right|_{\mathbf{p}}^2 + e^{2p_2} \sum_i \rho_i^{2p_3} |\!|\!| E_i^{(2)}(t) u \otimes \phi_f |\!|\!|_{\mathbf{p}}^2 + 3e^{2p_2} \sum_i \rho_i^{2p_3} |\!|\!| E_i^{(3)}(t) u \otimes \phi_f |\!|\!|_{\mathbf{p}}^2 + |\!|\!| E^{(4)}(t) u \otimes \phi_f |\!|\!|_{\mathbf{p}}^2. \quad (3.4)$$

Proof. By similar arguments of those used in [19] and [9] using the quantum Itô formula (2.1), for simple quadruple $\mathbf{E}$ with finite number of non-zero components we can compute that

$$\frac{\mathrm{d}}{\mathrm{dt}} |\!|\!| \Xi(t) u \otimes \phi_f |\!|\!|_{\mathbf{p}}^2 = 2\,\mathrm{Re} \left\{ \sum_{k=1}^5 S_k \right\} + \sum_i e^{2p_2} \rho_i^{2p_3} \left|\!\left|\!\left| \sum_j f_j(t) E_{ij}^{(1)}(t) u \otimes \phi_f \right|\!\right|\!\right|_{\mathbf{p}}^2 + \sum_i e^{2p_2} \rho_i^{2p_3} |\!|\!| E_i^{(3)}(t) u \otimes \phi_f |\!|\!|_{\mathbf{p}}^2, \quad (3.5)$$

where

$$S_1 = e^{2p_2} \sum_i \rho_i^{2p_3} \left\langle\!\!\left\langle \mathbf{A^p}(f_i(t) \Xi(t) u \otimes \phi_f),\ \overline{\mathbf{A^p}\left(\sum_j f_j(t) E_{ij}^{(1)}(t) u \otimes \phi_f \right)} \right\rangle\!\!\right\rangle,$$

$$S_2 = \sum_i e^{2p_2} \rho_i^{2p_3} \left\langle\!\!\left\langle \mathbf{A^p}(\bar{f}_i(t) \Xi(t) u \otimes \phi_f),\ \overline{\mathbf{A^p}(E_i^{(2)}(t) u \otimes \phi_f)} \right\rangle\!\!\right\rangle,$$

$$S_3 = \sum_i e^{2p_2} \rho_i^{2p_3} \left\langle\!\!\left\langle \mathbf{A^p}(f_i(t) \Xi(t) u \otimes \phi_f),\ \overline{\mathbf{A^p}(E_i^{(3)}(t) u \otimes \phi_f)} \right\rangle\!\!\right\rangle,$$

$$S_4 = \left\langle\!\!\left\langle \mathbf{A^p}(\Xi(t) u \otimes \phi_f),\ \overline{\mathbf{A^p}(E^{(4)}(t) u \otimes \phi_f)} \right\rangle\!\!\right\rangle,$$

$$S_5 = \sum_i e^{2p_2} \rho_i^{2p_3} \left\langle\!\!\left\langle \mathbf{A^p}(E_i^{(3)}(t) u \otimes \phi_f),\ \overline{\mathbf{A^p}\left(\sum_j f_j(t) E_{ij}^{(1)} u \otimes \phi_f \right)} \right\rangle\!\!\right\rangle.$$

By using the Cauchy–Schwarz inequality and the fact $2\,\mathrm{Re}\, \bar{a}b \leq |a|^2 + |b|^2$, we obtain from (3.5) that

$$\frac{\mathrm{d}}{\mathrm{dt}} |\!|\!| \Xi(t) u \otimes \phi_f |\!|\!|_{\mathbf{p}}^2 \leq (1 + 3e^{2p_2} |f(t)|_{p_3}^2) |\!|\!| \Xi(t) u \otimes \phi_f |\!|\!|_{\mathbf{p}}^2 + G(t),$$

where $G(t)$ is given as in (3.4). The inequality (3.3) can be obtained by applying Gronwall's lemma with the above inequality, as in [8] or [17]. Then the inequality (3.3) allows the extension of the integral to $\mathcal{A}_2(\widetilde{\mathcal{E}}, \mathcal{G}_{\mathbf{p}})$ satisfying the inequality (3.3). ■

4. Regular quantum martingales

An adapted processes $\{\Xi(t)\}_{t\geq 0} \subset L(\widetilde{\mathcal{E}}, \mathcal{G}_{\mathbf{p}})$ is called a *quantum martingale* if for any $0 \leq s \leq t$,

$$\langle\langle \Xi(t)(u \otimes \phi_{\mathbf{1}_{[0,s]}f}),\ v \otimes \phi_{\mathbf{1}_{[0,s]}g}\rangle\rangle = \langle\langle \Xi(s)(u \otimes \phi_{\mathbf{1}_{[0,s]}f}),\ v \otimes \phi_{\mathbf{1}_{[0,s]}g}\rangle\rangle$$

for any $u, v \in \mathcal{D}_0$ and $f, g \in M$. For each $1 \leq i, j < \infty$, the annihilation process $\{A_i(t)\}_{t\geq 0}$, creation process $\{A_i^*(t)\}_{t\geq 0}$ and conservation process $\{\Lambda_{ij}(t)\}_{t\geq 0}$ are quantum martingales which are called the basic martingales in quantum stochastic calculus.

In the following, for $\mathbf{p}, \mathbf{q} \in \mathbf{R}^3_+ \cup \mathbf{R}^3_-$ with $\mathbf{p}-\mathbf{q} \in \mathbf{R}^3_+$ we consider quantum martingales Ξ in $\mathcal{L}(\mathcal{G}_{\mathbf{p}}, \mathcal{G}_{\mathbf{q}})$. Thus, for any $0 \leq s \leq t$ and $\phi_{s]} \in \mathcal{G}_{\mathbf{p};s]}$, $\psi_{s]} \in \mathcal{G}_{-\mathbf{q};s]}$,

$$\langle\langle \Xi_t \phi_{s]},\ \psi_{s]}\rangle\rangle = \langle\langle \Xi_s \phi_{s]},\ \psi_{s]}\rangle\rangle.$$

The following definition of regular martingale is a simple modification of the definition of bounded regular martingale in [18] and [9].

