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Remarks on $B(H) \otimes B(H)$

GILLES PISIER

Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, USA
Université Paris VI, Equipe d'Analyse, Case 186, 75252 Paris Cedex 05, France

Abstract. We review the existing proofs that the min and max norms are different on $B(H) \otimes B(H)$ and give a shortcut avoiding the consideration of non-separable families of operator spaces.

Keywords. C^*-algebra; C^*-tensor products; nuclearity; amenability.

Let A, B be C^*-algebras. Let $\| \ \|_\alpha$ be a C^*-norm on their algebraic tensor product, denoted by $A \otimes B$; as usual, $A \otimes_\alpha B$ then denotes the C^*-algebra obtained by completing $A \otimes B$ with respect to $\| \ \|_\alpha$. By classical results (see [12]) the set of C^*-norms admits a minimal and a maximal element denoted respectively by $\| \cdot \|_{\min}$ and $\| \cdot \|_{\max}$. Then A is called nuclear if for any B we have $A \otimes_{\min} B = A \otimes_{\max} B$, or equivalently $\|x\|_{\min} = \|x\|_{\max}$ for any x in $A \otimes B$. We refer the reader to [7, 12] for more information on nuclear C^*-algebras. We note in particular that by results due to Connes and Haagerup [1, 2], a C^*-algebra is nuclear iff it is amenable as a Banach algebra (in B E Johnson's sense). See also [3, 4].

Although, it was known early [14] that $B(H)$ is not nuclear (assuming $\dim(H) = \infty$), the problem whether $A \otimes_{\min} B = A \otimes_{\max} B$ when $A = B = B(H)$ remained open until [6]. In the latter paper, several different proofs were given. Taking into account the most recent information from [5], we now know that

$$\sup\left\{ \frac{\|t\|_{\max}}{\|t\|_{\min}} \;\Big|\; t \in B(H) \otimes B(H),\ \operatorname{rk}(t) \le n \right\} \ge \frac{n}{2\sqrt{n-1}}. \tag{1}$$

This estimate is rather sharp asymptotically, since it can be shown that the supremum appearing in (1) is $\le \sqrt{n}$ (see [6] or p. 353 of [10]).

Remark. In (3) below, the complex conjugate $\bar{a}$ of a matrix a in M_N is meant in the usual way, i.e. $(\bar{a})_{ij} = \overline{a_{ij}}$. In general, we will need to consider the conjugate $\bar{A}$ of a C^*-algebra A. This is the same object but with the complex multiplication changed to $(\lambda, a) \to \bar{\lambda} a$, so that $\bar{A}$ is anti-isomorphic to A. For any $a \in A$, we denote by $\bar{a}$ the same element viewed as an element of $\bar{A}$. Note that $\bar{A}$ can also be identified with the opposite C^*-algebra A^{op} which is defined as the same object but with the product changed to $(a, b) \to ba$. It is easy to check that the mapping $a \to a^*$ (or more properly $\bar{a} \to a^*$) is a (linear) $*$-isomorphism from $\bar{A}$ to A^{op}.

The distinction between A and $\bar{A}$ is necessary in general, but not for $A = B(H)$ since in that case, using $H \simeq \overline{H}$, we have $\overline{B(H)} \simeq B(\overline{H}) \simeq B(H)$, and in particular $\overline{M_N} \simeq M_N$. As a consequence, it is easy to see that for any matrix a in $M_N(A)$ we have

$$\|[\bar{a}_{ij}]\|_{M_N(\bar{A})} = \|[a_{ij}]\|_{M_N(A)}.$$

Note however that $H \simeq \overline{H}$ depends on the choice of a basis so the isomorphism $\overline{B(H)} \simeq B(H)$ is not canonical. Nevertheless, this shows that the problem whether the min and max norms are the same is identical for $B(H) \otimes B(H)$ and for $B(H) \otimes \overline{B(H)}$.

Remark. Consider $a_1, \ldots, a_n$ in A and $b_1, \ldots, b_n$ in B. Using the preceding remark we have

$$\left\| \sum a_i \otimes \bar{b}_i \right\|_{A \otimes_\alpha \overline{B}} = \left\| \sum a_i \otimes b_i^* \right\|_{A \otimes_\alpha B^{\mathrm{op}}}$$

for any C^*-norm, in particular for $\alpha = \min$ or max. Moreover, we have

$$\left\| \sum a_i \otimes \bar{b}_i \right\|_{A \otimes_{\max} \overline{B}} = \left\| \sum a_i \otimes b_i^* \right\|_{A \otimes_{\max} B^{\mathrm{op}}} = \sup \left\{ \left\| \sum \pi(a_i)\sigma(b_i)^* \right\| \right\}$$

where the supremum runs over all commuting range pairs $\pi : A \to B(H)$, $\sigma : B \to B(H)$ with π a representation and σ an anti-representation on the same (arbitrary) Hilbert space H.

