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# Pro-torus actions on Poincaré duality spaces

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**Abstract.** In this paper, it is shown that some of the results of torus actions on Poincaré duality spaces, Borel's dimension formula and topological splitting principle to local weights, hold if 'torus' is replaced by 'pro-torus'.

**Keywords.** Pro-torus; Poincaré duality space; local weight.

## 1. Introduction

In the theory of linear representations of compact connected Lie groups, the crucial first step is restriction to the maximal torus. Hsiang [10] has suggested that the study of topological transformation groups proceed in the same way, and has offered the concepts of local weights and $F$-varieties to generalize the linear notions of weights and weight spaces.

It is well-known that locally compact groups can be approximated by Lie groups. This means if $G$ is a locally compact group, with finitely many components, then $G$ has arbitrarily small compact normal subgroup $N$ such that $G/N$ is a Lie group.

$G$ is called a $k$ dimensional pro-torus (i.e compact, connected, abelian group) if $G$ has a totally disconnected closed subgroup $N$ such that $G/N \simeq T^k$, a $k$ torus. Furthermore, the dimension of $G$ is infinite if $G$ has a totally disconnected closed subgroup $N$ such that $G/N$ is a infinite dimensional torus. If a $k$ dimensional pro-torus $G$ acts effectively on a Hausdorff space $X$, (all actions are assumed to be continuous), then there is an induced action of the $k$-torus $G/N$ on the orbit space $X/N$ and $X \to X/N$ induces a homeomorphism $X^G \approx (X/N)^{G/N}$. (Here $X^G$ denotes fixed point set of action $G$.) The orbit space of the action of $G/N$ on $X/N$ is homeomorphic to the orbit space $X/G$. On the other hand, the orbit space $X/N$ inherits global cohomological properties from the space $X$. Thus many questions about the cohomological properties of orbit spaces and fixed point sets of actions of pro-torus are reduced to questions about torus.

Let $X$ be a connected Hausdorff topological space such that $H^*(X, \mathbb{Q})$ finite dimensional. We say that $X$ is a Poincaré duality space over the rationals of formal dimension $n > 0$, and we write $fd(X) = n$, if $H^*(X, \mathbb{Q})$ is a finite dimensional vector space over $\mathbb{Q}$ and $H^*(X, \mathbb{Q}) \otimes H^{n-*}(X, \mathbb{Q}) \to H^n(X, \mathbb{Q}) = \mathbb{Q}$ is a nonsingular pairing. It is a well-known result proven by Chang and Skjelberg [7] and Bredon [4], that each component of the fixed point set of a torus action also satisfies Poincaré duality. Moreover, Borel formula also holds. In this paper these results are generalized for pro-torus actions using some techniques of Biller [2]. Biller showed that the fixed point set of pro-torus in a rational cohomology manifold is a rational cohomology manifold for even codimension.

From now on, $X$ will be a compact Poincaré duality space over the rationals of formal dimension $n$, and all cohomologies are sheaf cohomology over $\mathbb{Q}$.

**Theorem 1.1 [6, 11].** *Let $N$ be a totally disconnected compact group which acts on a locally compact space $X$. Then the orbit projection $X \to X/N$ induces an isomorphism*

$$H_c^*(X/N, \mathbb{Q}) \simeq H_c^*(X, \mathbb{Q})^N.$$

## 2. Components of fixed point sets and local weights

In this section, we define some cohomology classes for each components of fixed point set which we call local weights along each component.

**Theorem 2.1.** *Let $G$ be a compact connected abelian group which acts on a compact Poincaré duality space $X$ of formal dimension $n$. Then each connected component of $X^G$ is also a Poincaré duality space.*

*Proof.* Let $G$ be a compact connected abelian group of finite dimension which acts on a compact Poincaré duality space $X$ of formal dimension $n$. Let $N$ be a totally disconnected closed subgroup of $G$ such that $G/N \simeq T^k$. ($k$ is called the dimension of $G$.)