DEFINITION 4.1

A quantum martingale Ξ in $\mathcal{L}(\mathcal{G}_{\mathbf{p}}, \mathcal{G}_{\mathbf{q}})$ is said to be *regular with respect to a Radon measure* $\mathfrak{m}$ on $\mathbf{R}_+$, or simply *regular* if for any $0 \leq v < u$ and $\phi \in \mathcal{G}_{\mathbf{p};v]}$, $\psi \in \mathcal{G}_{-\mathbf{q};v]}$,

$$\begin{aligned} &|||\, (\Xi_u - \Xi_v)\phi\, |||^2_{\mathbf{q}} \leq |||\, \phi\, |||^2_{\mathbf{p}}\ \mathfrak{m}([v, u]),\\ &|||\, (\Xi^*_u - \Xi^*_v)\psi\, |||^2_{-\mathbf{p}} \leq |||\, \psi\, |||^2_{-\mathbf{q}}\ \mathfrak{m}([v, u]). \end{aligned} \tag{4.1}$$

PROPOSITION 4.2

Let Ξ be a quantum martingale in $\mathcal{L}(\mathcal{G}_{\mathbf{p}}, \mathcal{G}_{\mathbf{q}})$. If Ξ has the integral representation:

$$\mathrm{d}\,\Xi = \sum_{i,j} E_{ij}\mathrm{d}\Lambda_{ij} + \sum_i F_i^*\mathrm{d}A_i + \sum_i G_i \mathrm{d}A_i^*,$$

where the quadruples $(E_{ij}, F_i^*, G_i, 0)$ and $(E_{ij}^*, F_i, G_i^*, 0)$ belong to $\mathcal{A}_2(\widetilde{\mathcal{E}}, \mathcal{G}_{\mathbf{q}})$ and $\mathcal{A}_2(\widetilde{\mathcal{E}}, \mathcal{G}_{-\mathbf{p}})$, respectively, and E_{ij}, F_i^*, G_i are adapted processes in $\mathcal{L}(\mathcal{G}_{\mathbf{p}}, \mathcal{G}_{\mathbf{q}})$ such that

$$\sum_i \rho_i^{2q_3} G_i^\dagger(s)\mathbf{A}^{2\mathbf{q}} G_i(s) \quad \textit{and} \quad \sum_i \rho_i^{-2p_3} F_i^\dagger(s)\mathbf{A}^{-2\mathbf{p}} F_i(s)$$

converge weakly to self-adjoint operators $G(s) \in \mathcal{L}(\mathcal{G}_{\mathbf{p}}, \mathcal{G}_{-\mathbf{p}})$ and $F(s) \in \mathcal{L}(\mathcal{G}_{-\mathbf{q}}, \mathcal{G}_{\mathbf{q}})$, respectively, with the property that $\|G(s)\|_{\mathbf{p};-\mathbf{p}}$ and $\|F(s)\|_{-\mathbf{q};\mathbf{q}}$ are locally integrable, where $K^\dagger$ denotes the adjoint of the operator K with respect to $\langle\langle \cdot, \bar{\cdot}\rangle\rangle$ and $\|\Xi\|_{\mathbf{r};\mathbf{s}}$ is the operator norm of $\Xi \in \mathcal{L}(\mathcal{G}_{\mathbf{r}}, \mathcal{G}_{\mathbf{s}})$. Then Ξ is regular.

Proof. Let $\mathbf{p} = (p_1, p_2, p_3) \in \mathbf{R}^3_+ \cup \mathbf{R}^3_-$ and $\mathbf{q} = (q_1, q_2, q_3) \in \mathbf{R}^3_+ \cup \mathbf{R}^3_-$. Note that for any $0 \leq a < t$ and $\phi_{a]} \in \mathcal{G}_{\mathbf{p};a]}$, $\psi_{a]} \in \mathcal{G}_{\mathbf{q};a]}$,

$$\langle\langle \Xi(t)\phi_{a]},\ \psi_{a]}\rangle\rangle_{\mathbf{q}} = \langle\langle \Xi(s)\phi_{a]},\ \psi_{a]}\rangle\rangle_{\mathbf{q}}.$$

It follows that

$$|||\, (\Xi(t) - \Xi(a))\phi_{a]}\, |||^2_{\mathbf{q}} = |||\, \Xi(t)\phi_{a]}\, |||^2_{\mathbf{q}} - |||\, \Xi(a)\phi_{a]}\, |||^2_{\mathbf{q}}. \tag{4.2}$$

Therefore, by applying (3.5), we obtain that for any $0 \le a < t$ and $\phi_{a]} \in \widetilde{\mathcal{E}}_{a]}$,

$$
\begin{aligned}
||| (\Xi(t) - \Xi(a))\phi_{a]} |||_{\mathbf{q}}^2 &= e^{2q_2} \int_a^t \sum_i \rho_i^{2q_3} ||| G_i(s)\phi_{a]} |||_{\mathbf{q}}^2 \mathrm{d}s \\
&\le e^{2q_2} ||| \phi_{a]} |||_{\mathbf{p}}^2 \int_a^t \|G(s)\|_{\mathbf{p};-\mathbf{p}} \mathrm{d}s. \qquad (4.3)
\end{aligned}
$$

Similarly, for any $0 \le a < t$ and $\psi_{a]} \in \widetilde{\mathcal{E}}_{a]}$, we have

$$
||| (\Xi(t)^* - \Xi(a)^*)\psi_{a]} |||_{-\mathbf{p}}^2 \le e^{-2p_2} ||| \psi_{a]} |||_{-\mathbf{q}}^2 \int_a^t \|F(s)\|_{-\mathbf{q};\mathbf{q}}^2 \mathrm{d}s. \qquad (4.4)
$$

Now, we define a Radon measure $\mathfrak{m}$ on $\mathbf{R}_+$ by

$$
\mathfrak{m}([a,b]) = \int_a^b (e^{2q_2} \|G(s)\|_{\mathbf{p};-\mathbf{p}}^2 + e^{-2p_2} \|F(s)\|_{-\mathbf{q};\mathbf{q}}^2) \mathrm{d}s \quad \text{for all } 0 \le a \le b < \infty.
$$

Therefore, by (4.3), (4.4) and the density of $\widetilde{\mathcal{E}}_{a]}$ in $\mathcal{G}_{\mathbf{p};a]}$ and $\mathcal{G}_{-\mathbf{q};a]}$, we see that Ξ is regular with respect to the absolutely continuous Radon measure $\mathfrak{m}$. ■

Remark 4.3. Let Ξ be a martingale in $\mathcal{L}(\mathcal{G}_{\mathbf{p}}, \mathcal{G}_{\mathbf{q}})$ which is regular with respect to the Radon measure $\mathfrak{m}$. Then for any $t > a$,

$$
\begin{aligned}
\|\Xi(t)\|_{\mathbf{p};\mathbf{q}} &\ge \sup_{||| \phi_{a]} |||_{\mathbf{p}}=1} ||| \Xi(t)\phi_{a]} |||_{\mathbf{q}} \\
&\ge \sup_{||| \phi_{a]} |||_{\mathbf{p}}=1} ||| \Xi(a)\phi_{a]} |||_{\mathbf{q}} \ge \|\Xi(a)\|_{\mathbf{p};\mathbf{q}},
\end{aligned}
$$

where we used (4.2) for the second inequality. Therefore, $\|\Xi(\cdot)\|_{\mathbf{p};\mathbf{q}}$ is non-decreasing.