Remark. Let M be a C^*-algebra equipped with a tracial state τ. Then the GNS construction associated to (M, τ) produces a Hilbert space H, a cyclic unit vector ξ in H and commuting left and right actions of M on H (we denote the latter simply by $a \cdot h \cdot b$ for $h \in H, a, b \in A$) so that $x \cdot \xi = \xi \cdot x$ and $\tau(xy) = \langle x \cdot \xi \cdot y, \xi \rangle$ $(x, y \in A)$. If we denote $L(a)h = a \cdot h$ and $R(a)h = h \cdot a$ then L (resp. R) is a representation of A resp. (A^{op}) on $B(H)$ and the ranges of L and R commute.

We then have for any n-tuple $(u_1, \ldots, u_n)$ of unitaries in M,

$$\left\| \sum_1^n u_i \otimes \bar{u}_i \right\|_{M \otimes_{\max} \overline{M}} = \left\| \sum_1^n u_i \otimes u_i^* \right\|_{M \otimes_{\max} M^{\mathrm{op}}} = n. \tag{2}$$

Indeed, this is

$$\geq \left\| \sum_1^n L(u_i)R(u_i^*) \right\| \geq \left\| \sum_1^n u_i \cdot \xi \cdot u_i^* \right\|$$

but $u_i \cdot \xi \cdot u_i^* = \xi$ hence (2) is $\geq n$ and $\leq n$ is trivial by the triangle inequality. In particular, for any unitary matrices $u_1, \ldots, u_n$ in M_N we have

$$\left\| \sum_1^n u_i \otimes \bar{u}_i \right\|_{\max} = n.$$

In all the proofs in [6], a crucial role is played by a certain constant $C(n)$, defined as follows: $C(n)$ is the smallest constant C such that for each $m \geq 1$, there is $N_m \geq 1$ and an n-tuple $[u_1(m), \ldots, u_n(m)]$ of unitary $N_m \times N_m$ matrices such that

$$\sup_{m \neq m'} \left\| \sum_{i=1}^n u_i(m) \otimes \overline{u_i(m')} \right\|_{\min} \leq C. \tag{3}$$

The crucial fact to show that $B(H) \otimes_{\min} B(H) \neq B(H) \otimes_{\max} B(H)$ is that $C(n) < n$. Various improvements were given over the initial estimates of $C(n)$ in [6], notably by Valette [13] using Ramanujan graphs (see [10] for more details). The final word on this is now.

Theorem 1 [5]. $C(n) = 2\sqrt{n-1}$ *for any* $n \geq 2$.

The (much easier) lower bound $2\sqrt{n-1} \leq C(n)$ was proved in [11]. The connection of $C(n)$ to $B(H) \otimes B(H)$ goes through the next statement.

Theorem 2 [6]. *For any* $n \geq 1$ *and any* $C > C(n)$, *there is a tensor* t *of rank* n *in* $B(H) \otimes B(H)$ *such that*

$$\|t\|_{\max}/\|t\|_{\min} \geq n/C.$$

Our goal is to describe below a shortcut on the proof of this last result. Our 'shortcut' avoids any reference to the non-separability of the space of n-dimensional operator spaces (as was done in [6]) and uses instead a compactness argument for 'convergence in distribution' of n-tuples of operators.

More precisely, the notion of 'distribution' that we will use is the same as Voiculescu's definition in free probability but our terminology is slightly different. Let S be the set consisting of the disjoint union of the sets

$$S_k = [1, \ldots, n]^k \times \{1, *\}^k.$$

For any $w = ((i_1, \ldots, i_k), (\varepsilon_1, \ldots, \varepsilon_k))$ in S_k and any n-tuple $x = (x_1, \ldots, x_n)$ in $B(H)$ we denote

$$w(x) = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \ldots x_{i_k}^{\varepsilon_k}$$

(where $x^\varepsilon = x$ if $\varepsilon = 1$ and $x^\varepsilon = x^*$ if $\varepsilon = *$). Let $x = (x_1, \ldots, x_n)$ be an n-tuple in a von Neumann algebra M equipped with a tracial state τ. By 'the distribution of x', we mean the function

$$\mu_x : \ S \to \mathbb{C}$$

defined by

$$\mu_x(w) = \tau(w(x)).$$

When $x = (u_1, \ldots, x_n)$ is an n-tuple of unitary operators, we may as well consider that μ_x is a function defined on $\mathbb{F}_n$ (free group with generators $g_1, \ldots, g_n$) by setting $\mu_x(w) = \tau(\pi_x(w))$ where $\pi_x : \ \mathbb{F}_n \to M$ is the unitary representation defined by $\pi(g_i) = u_i$.

