

## Multiplicity of nontrivial solutions for elliptic equations with nonsmooth potential and resonance at higher eigenvalues

LESZEK GASIŃSKI<sup>1</sup>, DUMITRU MOTREANU<sup>2</sup>  
and NIKOLAOS S PAPAGEORGIOU<sup>3</sup>

<sup>1</sup>Institute of Computer Science, Jagiellonian University, ul. Nawojki 11,  
30072 Cracow, Poland

<sup>2</sup>Département de Mathématiques, Université de Perpignan, 66860 Perpignan, France

<sup>3</sup>Department of Mathematics, National Technical University, Zografou Campus,  
Athens 15780, Greece

E-mail: npapg@math.ntua.gr

MS received 15 March 2004; revised 22 February 2005

**Abstract.** We consider a semilinear elliptic equation with a nonsmooth, locally Lipschitz potential function (hemivariational inequality). Our hypotheses permit double resonance at infinity and at zero (double-double resonance situation). Our approach is based on the nonsmooth critical point theory for locally Lipschitz functionals and uses an abstract multiplicity result under local linking and an extension of the Castro–Lazer–Thews reduction method to a nonsmooth setting, which we develop here using tools from nonsmooth analysis.

**Keywords.** Double resonance; reduction method; eigenvalue; hemivariational inequality; locally Lipschitz function; Clarke subdifferential; critical point; local linking; nonsmooth Cerami condition.

### 1. Introduction

Let  $Z \subseteq \mathbb{R}^N$  be a bounded domain with a  $C^2$  boundary  $\Gamma$ . We study the following resonant semilinear elliptic differential equation with a nonsmooth potential (hemivariational inequality):

$$\begin{cases} -\Delta x(z) - \lambda_k x(z) \in \partial j(z, x(z)), & \text{for a.a. } z \in Z \\ x|_{\Gamma} = 0. \end{cases} \quad (\text{HVI})$$

Here  $k \geq 1$  is a fixed integer,  $\{\lambda_n\}_{n \geq 1}$  is the increasing sequence of distinct eigenvalues of the negative Laplacian with Dirichlet boundary condition (i.e. of  $(-\Delta, H_0^1(Z))$ ),  $j(z, \zeta)$  is a locally Lipschitz in the  $\zeta$ -variable integrand (in general it can be nonsmooth) and  $\partial j(z, \zeta)$  is the Clarke subdifferential with respect to the  $\zeta$ -variable.

For problem (HVI), we prove a multiplicity result, using a recent abstract theorem on the existence of multiple nontrivial critical points for a nonsmooth locally Lipschitz functional, proved by Kandilakis, Kourogenis and Papageorgiou [21]. Our approach is variational and is based on the nonsmooth critical point theory (see [9] and [22]). In particular, we develop and use a nonsmooth variant of the so-called ‘reduction method’. This method was first introduced for smooth problems by Castro and Lazer [7] and Thews [36] (see also [5, 6]).

Our hypotheses allow the nonsmooth potential to interact asymptotically at  $\pm\infty$  with two consecutive eigenvalues of higher order. Berestycki and de Figueiredo [2] were the first to consider such problems with resonance between  $\lambda_1$  and  $\lambda_2$  and they coined the term ‘double resonance problems’. In their analysis, the use of the interval  $[\lambda_1, \lambda_2]$  is crucial, since they exploit heavily the fact that the principal eigenfunction  $u_1$  is strictly positive and  $\partial u_1/\partial n < 0$  with  $n$  being the outward unit normal on the boundary (this is a consequence of the strong maximum principle). It is well-known that in higher parts of the spectrum this is no longer true. Recall that the principal eigenfunction  $u_1$  is the only one with constant sign. So in higher parts of the spectrum the analysis is more delicate. Recently for smooth problems, the issue was investigated by Cac [4], Hirano and Nishimura [16], Robinson [31], Costa-Silva [11], Landesman, Robinson and Rumbos [23], Iannacci and Nkashama [19], Tang and Wu [34], Su and Tang [33] and Su [32].

For problems with nonsmooth potential (known in the literature as hemivariational inequalities), equations resonant at higher eigenvalues were investigated by Goeleven, Motreanu and Panagiotopoulos [15] and Gasiński and Papageorgiou [14]. However, they did not allow for the situation of double resonance.

We should mention that hemivariational inequalities arise in physical problems, when one wants to consider more realistic models with a nonsmooth and nonconvex energy functionals. For concrete applications we refer to the book of Naniewicz and Panagiotopoulos [27]. For the mathematical theory of hemivariational inequalities we refer to the work of Gasiński and Papageorgiou [12, 13], Motreanu and Panagiotopoulos [25, 26], Niculescu and Radulescu [28], Radulescu [29], Radulescu and Panagiotopoulos [30] and the references therein.

## 2. Mathematical background

As we have already mentioned, our approach is based on the theory of the nonsmooth critical point theory for locally Lipschitz functionals.

Let  $X$  be a Banach space and  $X^*$  its topological dual. By  $\|\cdot\|_X$  we denote the norm of  $X$  and by  $\langle \cdot, \cdot \rangle_X$  the duality pairing for the pair  $(X, X^*)$ .

We will be dealing with locally Lipschitz functions  $\varphi: X \rightarrow \mathbb{R}$ .