Let P denote the probability measure of an independent identically distributed sequence $\{B_1, B_2, \dots\}$ of standard Brownian motions. Then the Hilbert space $L_2(P)$ is identified with $\Gamma(L_2(\mathbf{R}_+, \mathbf{R}) \otimes \ell_2)$ by the following correspondence:

$$
\phi_f \longleftrightarrow \exp \sum_i \left(\int_0^\infty f_i \,\mathrm{d}B_i - \frac{1}{2} \int_0^\infty f_i^2 \,\mathrm{d}t \right),
$$

where $f = (f_1, f_2, \dots) \in \bigoplus_{i=1}^\infty L_2(\mathbf{R}_+, \mathbf{R}) \cong L_2(\mathbf{R}_+, \mathbf{R}) \otimes \ell_2$. Put

$$
M_0 = \{ f = (f_1, \dots, f_i, \dots) \in H_\infty ; \; f_i = 0 \text{ for all but a finite number of } i'\text{s} \}.
$$

Then $\mathcal{E}_0 = \mathcal{E}(M_0)$ and $\widetilde{\mathcal{E}}_0 = \mathcal{I}_\infty \otimes_{\mathrm{al}} \mathcal{E}(M_0)$ are total in H and $\mathcal{G}$, respectively, where $\mathcal{I}_\infty$ is the Fréchet space constructed by the standard manner with $\mathcal{I}$ and the positive operator A, and then we have

$$
\phi_{f\mathbf{1}_{[0,t]}} - 1 = \sum_i \int_0^t f_i(s) \phi_{f\mathbf{1}_{[0,s]}} \,\mathrm{d}B_i(s)
$$

for $\phi_f \in \mathcal{E}_0$. In general, we have the following proposition which is an extension of the classical martingale representation theorem of Kunita–Watanabe for L^2-martingales adapted to one Brownian motion to an $\mathcal{I}$-valued L^2-martingale adapted to a countable family of independent Brownian motion.

PROPOSITION 4.4 [19]

Let $\{X(t)\}_{t\geq 0}$ be an $\mathcal{I}$-valued square integrable martingale adapted to $\{B_i\}$ which is an independent identically distributed sequence of standard Brownian motions. Then

$$X(t) = X(0) + \sum_i \int_0^t \xi_i \,\mathrm{d}B_i,$$

where $\{\xi_i\}_{i\geq 1}$ is a sequence of adapted processes satisfying

$$\int_0^t \sum_i \mathbf{E}[\|\xi_i(s)\|_{\mathcal{I}}^2]\mathrm{d}s < \infty, \quad t \in \mathbf{R}_+.$$

Our aim is to prove the converse of Proposition 4.2 generalizing the main result in [9] and [19]. For the proof we use similar arguments to those used in [19].

Theorem 4.5. *Let $\mathbf{p}, \mathbf{q} \in \mathbf{R}_+^3 \cup \mathbf{R}_-^3$ with $\mathbf{p} - \mathbf{q} \in \mathbf{R}_+^3$. Let Ξ be a martingale in $\mathcal{L}(\mathcal{G}_{\mathbf{p}}, \mathcal{G}_{\mathbf{q}})$ which is regular with respect to a Radon measure $\mathfrak{m}$ on $\mathbf{R}_+$. Then there exist three unique families of adapted processes $\{E_{ij}\}$, $\{F_i^*\}$, $\{G_i\}$ in $\mathcal{L}(\mathcal{G}_{\mathbf{p}}, \mathcal{G}_{\mathbf{q}})$ such that*

$$\mathrm{d}\Xi = \sum_{i,j} E_{ij}\mathrm{d}\Lambda_{ij} + \sum_i F_i^*\mathrm{d}A_i + \sum_i G_i\mathrm{d}A_i^*$$

on $\widetilde{\mathcal{E}}_{00}$ (see eq. (4.15)). Furthermore,

$$\sum_i \rho_i^{2q_3} G_i^\dagger(s)\mathbf{A}^{2\mathbf{q}}G_i(s) \quad \text{and} \quad \sum_i \rho_i^{-2p_3} F_i^\dagger(s)\mathbf{A}^{-2\mathbf{p}}F_i(s)$$

converge weakly to operators $G(s) \in \mathcal{L}(\mathcal{G}_{\mathbf{p}}, \mathcal{G}_{-\mathbf{p}})$ and $F(s) \in \mathcal{L}(\mathcal{G}_{-\mathbf{q}}, \mathcal{G}_{\mathbf{q}})$, respectively, with

$$\max\{\|G(s)\|_{\mathbf{p};-\mathbf{p}}, \|F(s)\|_{-\mathbf{q};\mathbf{q}}\} \leq \mathfrak{m}_{\mathrm{ac}}'(s), \qquad s \in \mathbf{R}_+,$$

where $\mathfrak{m}_{\mathrm{ac}}$ denotes the absolutely continuous part of $\mathfrak{m}$.

Proof. This follows from the identity (4.8) and Lemma 4.11 below. ∎

Lemma 4.6. *Let $\mathbf{p} = (p_1, p_2, p_3)$ and $\mathbf{q} = (q_1, q_2, q_3)$. Let Ξ be a martingale in $\mathcal{L}(\mathcal{G}_{\mathbf{p}}, \mathcal{G}_{\mathbf{q}})$ which is regular with respect to a Radon measure $\mathfrak{m}$ on $\mathbf{R}_+$. Then*

(i) $\mathfrak{m}$ *can be replaced by its absolutely continuous part*;
(ii) *there exist two countable families of adapted processes $\{F_i^*(t)\}$ and $\{G_i(t)\}$ in $\mathcal{L}(\mathcal{G}_{\mathbf{p}}, \mathcal{G}_{\mathbf{q}})$ such that for any $\varphi \in \mathcal{G}_{\mathbf{p};a]}$ and $\psi \in \mathcal{G}_{-\mathbf{q};a]}$, $t > a \geq 0$,*

$$(\Xi(t) - \Xi(a))\varphi = \int_a^t \sum_i G_i(s)\varphi \,\mathrm{d}B_i(s),$$

$$(\Xi^*(t) - \Xi^*(a))\psi = \int_a^t \sum_i F_i(s)\psi \,\mathrm{d}B_i(s),$$

where $\{B_i(s)\}$ is the countable family of Brownian motions in Proposition 4.4;

(iii) *the series*

$$\sum_i \rho_i^{2q_3} G_i^\dagger(s)\mathbf{A}^{2\mathbf{q}}G_i(s) \quad \textit{and} \quad \sum_i \rho_i^{-2p_3} F_i^\dagger(s)\mathbf{A}^{-2\mathbf{p}}F_i(s)$$

converge weakly to operators $G(s) \in \mathcal{L}(\mathcal{G}_{\mathbf{p}}, \mathcal{G}_{-\mathbf{p}})$ *and* $F(s) \in \mathcal{L}(\mathcal{G}_{-\mathbf{q}}, \mathcal{G}_{\mathbf{q}})$, *respectively, with the property that* $\|G(s)\|_{\mathbf{p};-\mathbf{p}}$ *and* $\|F(s)\|_{-\mathbf{q};\mathbf{q}}$ *are locally integrable.*