Since $G$ is connected, its action (hence that of $N$) on $H^*(X)$ is trivial (see II.10.6, II.11.11 of [5]). The orbit space of $X/N$ is a Poincaré duality space such that $fd(X/N) = fd(X) = n$ by Theorem 1.1. Let $F' \subset X^G$ be a nonempty connected component of fixed point set $X^G$. Since the space $(X/N)^{G/N}$ is homeomorphic to $X^G$, $F'$ can be considered to be a connected component of $(X/N)^{G/N}$ by identifying $F'$ with its homeomorphic image. So $F'$ is a Poincaré duality space such that $fd(F') = r$.

If the dimension of $G$ is infinite then we find a totally disconnected closed subgroup $N$ of $G$ such that $G/N$ is an infinite dimensional torus and every sub-torus of $G/N$ is of the form $HN/N$ for a pro-torus $H \le G$ [9]. Let $F'$ be a connected component of $X^G$. Choose a finite dimensional pro-torus $H_1 \le G$ whose action on $X$ is not trivial. Let $F_1$ be the connected component of $X^{H_1}$ which contains $F'$. If $F' \neq F_1$ then the induced action of $G/H_1$ on $F_1$ is not trivial, hence we may choose a finite dimensional pro-torus $H_2 \le G$ which contains $H_1$ and acts non-trivially on $F_1$. So replacing $F_1$ and $H_1$ by $X$ and $G$ we can find a pro-torus $H_2$ such that $H_1 \le H_2$ and $H_2$ acts non-trivially on $F_1$. Thus we have constructed a properly descending sequence of Poincaré duality spaces $F_j$ by induction. Since formal dimensions decrease strictly, we reach a finite dimensional pro-torus $H_m \le G$ such that $F'$ is a component of the fixed point set $X^{H_m}$. Thus $F'$ is also a Poincaré duality space. ■

**Theorem 2.2.** *Let $S$ be the set of all closed, connected, codimension 1 subgroups of $G$. Then the formula*

$$n - r = \sum_{H \in S} (n(H) - r)$$

*holds where $n(H)$ denotes the formal dimension of connected component of $X^H$ containing $F'$. We will denote this set by $F'(H)$.*

*Proof.* If the dimension of $G$ is infinite then $G$ has closed connected subgroups of arbitrarily high finite dimension [9]. We may therefore assume that $G$ is finite dimensional. Choose $H \in S$. Then fixed point set, $X^H$, is invariant under $G$. So $H^*(X^H) \simeq H^*(X^H/N)$ by Theorem 1.1. Moreover $X^H/N = (X/N)^H = (X/N)^{HN/N}$. Since $H^*(X^H) \simeq H^*((X/N)^{HN/N})$ there is a one-to-one correspondence between connected components of $X^H$ and $(X/N)^{HN/N}$. Let $F' \subset X^G$ be a connected component. Then there exists $F' \subset F'(HN/N)$ such that $F'(HN/N)$ is a connected component of $(X/N)^{HN/N}$. Moreover, every codimension 1 sub-torus of $G/N$ is of the form $HN/N$ for a unique $H \in S$ [9]. So $F'(HN/N)$ is a Poincaré duality space such that $fd(F'(HN/N)) = n(HN/N)$.

Let $F'(H)$ be the connected component of $X^H$ containing $F'$ and assume it corresponds to $F'(HN/N)$ under the isomorphism $H^*(X^H) \simeq H^*((X/N)^{HN/N})$. Then it is clear that $F'(H)$ is a Poincaré duality space and $fd(F'(H)) = fd(F'(HN/N))$. Finally, consider the action of $G/N \simeq T^k$ on $X/N$ which is also a Poincaré duality space. Then we can deduce the theorem from Borel's formula for torus actions on Poincaré duality spaces:

$$n - r = \sum_{HN/N} (fd(F'(HN/N)) - r) = \sum_{H\in S} (n(H) - r).$$

■

If $n(H) > r$ then we will say that $H$ is a local weight at $F'$.