The following is elementary and well known.

Lemma 3. *Fix n. Let* $(M(m), \tau_m)$ *be a sequence of von Neumann algebras equipped with (tracial) states. Let* $x(m) = (x_1(m), \ldots, x_n(m))$ *be a bounded sequence of n-tuples with* $x(m) \in M(m)^n$. *Then there is a subsequence* $\{m_k\}$ *such that the distributions of* $x(m_k)$ *converge pointwise on S when* $k \to \infty$.

To identify the limit of a sequence of distributions, it will be convenient to use von Neumann algebra ultraproducts. We briefly recall how those are constructed.

Let $(M(m), \tau_m)$, $m \in \mathbb{N}$ be as before. Let $\mathcal{U}$ be a non-trivial ultrafilter on $\mathbb{N}$. Let $\mathcal{B} = \bigoplus_m M(m)$. We equip the von Neumann algebra $\mathcal{B}$ with the state $\tau_{\mathcal{U}}$ defined by

$$\tau_{\mathcal{U}}(y) = \lim_{\mathcal{U}} \tau_m(y_m), \forall y = (y_m) \in \mathcal{B}.$$

The GNS-construction applied to $\mathcal{B}$ and the state $\tau_{\mathcal{U}}$ produces a Hilbert space $H_{\mathcal{U}}$ and left and right actions of $\mathcal{B}$ on $H_{\mathcal{U}}$, each with kernel equal to

$$I_{\mathcal{U}} = \{y = (y_m) | \lim_{\mathcal{U}} \tau_m(y_m^* y_m) = 0\}.$$

We then set

$$M(\mathcal{U}) = \mathcal{B}/I_{\mathcal{U}}.$$

Thus, after passing to the quotient by $I_{\mathcal{U}}$ we obtain an isometric representation:

$$a \to L(a) \in B(H_{\mathcal{U}})$$

of $M(\mathcal{U})$ on $H_{\mathcal{U}}$ and an isometric representation $a \to R(a) \in B(H_{\mathcal{U}})$ of $M(\mathcal{U})^{\mathrm{op}}$.

It is well-known (see e.g. p. 211 of [10]) that $M(\mathcal{U})$ is a von Neumann algebra, that $L(M(\mathcal{U}))$ is a von Neumann subalgebra of $B(H_{\mathcal{U}})$, and that we have

$$L(M(\mathcal{U}))' = R(M(\mathcal{U})), \quad R(M(\mathcal{U}))' = L(M(\mathcal{U})).$$

Let $x = \{x(m) | m \in \mathbb{N}\}$ be a bounded sequence of n-tuples with $x(m) \in M(m)^n$ as before. Equivalently, x can be viewed as an n-tuple of elements of $\mathcal{B}$ (i.e. as an element of $\mathcal{B}^n$). Let $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)$ be the associated n-tuple in $(\mathcal{B}/I_{\mathcal{U}})^n$. Then, for any 'word' w in S, we clearly have

$$\tau_{\mathcal{U}}(w(\hat{x})) = \lim_{\mathcal{U}} \tau_m(w(x(m))).$$

Hence the distribution of $x(m)$ tends pointwise to that of $\hat{x}$ along $\mathcal{U}$, so we can write $\lim_{\mathcal{U}} \mu_{x(m)} = \mu_{\hat{x}}$. The next (again elementary and well-known) lemma connects limits in distribution with ultrafilters.

Lemma 4. *Let* $\{(x(m) | m \in \mathbb{N}\}$ *be a sequence of* n*-tuples as in the preceding lemma. The following are equivalent*:

- (i) *The distributions of* $x(m)$ *converge pointwise when* $m \to \infty$.
- (ii) *For any non-trivial ultrafilter* $\mathcal{U}$ *on* $\mathbb{N}$, *the associated* n*-tuple* $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)$ *in* $(M(\mathcal{U}), \tau_{\mathcal{U}})$ *has the same distribution* (*i.e. its distribution does not depend on* $\mathcal{U}$).
- (iii) *There is a von Neumann algebra* (M, τ) *equipped with a tracial state and* $y = (y_1, \dots, y_n)$ *in* M^n *such that* $x(m) \to y$ *in distribution.*