Proof.
(i) Let $\varphi \in \mathcal{G}_{\mathbf{p};a]}$ be fixed. Since $\{\mathbf{A}^{\mathbf{q}}\Xi(t)\varphi\}_{t\geq a}$ is a classical $\mathcal{I}$-valued square integrable martingale in $[a, \infty)$ adapted to the countable family $\{B_i(s)\}$ of independent Brownian motions, by Proposition 4.4 there exists a countable family of $\mathcal{I}$-valued adapted square integrable process $\{\xi_i(s, \varphi)\}_{s\geq a}$ such that

$$\mathbf{A}^{\mathbf{q}}\Xi(t)\varphi - \mathbf{A}^{\mathbf{q}}\Xi(a)\varphi = \int_a^t \sum_i \xi_i(s, \varphi)\mathrm{d}B_i(s).$$

By the Itô isometry and (4.1), we have for all $0 \leq a < b < t < \infty$,

$$\int_b^t \mathbf{E}\left[\sum_i \|\xi_i(s, \varphi)\|_{\mathcal{I}}^2\right]\mathrm{d}s = ||| [\mathbf{A}^{\mathbf{q}}\Xi(t) - \mathbf{A}^{\mathbf{q}}\Xi(b)]\varphi |||_0^2 \leq ||| \varphi |||_{\mathbf{p}}^2 \,\mathfrak{m}([b, t]). \tag{4.5}$$

Similarly, we prove that for fixed $\psi \in \mathcal{G}_{-\mathbf{q};a]}$ there exists a countable family of $\mathcal{I}$-valued adapted square integrable process $\{\eta_i(s, \psi)\}_{s\geq a}$ such that

$$\mathbf{A}^{-\mathbf{p}}\Xi(t)^*\psi - \mathbf{A}^{-\mathbf{p}}\Xi(a)^*\psi = \int_a^t \sum_i \eta_i(s, \psi)\mathrm{d}B_i(s)$$

and for all $0 \leq a < b < t < \infty$,

$$\begin{aligned}\int_b^t \mathbf{E}\left[\sum_i \|\eta_i(s, \psi)\|_{\mathcal{I}}^2\right]\mathrm{d}s &= ||| [\mathbf{A}^{-\mathbf{p}}\Xi(t)^* - \mathbf{A}^{-\mathbf{p}}\Xi(b)^*]\psi |||_0^2 \\ &\leq ||| \psi |||_{-\mathbf{q}}^2 \,\mathfrak{m}([b, t]). \end{aligned} \tag{4.6}$$

From (4.5) and (4.6) we see that $\mathfrak{m}$ can be replaced by its absolutely continuous part $\mathfrak{m}_{\mathrm{ac}}$.
(ii)–(iii). From (i) we assume that $\mathfrak{m}$ is an absolutely continuous Radon measure. By similar arguments of those used in the proof of Proposition 7.5 in [9] we see that $\{\xi_i(s, \varphi)\}_{s\geq a}$ does not depend on the end point a and we put

$$G_i(s)\varphi = e^{-2q_2}\rho_i^{-q_3}\mathbf{A}^{-\mathbf{q}}\xi_i(s, \varphi) \quad \text{a.e.} \quad s > a, \quad \varphi \in \mathcal{G}_{\mathbf{p};a]}.$$

This gives an adapted operator family $\{G_i(s)\}$ (for details see the proof of the Proposition 7.5 in [9]). Hence by (4.5) for any $\varphi \in \mathcal{G}_{\mathbf{p};a]}$ we have

$$\begin{aligned}\int_b^t \sum_i \rho_i^{2q_3} ||| \mathbf{A}^{\mathbf{q}}G_i(s)\varphi |||_0^2\mathrm{d}s &= e^{-2q_2}\int_b^t \mathbf{E}\left[\sum_i \|\xi_i(s, \varphi)\|_{\mathcal{I}}^2\right]\mathrm{d}s \\ &\leq e^{-2q_2} ||| \varphi |||_{\mathbf{p}}^2 \,\mathfrak{m}([b, t])\end{aligned}$$

which implies that

$$\sum_i \rho_i^{2q_3} \| \mathbf{A}^{\mathbf{q}} G_i(s)\varphi \|_0^2 \le e^{-2q_2} \mathfrak{m}'(s) \| \varphi \|_{\mathbf{p}}^2 \quad \text{for all } s. \tag{4.7}$$

This shows that each $G_i(s)$ is an adapted process in $\mathcal{L}(\mathcal{G}_{\mathbf{p}}, \mathcal{G}_{\mathbf{q}})$ and that $\sum_i \rho_i^{2q_3} G_i^\dagger(s)\mathbf{A}^{2\mathbf{q}} G_i(s)$ converges strongly to an operator $G(s) \in \mathcal{L}(\mathcal{G}_{\mathbf{p}}, \mathcal{G}_{-\mathbf{p}})$ such that $\|G(s)\|_{\mathbf{p};-\mathbf{p}}$ is locally integrable. In fact, we prove that

$$\left\| \sum_i \rho_i^{2q_3} G_i^\dagger(s)\mathbf{A}^{2\mathbf{q}} G_i(s) \right\|_{\mathbf{p};-\mathbf{p}} \le e^{-2q_2} \mathfrak{m}'(s) \quad \text{a.e.}$$

The remainder of the proof is similar. ■

Now, we put

$$S(t) = \int_0^t \sum_i \left(G_i(s)\mathrm{d}A_i^*(s) + F_i^*(s)\mathrm{d}A_i(s) \right),$$

$$S^*(t) = \int_0^t \sum_i \left(F_i(s)\mathrm{d}A_i^*(s) + G_i^*(s)\mathrm{d}A_i(s) \right),$$

$$Z(t) = \Xi(t) - S(t) \quad \text{and} \quad Z^*(t) = \Xi^*(t) - S^*(t). \tag{4.8}$$