**Theorem 2.3.** *$H$ is a local weight at $F'$ if and only if $i^*\colon H^*(F'(H)) \to H^*(F')$ (induced by inclusion $i\colon F' \hookrightarrow F'(H)$) is not an isomorphism.*

*Proof.* If $H$ is a local weight at $F'$ then it is trivial that $i^*\colon H^*(F'(H)) \to H^*(F')$ is not an isomorphism. Conversely, assume that $H$ is not a local weight at $F'$. Since $fd(F'(HN/N)) = fd(F'(H))$, $HN/N$ is not a local weight at $F'$ (considering the action of $G/N$ on $X/N$). Thus $j^*\colon H^*(F'(HN/N)) \to H^*(F')$, where $j\colon F' \hookrightarrow F'(HN/N)$ is an isomorphism. Since $H^*(F'(H)) \simeq H^*(F'(HN/N))$, $i^*\colon H^*(F'(H)) \to H^*(F')$ is an isomorphism. ■

The Borel theory is reflected in the algebraic properties of the equivariant cohomology ring $H_G^*(X)$ which is defined below.

Let $B_G$ be a classifying space of $G$ and $X_G = (X \times E_G)/G$ be the balanced product, where $E_G \to B_G$ is a universal principal $G$-bundle. Then the equivariant cohomology ring of $X$ is $H_G^*(X) = H^*(X_G)$. Let $B_\pi\colon B_G \to B_{G/N}$ be the mapping induced by the canonical epimorphism $\pi\colon G \to G/N$, since $G/N \simeq T^k$, $H^*(B_{G/N})$ is a polynomial algebra over $\mathbb{Q}$. Let $R = \operatorname{Im} B_\pi^*$ where $B_\pi^*\colon H^*(B_{G/N}) \to H^*(B_G)$. $R$ is a subring of $H^*(B_G)$.

Let $f_1$ be a generator of $H^r(F')$ where $f_1 \in H^*(F') \subset H^*(X^G) \subset H_G^*(X^G) = H^*(B_G) \otimes H^*(X^G)$. For $H \in S$, let us define the ideals:

$$I_{f_1}^G = \{a \in R\colon a \otimes f_1 \in \operatorname{Im}\{H_G^*(X) \to H_G^*(X^G)\}\}.$$

and

$$I_{f_1}^H = \{a \in R\colon a \otimes f_1 \in \operatorname{Im}\{H_G^*(X^H) \to H_G^*(X^G)\}\}.$$

We say that $X/N$ is totally nonhomologous to zero in $(X/N)_{G/N} \to B_{G/N}$ with respect to rational cohomology if

$$j^*\colon H_{G/N}^*(X/N) \to H^*(X/N)$$

is surjective. It is a well-known fact that $\dim_{\mathbb{Q}} H^*((X/N)^{G/N}) = \dim_{\mathbb{Q}} H^*(X/N)$ if and only if Leray–Serre spectral sequence associated to $(X/N)_{G/N} \to B_{G/N}$ degenerates, i.e. $X/N$ is totally non-homologous to zero in $(X/N)_{G/N} \to B_{G/N}$ with respect to rational cohomology.

**Theorem 2.4.** *If $X/N$ is totally nonhomologous to zero in $(X/N)_{G/N} \to B_{G/N}$, then $I^G_{f_1}$ is a principal ideal in $R$, generated by an element $\xi \in H^{n-r}(B_G)$. Furthermore, $\xi$ splits as a product of homogeneous linear factors in $H^*(B_G)$, such that each linear factor corresponds to a local weight at $F'$ and has multiplicity $\frac{n(H)-r}{2}$.*

*Proof.* Let $I^{G/N}_{f_1} = \{a \in H^*(B_{G/N})\colon a \otimes f_1 \in \mathrm{Im}\, H^*_{G/N}(X/N) \to H^*_{G/N}(X/N)^{G/N}\}$. If $a \in I^{G/N}_{f_1}$, then $a \otimes f_1 \in \mathrm{Im}\, H^*_{G/N}(X/N) \to H^*_{G/N}(X/N)^{G/N}$. Consider the commutative diagram

$$\begin{array}{ccccccc} & & X & \to & X/N & & \\ & & \downarrow & & \downarrow & & \\ X^G \times B_G & \hookrightarrow & X_G & \to & (X/N)_{G/N} & \hookleftarrow & (X/N)^{G/N} \times B_{G/N} \\ & & \downarrow & & \downarrow & & \\ & & B_G & \to & B_{G/N} & & \end{array}.$$