Theorem 5. *Let* $\{[u_1(m), \dots, u_n(m)], m \in \mathbb{N}\}$ *a sequence of* n*-tuples of unitary matrices satisfying* (3) (*recall* $u_1(m), \dots, u_n(m)$ *are of size* $N_m \times \mathbb{N}_m$). *Suppose that* $[u_1(m), \dots, u_n(m)]$ *converges in distribution when* $m \to \infty$. *Let* $\mathbb{N} = \alpha \cup \beta$ *be any disjoint partition of* $\mathbb{N}$ *into two infinite subsets, let* $u_i(\alpha) = \bigoplus_{m\in\alpha} u_i(m)$, $u_i(\beta) = \bigoplus_{m'\in\beta} u_i(m')$, *and finally let*

$$t = \sum_{i=1}^{n} u_i(\alpha) \otimes \overline{u_i(\beta)}.$$

We have then

$$\|t\|_{\min} \le C \quad \text{and} \quad \|t\|_{\max} = n.$$

Hence $\|t\|_{\max} / \|t\|_{\min} \ge n/C$, *where the min and max norms are relative to* $(\bigoplus_{m\in\alpha} M_{N_m}) \otimes \overline{(\bigoplus_{m\in\beta} M_{N_m})}$.

Proof. We have obviously

$$\|t\|_{\min} = \sup_{\substack{m\in\alpha \\ m'\in\beta}} \left\| \sum u_i(m) \otimes \overline{u_i(m')} \right\|,$$

hence $\|t\|_{\min} \le C$. We now turn to $\|t\|_{\max}$. Let $\mathcal{U}$ be a nontrivial ultrafilter on α and let $\mathcal{V}$ be one on β. We construct the ultraproducts $M(\mathcal{U})$ and $M(\mathcal{V})$ as above. Since the quotient mappings $\bigoplus_{m\in\alpha} M(N_m) \to M(\mathcal{U})$ and $\bigoplus_{m\in\beta} M(N_m) \to M(\mathcal{V})$ are $*$-homomorphisms, we have

$$\|t\|_{\max} \ge \left\| \sum u_i \otimes \bar{v}_i \right\|_{M(\mathcal{U})\otimes_{\max}\overline{M(\mathcal{V})}},$$

where u_i (resp. v_i) is the equivalence class modulo $\mathcal{U}$ (resp. $\mathcal{V}$) of $\bigoplus_{m\in\alpha} u_i(m)$ (resp. $\bigoplus_{m\in\beta} u_i(m)$).

Now, since we assume that $[u_1(m), \ldots, u_n(m)]$ converges in distribution, $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ must have the same distribution relative respectively to $\tau_{\mathcal{U}}$ and $\tau_{\mathcal{V}}$. But this implies that there is a $*$-isomorphism π from the von Neumann algebra M_u generated by $(u_1, \ldots, u_n)$ to the one M_v generated by $(v_1, \ldots, v_n)$, defined simply by $\pi(u_i) = v_i$. Moreover, since we are dealing here with *finite* traces, there is a conditional expectation P from $M(\mathcal{U})$ onto M_u. Therefore the composition $T = \pi P$ is a unital completely positive map such that $T(u_i) = v_i$. Hence we have

$$\left\| \sum u_i \otimes \bar{v}_i \right\|_{\max} \ge \left\| \sum T(u_i) \otimes \bar{v}_i \right\|_{M(v)\otimes_{\max}\overline{M(v)}}$$

$$= \left\| \sum v_i \otimes \bar{v}_i \right\|_{M(v)\otimes_{\max}\overline{M(v)}}.$$

But then by the remark preceding Theorem 1 we conclude that $\|t\|_{\max} \ge n$.

Proof of Theorem 2. Fix any number $C > C(n)$. Then there is a sequence $\{[u_1(m), \ldots, u_n(m)], m \in \mathbb{N}\}$ satisfying (3). By Lemma 3, there is a subsequence that converges in distribution. Let $\{N(m)\}$ be the sequence of sizes for this subsequence and let $\mathcal{B} = \oplus_m M_{N(m)}$. Applying Theorem 5 to this subsequence we find t in $\mathcal{B} \otimes \bar{\mathcal{B}}$ such that

$$\|t\|_{\max}/\|t\|_{\min} \ge n/C.$$

But since $\mathcal{B} \subset B(H)$ and there is a unital completely positive projection from $B(H)$ onto $\mathcal{B}$, the min and max norms of t viewed as sitting in $B(H) \otimes B(\bar{H})$ are the same as when computed in $\mathcal{B} \otimes \bar{\mathcal{B}}$ (this essentially goes back to [8]) (see also ch. 14 of [10]). Thus we obtain Theorem 2.

Remark. The same shortcut applies to the proof of the main result in [9].

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