Remark 4.7. By (3.1) and (4.7), the integrals

$$\int_0^t \sum_i G_i(s)\mathrm{d}A_i^*, \quad \int_0^t \sum_i F_i(s)\mathrm{d}A_i^*$$

are well-defined on $\widetilde{\mathcal{E}}$ with $M = H$. But, in general, the other two integrals $\int_0^t \sum_i F_i^*(s) \mathrm{d}A_i(s)$ and $\int_0^t \sum_i G_i^*(s)\mathrm{d}A_i(s)$ are not well-defined on $\widetilde{\mathcal{E}}$ with $M = H$ since we have no estimates for $\sum_i \rho_i^{-2p_3} \| \mathbf{A}^{-\mathbf{p}} G_i^*(s)\varphi \|_0^2$ and $\sum_i \rho_i^{2q_3} \| \mathbf{A}^{\mathbf{q}} F_i^*(s)\varphi \|_0^2$. If we consider the integrals on $\widetilde{\mathcal{E}}_0$, then the infinite series reduce to finite sums and hence the stochastic integrals are well-defined on $\widetilde{\mathcal{E}}_0$ by (3.1). Then from (3.2) and the definitions it is immediate that the processes $\{S, S^*\}$ and $\{Z, Z^*\}$ are adjoint pairs on $\widetilde{\mathcal{E}}_0$. Also, we can easily see that for all $u \in \mathcal{I}_\infty$ and $f \in M_0$, $\{\mathbf{A}^{\mathbf{q}} Z(t)u \otimes \phi_{\mathbf{1}_{[0,t]}f}\}$ and $\{\mathbf{A}^{-\mathbf{p}} Z^*(t)u \otimes \phi_{\mathbf{1}_{[0,t]}f}\}$ are classical $\mathcal{I}$-valued martingales adapted to the countable family of Brownian motions $\{B_i\}$ in Proposition 4.4. Moreover, for all $t > a$,

$$Z(t)u \otimes \phi_{\mathbf{1}_{[0,a]}f} = Z(a)u \otimes \phi_{\mathbf{1}_{[0,a]}f},\ Z^*(t)u \otimes \phi_{\mathbf{1}_{[0,a]}f} = Z^*(a)u \otimes \phi_{\mathbf{1}_{[0,a]}f}. \tag{4.9}$$

Lemma 4.8. *Let $u \in \mathcal{I}_\infty$ and $f \in M_0$. Then*

(i) *there exists a $\mathcal{I}$-valued square integrable classical process $\{\xi_i(\cdot, u, f)\}$ such that*

$$\mathbf{A}^{\mathbf{q}} Z(t)\mathbf{A}^{-\mathbf{p}} u \otimes \phi_{\mathbf{1}_{[0,t]}f} = \mathbf{A}^{\mathbf{q}} \Xi(0)\mathbf{A}^{-\mathbf{p}} u \otimes \phi_0 + \int_0^t \sum_i \xi_i(s, u, f)\mathrm{d}B_i(s);$$

(ii) *there exists a $\mathcal{I}$-valued square integrable classical process $\{\eta_i(\cdot, u, f)\}$ such that*

$$\mathbf{A}^{-\mathbf{p}} Z^*(t)\mathbf{A}^{\mathbf{q}} u \otimes \phi_{\mathbf{1}_{[0,t]}f} = \mathbf{A}^{-\mathbf{p}} \Xi^*(0)\mathbf{A}^{\mathbf{q}} u \otimes \phi_0 + \int_0^t \sum_i \eta_i(s, u, f)\mathrm{d}B_i(s).$$

Proof. The proofs of (i) and (ii) are simple applications of Proposition 4.4. ■

Now, we prove that $\{Z\}_{t\geq 0}$ can be represented by a stochastic integral with respect to $\{\Lambda_{ij}\}$. For the proof, we use similar arguments to those used in [19] by using a special martingale $U^{(i)}$ related to the Weyl representation.

Lemma 4.9. [19]. *For each* $i = 1, 2, \dots$, *let* $U^{(i)}$ *be the unique bounded martingale satisfying*

$$\mathrm{d}U^{(i)} = (\mathrm{d}A_i^* - \mathrm{d}A_i)U^{(i)}, \quad U^{(i)}(0) = I.$$

Then

(i) $e^{-t/2}U^{(i)}(t) = I_0 \otimes W(\mathbf{1}_{[0,t]}e_i, I)$, *where* I_0 *is the identity in* $\mathcal{B}(\mathcal{I})$ *and* W *is the Weyl representation defined in* [8];
(ii) $U^{(i)}(t)$ *leaves* $\widetilde{\mathcal{E}}_0$ *invariant.*

Lemma 4.10. *Let* Ξ *be a regular martingale in* $\mathcal{L}(\mathcal{G}_{\mathbf{p}}, \mathcal{G}_{\mathbf{q}})$ *and let* $\{G_i\}$, $\{F_i\}$ *be the associated families of adapted processes defined in Lemma* 4.6. *For each* $i = 1, 2, \dots$, *put*

$$Y^{(i)}(t) = \mathbf{A}^{\mathbf{q}}\left(\Xi(t)\mathbf{A}^{-\mathbf{p}}U^{(i)}(t) - e^{-p_2}\rho_i^{-p_3}\int_0^t F_i^*(s)\mathbf{A}^{-\mathbf{p}}U^{(i)}(s)\mathrm{d}s\right).$$

Then

(i) *for each* i, $\{Y^{(i)}\}_{t\geq 0}$ *is a bounded regular martingale in* $\mathcal{L}(\mathcal{G}, \mathcal{G})$;
(ii) *for each* i, *there exists a unique family* $\{M_j^{(i)}\}$ *of bounded adapted processes such that for all* $t > a > 0$ *and* $\varphi \in \mathcal{G}_{a]}$,

$$[Y^{(i)}(t) - Y^{(i)}(a)]\varphi = \int_a^t \sum_j M_j^{(i)}(s)\varphi\mathrm{d}B_j(s). \tag{4.10}$$

Proof.
(i) It is clear that $\mathbf{A}^{\mathbf{q}}\Xi(t)\mathbf{A}^{-\mathbf{p}}U^{(i)}(t)$ is bounded. By similar arguments of those used to get (4.7) we prove that for any $t > 0$,

$$\int_0^t \|\mathbf{A}^{-\mathbf{p}}F_i(s)\mathbf{A}^{\mathbf{q}}U^{(i)}(s)\|_{0;0}\mathrm{d}s \leq e^{p_2}\int_0^t e^{s/2}\sqrt{\mathfrak{m}'(s)}\mathrm{d}s < \infty \tag{4.11}$$

which implies that $Y^{(i)}(t)$ is bounded. Since $\widetilde{\mathcal{E}}_0$ is invariant by $U^{(i)}(t)$, the relation:

$$Y^{(i)}(t) = \mathbf{A}^{\mathbf{q}}Z(t)\mathbf{A}^{-\mathbf{p}}U^{(i)}(t) + W^{(i)}(t) \tag{4.12}$$

holds on $\widetilde{\mathcal{E}}_0$, where

$$W^{(i)}(t) = \mathbf{A}^{\mathbf{q}}\left(S(t)\mathbf{A}^{-\mathbf{p}}U^{(i)}(t) - e^{-p_2}\rho_i^{-p_3}\int_0^t F_i^*(s)\mathbf{A}^{-\mathbf{p}}U^{(i)}(s)\mathrm{d}s\right), \quad t \geq 0.$$

Note that $\widetilde{\mathcal{E}}_0$ is invariant by $\mathbf{A}^{\mathbf{p}}$. Therefore, by (4.9) and the martingale property of $U^{(i)}$, we prove that $\{\mathbf{A}^{\mathbf{q}} Z(t)\mathbf{A}^{-\mathbf{p}} U^{(i)}(t)\}$ is a martingale on $\widetilde{\mathcal{E}}_0$. Now let $u, v \in \mathcal{I}_\infty$ and $f, g \in M_0$. Then by applying Ito's product formula (2.1) (or see Theorem 6.2 in [9]) to $\mathbf{A}^{\mathbf{q}} S(t)\mathbf{A}^{-\mathbf{p}} U^{(i)}(t)$, we can easily see that $\{W^{(i)}(t)\}_{t\geq 0}$ is a martingale.