This diagram induces the following commutative diagram:

$$\begin{array}{ccccccc} & & H^*(X/N) & \to & H^*(X) & & \\ & & \uparrow & & \uparrow & & \\ H^*_{G/N}((X/N)^{G/N}) & \leftarrow & H^*_{G/N}(X/N) & \to & H^*_G(X) & \to & H^*_G(X^G) \\ & & \uparrow & & \uparrow & & \\ & & H^*(B_{G/N}) & \to & H^*(B_G) & & \end{array}.$$

It is clear that $B^*_\pi(a) \in \mathrm{Im}\,\{H^*_G(X) \to H^*_G(X^G)\}$. On the other hand, let $b = B^*_\pi(a) \in I^G_{f_1}$. Since $X/N$ is totally nonhomologous to zero in $(X/N)_{G/N} \to B_{G/N}$, it is clear that $a \in I^{G/N}_{f_1}$. So restriction of $B^*_\pi$ to $I^{G/N}_{f_1}$ induces an epimorphism $I^{G/N}_{f_1} \to I^G_{f_1}$. Let $H$ be a local weight at $F' \subset X^G$. Then $HN/N$ is a local weight at $F' \subset (X/N)^{G/N}$. Let $w_{HN/N} \in H^2(B_{G/N})$ be the corresponding cohomology class. Then $w_H = B^*_\pi(w_{HN/N}) \in H^2(B_G)$ is the corresponding cohomology class of $H$. It is well-known that $I^{G/N}_{f_1}$ is a principal ideal in $H^*(B_{G/N})$ generated by $\prod(w_{HN/N})^{m_H} \in H^{n-r}(B_{G/N})$ [1], where $\frac{n(H)-r}{2}$. Then it is easy to see that $I^G_{f_1}$ is a principal ideal in $R$ generated by $\prod(w_H)^{m_H}$. This finishes the proof. ■

COROLLARY 2.5

*If $H$ is a local weight at $F'$ then the ideal $I^H_{f_1}$ is a principal maximal ideal with respect to the property $I^H_{f_1} \neq R$ which is generated by $w_H$.*

*Proof.* If $H$ is a local weight at $F'$, then $HN/N$ is a local weight at $F'$ (considering $G/N$ action on $X/N$). Then the ideal

$$\begin{aligned} I^{HN/N}_{f_1} &= \{a \in H^*(B_{G/N})\colon a \otimes f_1 \in \mathrm{Im}\,\{H^*_{G/N}((X/N)^{HN/N}) \\ &\quad \to H^*_{G/N}((X/N)^{G/N})\}\} \end{aligned}$$

is a maximal ideal with respect to the property $I_{f_1}^{HN/N} \neq H^*(B_{G/N})$ which is generated by $w_{HN/N}$ [1]. Let us replace $X^H$ by $X$ and consider the $G/N$ action on $X^H/N = (X/N)^{HN/N}$. We have $\dim_{\mathbb{Q}} H^*((X/N)^{G/N}) = \dim_{\mathbb{Q}} H^*(X/N)$, since $X/N$ is totally nonhomologous to zero in $(X/N)_{G/N} \to B_{G/N}$. Let $G/N = S^1 \times HN/N$. Then

$$\begin{aligned}\dim_{\mathbb{Q}} H^*((X/N)^{G/N}) = \dim_{\mathbb{Q}} H^*((X^H/N)^{S^1}) &\leq \dim_{\mathbb{Q}} H^*(X^H/N) \\ &\leq \dim_{\mathbb{Q}} H^*(X/N).\end{aligned}$$

Thus $\dim_{\mathbb{Q}} H^*(X^H/N) = \dim_{\mathbb{Q}} H^*(X/N)$. This implies that $X^H/N$ is totally nonhomologous to zero in $(X^H/N)_{G/N} \to B_{G/N}$. Consider the commutative diagram:

$$\begin{array}{ccccccc} & & X^H & \to & X^H/N & & \\ & & \downarrow & & \downarrow & & \\ X^G \times B_G & \hookrightarrow & X_G^H & \to & (X^H/N)_{G/N} & \hookleftarrow & (X/N)^{G/N} \times B_{G/N} \\ & & \downarrow & & \downarrow & & \\ & & B_G & \to & B_{G/N} & & \end{array}.$$

It is clear that $B_\pi^*$ induces the epimorphism $I_{f_1}^{HN/N} \to I_{f_1}^H$. This ends the proof. ■