In fact, for any $t \geq 0$ we have

$$\begin{aligned}\mathbf{A}^{\mathbf{q}} S(t)\mathbf{A}^{-\mathbf{p}} U^{(i)}(t) = {} & e^{-p_2}\rho_i^{-p_3}\int_0^t \mathbf{A}^{\mathbf{q}} S(s)\mathbf{A}^{-\mathbf{p}} U^{(i)}(s)\mathrm{d}A_i^*(s) \\ & - \int_0^t \mathbf{A}^{\mathbf{q}} S(s)\mathbf{A}^{-\mathbf{p}} U^{(i)}(s)\mathrm{d}A_i(s) \\ & + e^{q_2}\rho_i^{q_3}\int_0^t \mathbf{A}^{\mathbf{q}} G_i(s)\mathbf{A}^{-\mathbf{p}} U^{(i)}(s)\mathrm{d}A_i^*(s) \\ & + \int_0^t \mathbf{A}^{\mathbf{q}} F_i^*(s)\mathbf{A}^{-\mathbf{p}} U^{(i)}(s)\mathrm{d}A_i(s) \\ & + e^{-p_2}\rho_i^{-p_3}\int_0^t \mathbf{A}^{\mathbf{q}} F_i^*(s)\mathbf{A}^{-\mathbf{p}} U^{(i)}(s)\mathrm{d}s \end{aligned} \tag{4.13}$$

on $\widetilde{\mathcal{E}}_0$. The proof of regularity is similar to that in [18] and [19]. By similar arguments of those used to get (4.11) we first show that for $t > a > 0$ and $\varphi \in \mathcal{G}_{a]}$,

$$\begin{aligned} & \left|\!\left|\!\left| e^{-p_2}\rho_i^{-p_3}\int_a^t F_i^*(s)\mathbf{A}^{-\mathbf{p}} U^{(i)}(s)\varphi\mathrm{d}s \right|\!\right|\!\right|_0^2 \\ & \quad \leq \rho_i^{-2p_3} |\!|\!| \varphi |\!|\!|_0^2 \left(\int_a^t e^{s/2}\sqrt{\mathfrak{m}'(s)}\mathrm{d}s\right)^2 \\ & \quad \leq \rho_i^{-2p_3} |\!|\!| \varphi |\!|\!|_0^2 (e^t - e^a)\mathfrak{m}([a,t]). \end{aligned}$$

On the other hand, for $t > a > 0$ and $\varphi \in \mathcal{G}_{a]}$ we have

$$\begin{aligned} & |\!|\!| (\Xi(t)\mathbf{A}^{-\mathbf{p}} U^{(i)}(t) - \Xi(a)\mathbf{A}^{-\mathbf{p}} U^{(i)}(a))\varphi |\!|\!|_q^2 \\ & \quad \leq 2|\!|\!| \Xi(t)\mathbf{A}^{-\mathbf{p}}(U^{(i)}(t) - U^{(i)}(a))\varphi |\!|\!|_q^2 \\ & \qquad + 2|\!|\!| (\Xi(t) - \Xi(a))\mathbf{A}^{-\mathbf{p}} U^{(i)}(a)\varphi |\!|\!|_q^2 \\ & \quad \leq 2\|\Xi(t)\|_{\mathbf{p};\mathbf{q}}^2 (e^t - e^a) |\!|\!| \varphi |\!|\!|_0^2 + 2e^a |\!|\!| \varphi |\!|\!|_0^2\, \mathfrak{m}([a,t]). \end{aligned}$$

Therefore, since $\|\Xi(t)\|_{\mathbf{p};\mathbf{q}}$ is non-decreasing by Remark 4.3, for $t > a > 0$ and $\varphi \in \mathcal{G}_{a]}$ we have

$$\begin{aligned} & |\!|\!| (Y^{(i)}(t) - Y^{(i)}(a))\varphi |\!|\!|_0^2 \\ & \quad \leq 2C\, |\!|\!| \varphi |\!|\!|_0^2 \left((e^t - e^a)\mathfrak{m}([a,t]) + (e^t - e^a)\|\Xi(t)\|_{\mathbf{p};\mathbf{q}}^2 + e^a\mathfrak{m}([a,t])\right) \\ & \quad \leq 2C\, |\!|\!| \varphi |\!|\!|_0^2 \left[e^t(\|\Xi(t)\|_{\mathbf{p};\mathbf{q}}^2 + \mathfrak{m}([0,t])) - e^a(\|\Xi(a)\|_{\mathbf{p};\mathbf{q}}^2 + \mathfrak{m}([0,a]))\right] \\ & \quad \equiv |\!|\!| \varphi |\!|\!|_0^2\, \mathfrak{n}([a,t]), \end{aligned} \tag{4.14}$$

where $C = \max\{\rho_i^{-2p_3}, 2\}$ and $\mathfrak{n}$ is the Radon measure defined by (4.14). Hence we prove (i). The proof of (ii) is similar to the proof of Lemma 4.6 by applying Proposition 4.4 to the bounded regular martingale $\{Y^{(i)}(t)\}_{t\geq 0}$. ■

Let

$$M_{00} = \{f \in M_0 \subset H_\infty\, ;\ \|f(t)\| \text{ is a locally bounded function of } t\}$$

and

$$\widetilde{\mathcal{E}}_{00} = \mathcal{I}_\infty \otimes_{\mathrm{al}} \mathcal{E}(M_{00}). \tag{4.15}$$

Lemma 4.11. *Let* Ξ *be a regular martingale in* $\mathcal{L}(\mathcal{G}_{\mathbf{p}}, \mathcal{G}_{\mathbf{q}})$ *and let* $\{F_i^*\}$, $\{G_i\}$, $\{S, S^*\}$, $\{Z, Z^*\}$, $U^{(i)}$ *and* $\{M_j^{(i)}\}$ *be as defined in Lemmas* 4.6–4.10. *Put*

$$\begin{aligned} L_{ij}(t) = {} & \mathbf{A}^{-\mathbf{q}} M_j^{(i)}(t) U^{(i)}(t)^{-1} \mathbf{A}^{\mathbf{p}} \\ & - e^{q_2} \rho_i^{q_3} G_i(t) - e^{-p_2} \rho_i^{-p_3} S(t)\delta_{ij} - Z(t)\delta_{ij}. \end{aligned} \tag{4.16}$$

Then the processes $\{\eta_i(t, u, f)\}$ *defined in* (ii) *of Lemma* 4.8 *satisfies the relation for any* $u \in \mathcal{I}_\infty$ *and* $f \in M_0$:

$$\eta_i(t, u, f) = \sum_j f_j(t) \mathbf{A}^{-\mathbf{p}} [L_{ij}^*(t) + Z^*(t)\delta_{ij}] \mathbf{A}^{\mathbf{q}} u \otimes \phi_{\mathbf{1}_{[0,t]} f} \quad \text{a.e. } t. \tag{4.17}$$

Moreover, we have

$$Z(t) = \Xi(0) + \int_0^t \sum_{i,j} E_{ij}(s) \mathrm{d}\Lambda_{ij}(s) \tag{4.18}$$

defined on $\widetilde{\mathcal{E}}_{00}$, *where* $E_{ij} = e^{p_2 - q_2} \rho_i^{-q_3} \rho_j^{p_3} L_{ij}$ *for each* $i, j = 1, 2, \ldots$.

Proof. Let $u, v \in \mathcal{I}_\infty$, $f, g \in M_0$ and $t > a$. Then by (i) in Lemma 4.9 we have

$$U^{(i)}(t) v \otimes \phi_{\mathbf{1}_{[0,a]} g} = \mathrm{e}^{-\int_0^a g_i(s)\mathrm{d}s} v \otimes \phi_{\mathbf{1}_{[0,a]} g + \mathbf{1}_{[0,t]} e_i}.$$

Thus by (ii) in Lemma 4.8 and the Itô isometry, we have

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{dt}} \langle\langle \mathbf{A}^{-\mathbf{p}} Z^*(t) \mathbf{A}^{\mathbf{q}} u \otimes \phi_{\mathbf{1}_{[0,t]} f},\ U^{(i)}(t) v \otimes \phi_{\mathbf{1}_{[0,a]} g} \rangle\rangle \\ & \quad = \mathrm{e}^{-\int_0^a g_i(s)\mathrm{d}s} \frac{\mathrm{d}}{\mathrm{dt}} \int_0^t \sum_j (\mathbf{1}_{[0,a]}(s) g_i(s) + \delta_{ij}) \langle\langle \eta_j(s, u, f),\ v \otimes \phi_{\mathbf{1}_{[0,a]} g + \mathbf{1}_{[0,t]} e_i} \rangle\rangle \mathrm{d}s \\ & \quad = \langle\langle \eta_i(t, u, f),\ U^{(i)}(t) v \otimes \phi_{\mathbf{1}_{[0,a]} g} \rangle\rangle \quad \text{a.e. } t > a. \end{aligned} \tag{4.19}$$

On the other hand, from (4.12), (4.13), (4.10) and Itô isometry, for $t > a$ we obtain that

$$\begin{aligned} & \langle\langle u \otimes \phi_{\mathbf{1}_{[0,t]} f},\ [\mathbf{A}^{\mathbf{q}} Z(t) \mathbf{A}^{-\mathbf{p}} U^{(i)}(t) - \mathbf{A}^{\mathbf{q}} Z(a) \mathbf{A}^{-\mathbf{p}} U^{(i)}(a)] v \otimes \phi_{\mathbf{1}_{[0,a]} g} \rangle\rangle \\ & \quad = \langle\langle u \otimes \phi_{\mathbf{1}_{[0,t]} f},\ [Y^{(i)}(t) - Y^{(i)}(a)] v \otimes \phi_{\mathbf{1}_{[0,a]} g} \rangle\rangle \\ & \qquad - \langle\langle u \otimes \phi_{\mathbf{1}_{[0,t]} f},\ [W^{(i)}(t) - W^{(i)}(a)] v \otimes \phi_{\mathbf{1}_{[0,a]} g} \rangle\rangle \end{aligned}$$

$$= \int_a^t \sum_j f_j(s)\langle\langle u \otimes \phi_{\mathbf{1}_{[0,t]}f},\ M_j^{(i)}(s)v \otimes \phi_{\mathbf{1}_{[0,a]}g}\rangle\rangle \mathrm{d}s - \int_a^t \sum_j f_j(s)$$

$$\times \langle\langle u \otimes \phi_{\mathbf{1}_{[0,t]}f},\ \mathbf{A}^{\mathbf{q}}[e^{q_2}\rho_i^{q_3}G_i(s) + e^{-p_2}\rho_i^{-p_3}S(s)\delta_{ij}]\mathbf{A}^{-\mathbf{p}}U^{(i)}(s)v \otimes \phi_{\mathbf{1}_{[0,a]}g}\rangle\rangle \mathrm{d}s$$

$$= \int_a^t \sum_j f_j(s)\langle\langle u \otimes \phi_{\mathbf{1}_{[0,t]}f},\ \mathbf{A}^{\mathbf{q}}[L_{ij}(s) + Z(s)\delta_{ij}]\mathbf{A}^{-\mathbf{p}}U^{(i)}(s)v \otimes \phi_{\mathbf{1}_{[0,a]}g}\rangle\rangle \mathrm{d}s. \tag{4.20}$$

Therefore, by comparing (4.19) and (4.20) using the totality of the set $\{v \otimes \phi_{\mathbf{1}_{[0,a]}g} \mid v \in \mathcal{I}_\infty,\ g \in M_0,\ 0 < a < t\}$ in $\mathcal{G}_{t]}$, we have (4.17). Hence by (ii) in Lemma 4.8 and (4.17) we obtain that

$$\mathbf{A}^{-\mathbf{p}}Z^*(t)\mathbf{A}^{\mathbf{q}}u \otimes \phi_{\mathbf{1}_{[0,t]}f}$$
$$= \mathbf{A}^{-\mathbf{p}}\Xi^*(0)\mathbf{A}^{\mathbf{q}}u \otimes \phi_0 + \int_0^t \sum_{i,j} f_j(s)\mathbf{A}^{-\mathbf{p}}[L_{ij}^*(s) + Z^*(s)\delta_{ij}]\mathbf{A}^{\mathbf{q}}u \otimes \phi_{\mathbf{1}_{[0,s]}f}\mathrm{d}B_i(s). \tag{4.21}$$