## 3. Applications

*Example* 3.1. Let $X$ be a $G$-space, $G$ finite-dimensional compact connected abelian group and $H^*(X) = H^*(S^n)$. Let $N$ be a totally disconnected closed subgroup of $G$ such that $G/N \simeq T^k$ is a torus group. Since the action of $G$, and hence that of $N$, on $H^*(X)$ is trivial, $H^*(X/N) = H^*(S^n)$ by Theorem 1.1. So the space $X^G$, which is homeomorphic to $(X/N)^{G/N}$, has the rational cohomology of $S^r$ for some $r \in -1, \ldots, n$ such that $n - r$ is even. Assume that $X^G \neq \emptyset$. Therefore, $\dim_{\mathbb{Q}} H^*(X^G) = \dim_{\mathbb{Q}} H^*(X) = 2$. Let $T'$ be a geometric weight for the $G/N$ space $X/N$ (see [10]). Then $T' = HN/N$ for a unique $H \in S$. Since $\dim_{\mathbb{Q}} H^*((X/N)^{G/N}) = \dim_{\mathbb{Q}} H^*((X^H)/N)$, $X^H/N$ is totally nonhomologous to zero in $(X^H/N)_{G/N} \to B_{G/N}$.

COROLLARY 3.2

*Let $A$ be a closed invariant subspace of $X$ such that inclusion $i$: $A \hookrightarrow X$ induces an isomorphism $H^*(X) \to H^*(A)$. Also assume that $X/N$ is totally nonhomologous to zero in $(X/N)_{G/N} \to B_{G/N}$. Then the local weights of $A$ are equal to the local weights of $X$.*

*Proof.* Recall that a space $X$ is called finitistic if every open covering of $X$ has a finite dimensional open refinement. (The dimension of a covering is the dimension of its nerve, which is one less than the maximum number of members of the covering which intersect nontrivially.) Clearly every compact space is finitistic. Consider $G/N \simeq T^k$ space $X/N$ and its closed invariant subspace $A/N$. It is easy to see that inclusion $A/N \hookrightarrow X/N$ induces an isomorphism $H^*(X/N) \to H^*(A/N)$ by Theorem 1.1. So $(X/N, A/N)$ is totally nonhomologous to zero in $(X/N, A/N) \to ((X/N)_{G/N}, (A/N)_{G/N}) \to B_{G/N}$. Thus inclusion $(A/N)^{G/N} \hookrightarrow (X/N)^{G/N}$ induces isomorphism $H^*((X/N)^{G/N}) \simeq H^*((A/N)^{G/N})$. This is essentially Theorem 1.6, ch. VII of [3]. The proof in Bredon is for the case where $G = S^1$ or $Z_p$. But one gets the result for higher-rank tori and $p$-tori using induction. We must assume finitistic orbit space for $S^1$ action but this is now known by Deo and Tripathi [8]. Let us consider the Leray–Serre spectral sequences of $A/N \to (A/N)_{G/N} \to B_{G/N}$ and $X/N \to (X/N)_{G/N} \to B_{G/N}$. It is clear that $H^*_{G/N}(X/N) \simeq H^*_{G/N}(A/N)$ by Zeeman's comparison theorem. So $I_{f_1}^{G/N} = J_{f_1}^{G/N}$ where

$$J_{f_1}^{G/N} = \{a \in H^*(B_{G/N})\colon a \otimes f_1 \in \operatorname{Im} H^*_{G/N}(A/N) \to H^*_{G/N}(A/N)^{G/N}\}.$$

Similarly, consider $G/N$ space $(X/N)^{HN/N}$ and its invariant subspace $(A/N)^{HN/N}$ for $H \in S$. It is easy to see that $I_{f_1}^{HN/N} = J_{f_1}^{HN/N}$ where

$$J_{f_1}^{HN/N} = \{a \in H^*(B_{G/N})\colon a \otimes f_1 \in \operatorname{Im}\{H^*_{G/N}((A/N)^{HN/N}) \to H^*_{G/N}((A/N)^{G/N})\}\}.$$

This shows that if $X/N$ is totally nonhomologous to zero in $(X/N)_{G/N} \to B_{G/N}$, then the local weights of $A$ are equal to the local weights of $X$. ■

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