It is obvious that $\{L_{ij}(t)\}_{t\geq 0}$ defined by (4.16) are adapted processes in $\mathcal{L}(\mathcal{G}_{\mathbf{p}}, \mathcal{G}_{\mathbf{q}})$ and a simple estimate shows that the integral $\int_0^t \sum_{i,j} L_{ij}(s)\mathrm{d}\Lambda_{ij}(s)$ is well-defined on $\widetilde{\mathcal{E}}_{00}$ since the integrability condition (3.1) is satisfied for any $f \in M_{00}$. Now, by (4.21) and the Itô isometry, for $f, g \in M_{00}$ we have

$$\langle\langle u \otimes \phi_f,\ \mathbf{A}^{\mathbf{q}}Z(t)\mathbf{A}^{-\mathbf{p}}v \otimes \phi_g\rangle\rangle$$
$$= \langle\langle \mathbf{A}^{-\mathbf{p}}Z^*(t)\mathbf{A}^{\mathbf{q}}u \otimes \phi_{\mathbf{1}_{[0,t]}f},\ v \otimes \phi_{\mathbf{1}_{[0,t]}g}\rangle\rangle \mathrm{e}^{\int_t^\infty \langle f(s),\, g(s)\rangle \mathrm{d}s}$$
$$= \mathrm{e}^{\int_t^\infty \langle f(s),\, g(s)\rangle \mathrm{d}s}\Bigg\{\langle\langle u \otimes \phi_0,\ \mathbf{A}^{\mathbf{q}}\Xi(0)\mathbf{A}^{-\mathbf{p}}v \otimes \phi_0\rangle\rangle$$
$$+ \int_0^t \sum_{i,j} f_j(s)g_i(s)\langle\langle u \otimes \phi_{\mathbf{1}_{[0,s]}f},\ \mathbf{A}^{\mathbf{q}}[L_{ij}(s) + Z(s)\delta_{ij}]\mathbf{A}^{-\mathbf{p}}v \otimes \phi_{\mathbf{1}_{[0,s]}g}\rangle\rangle \mathrm{d}s\Bigg\}.$$

By differentiation we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\langle u \otimes \phi_f,\ \mathbf{A}^{\mathbf{q}}Z(t)\mathbf{A}^{-\mathbf{p}}v \otimes \phi_g\rangle\rangle$$
$$= \sum_{i,j} f_j(t)g_i(t)\langle\langle u \otimes \phi_f,\ \mathbf{A}^{\mathbf{q}}L_{ij}(t)\mathbf{A}^{-\mathbf{p}}v \otimes \phi_g\rangle\rangle$$

which by (3.2) and (ii) in Lemma 4.8, proves that

$$Z(t) = \Xi(0) + \mathbf{A}^{-\mathbf{q}}\left[\int_0^t \sum_{i,j} \mathbf{A}^{\mathbf{q}}L_{ij}(s)\mathbf{A}^{-\mathbf{p}}\mathrm{d}\Lambda_{ij}(s)\right]\mathbf{A}^{\mathbf{p}} \tag{4.22}$$

on $\widetilde{\mathcal{E}}_{00}$. On the other hand, by direct computation we prove that

$$\begin{aligned}\mathbf{A}^{-\mathbf{q}}&\left[\int_0^t \sum_{i,j} \mathbf{A}^{\mathbf{q}} L_{ij}(s)\mathbf{A}^{-\mathbf{p}} \mathrm{d}\Lambda_{ij}(s)\right]\mathbf{A}^{\mathbf{p}}\\ &= \int_0^t \sum_{i,j} \mathrm{e}^{p_2-q_2}\rho_i^{-q_3}\rho_j^{p_3} L_{ij}(s)\mathrm{d}\Lambda_{ij}(s)\end{aligned}$$

on $\widetilde{\mathcal{E}}_{00}$. Thus by (4.22) we prove (4.18). ■

Acknowledgement

This work was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) (No. R05-2004-000-11346-0).

References

[1] Accardi L and Frigerio A, Markovian cocycles, *Proc. R. Irish Acad. Sect.* **A83** (1983) 251–263
[2] Barnett C, Streater R F and Wilde I F, The Itô-Clifford integral, *J. Funct. Anal.* **48** (1982) 172–212
[3] Belavkin V P, A quantum nonadapted Ito formula and stochastic analysis in Fock scale, *J. Funct. Anal.* **102** (1991) 414–447
[4] Goswami D, Lindsay J M, Sinha K B and Wills S J, Dilation of Markovian cocycles on a von Neumann algebra, *Pacific J. Math.* **211** (2003) 221–247
[5] Hida T, Analysis of Brownian Functionals, Carleton Math. Lect. Notes vol. 13 (Ottawa: Carleton University) (1975)
[6] Hudson R L and Lindsay J M, A non-commutative martingale representation theorem for non-Fock quantum Brownian motion, *J. Funct. Anal.* **61** (1985) 202–221
[7] Hudson R L, Lindsay J M and Parthasarathy K R, Stochastic integral representation of some quantum martingales in Fock space, in: From local times to global geometry, control and physics, *Proc. Warwick Symposium 1984/1985* (Pitman RNM) (1986) pp. 121–131
[8] Hudson R L and Parthasarathy K R, Quantum Ito's formula and stochastic evolutions, *Commun. Math. Phys.* **93** (1984) 301–323
[9] Ji U C, Stochastic integral representation theorem for quantum semimartingales, *J. Funct. Anal.* **201** (2003) 1–29
[10] Ji U C and Sinha K B, Integral representation of quantum martingales, *Infin. Dimen. Anal. Quantum Probab. Rel. Top.* **8** (2005) 55–72
[11] Lindsay J M, Fermion martingales, *Probab. Theory Related Fields* **71** (1986) 307–320
[12] Lindsay J M and Parthasarathy K R, Cohomology of power sets with applications in quantum probability, *Commun. Math. Phys.* **124** (1989) 337–364
[13] Lindsay J M and Wills S J, Markovian cocycles on operator algebras adapted to a Fock filtration, *J. Funct. Anal.* **178** (2000) 269–305
[14] Meyer P-A, Quantum Probability for Probabilists, Lecture Notes in Math., vol. 1538 (Springer-Verlag) (1993)
[15] Meyer P-A, Représentation de martingales d'opérateurs, in: Séminaire de probabilités XXVII, Lecture Notes in Math., vol. 1557 (Springer-Verlag) (1994) pp. 97–105
[16] Obata N, White noise calculus and Fock space, Lecture Notes in Math., vol. 1577 (Springer-Verlag) (1994)

[17] Parthasarathy K R, An introduction to quantum stochastic calculus (Birkhäuser) (1992)
[18] Parthasarathy K R and Sinha K B, Stochastic integral representation of bounded quantum martingales in Fock space, *J. Funct. Anal.* **67** (1986) 126–151
[19] Parthasarathy K R and Sinha K B, Representation of a class of quantum martingales II, in: Quantum Probability and Applications III (eds) L Accardi and W von Waldenfels, Lecture Notes in Math., vol. 1303 (Springer-Verlag) (1988) pp. 232–250