Recall that a continuous convex function is locally Lipschitz. For a locally Lipschitz function  $\varphi: X \rightarrow \mathbb{R}$ , we introduce the *generalized directional derivative* of  $\varphi$  at  $x \in X$  in the direction  $h \in X$ , defined by

$$\varphi^0(x; h) \stackrel{df}{=} \limsup_{\substack{x' \rightarrow x \\ t \searrow 0}} \frac{\varphi(x' + th) - \varphi(x')}{t}$$

(see [10]). It is easy to check that the function  $X \ni h \rightarrow \varphi^0(x; h) \in \mathbb{R}$  is sublinear, continuous, and so by the Hahn–Banach theorem,  $\varphi^0(x; \cdot)$  is the support function of a nonempty, convex and  $w^*$ -compact set  $\partial\varphi(x)$ , defined by

$$\partial\varphi(x) \stackrel{df}{=} \{x^* \in X^*: \langle x^*, h \rangle_X \leq \varphi^0(x; h) \text{ for all } h \in X\}.$$

The multifunction  $\partial\varphi: X \rightarrow 2^{X^*} \setminus \{\emptyset\}$  is known as the *generalized* (or *Clarke*) *subdifferential* of  $\varphi$ . If  $\varphi \in C^1(X)$ , then  $\partial\varphi(x) = \{\varphi'(x)\}$  and if  $\varphi$  is convex, then  $\partial\varphi(x)$  coincides with

the convex subdifferential. The generalized subdifferential has a rich calculus which can be found in [10].

A point  $x \in X$  is a critical point of the locally Lipschitz function  $\varphi$ , if  $0 \in \partial\varphi(x)$ . If  $x \in X$  is a critical point, the value  $c = \varphi(x)$  is a critical value of  $\varphi$ . It is easy to check that if  $x \in X$  is a local extremum of  $\varphi$  (i.e. a local minimum or a local maximum), then  $0 \in \partial\varphi(x)$  (i.e.  $x \in X$  is a critical point).

We will use the following compactness-type condition:

A locally Lipschitz function  $\varphi: X \rightarrow \mathbb{R}$  satisfies the *nonsmooth Palais–Smale condition* (*nonsmooth PS-condition* for short) if any sequence  $\{x_n\}_{n \geq 1} \subseteq X$  such that  $\{\varphi(x_n)\}_{n \geq 1}$  is bounded and

$$m_\varphi(x_n) \stackrel{df}{=} \min\{\|x^*\|_{X^*} : x^* \in \partial\varphi(x_n)\} \longrightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

has a strongly convergent subsequence.

Since for  $\varphi \in C^1(X)$  we have  $\partial\varphi(x) = \{\varphi'(x)\}$ , we see that the above definition is an extension of the smooth PS-condition. It was shown by Cerami [8] and Bartolo, Benci and Fortunato [1] that a slightly more general condition in the smooth setting, suffices to prove the main minimax principles. In the present nonsmooth setting this condition has the following form:

A locally Lipschitz function  $\varphi: X \rightarrow \mathbb{R}$  satisfies the *nonsmooth Cerami condition* (*nonsmooth C-condition* for short) if any sequence  $\{x_n\}_{n \geq 1} \subseteq X$  such that  $\{\varphi(x_n)\}_{n \geq 1}$  is bounded and

$$(1 + \|x_n\|_X)m_\varphi(x_n) \longrightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

has a strongly convergent subsequence.

Recently Kandilakis, Kourogenis and Papageorgiou [21], proved the following multiplicity result extending a corresponding theorem of Brezis and Nirenberg [3].

**Theorem 2.1.** *If  $X$  is a reflexive Banach space,  $X = Y \oplus V$  with  $\dim V < +\infty$ ,  $\varphi: X \rightarrow \mathbb{R}$  is locally Lipschitz, bounded below, satisfies the nonsmooth C-condition,  $\inf_X \varphi < 0$  and there exists  $\varrho > 0$  such that*

$$\begin{cases} \varphi(x) \leq 0 & \text{if } x \in V, \|x\|_X \leq \varrho \\ \varphi(x) \geq 0 & \text{if } x \in Y, \|x\|_X \leq \varrho \end{cases} \tag{2.1}$$

*then  $\varphi$  has at least two nontrivial critical points.*

Condition (2.1) implies  $\varphi(0) = 0$ . We call condition (2.1) the local linking condition.

Recall that, if  $\{\lambda_n\}_{n \geq 1}$  are the distinct eigenvalues of  $(-\Delta, H_0^1(Z))$ , then  $\lambda_n \rightarrow +\infty$  and  $\lambda_1$  is positive, simple and isolated. Also there is an orthonormal basis  $\{u_n\}_{n \geq 1} \subseteq H_0^1(Z) \cap C^\infty(Z)$  of  $L^2(Z)$ , which are eigenfunctions corresponding to the eigenvalues  $\{\lambda_n\}_{n \geq 1}$ , i.e.

$$\begin{cases} -\Delta u_n(z) = \lambda_n u_n(z) & \forall z \in Z, \\ u_n|_\Gamma = 0, \end{cases}$$

for  $n \geq 1$ . If the boundary  $\Gamma$  of  $Z$  is a  $C^k$ -manifold (respectively a  $C^\infty$ -manifold) then  $u_n \in C^k(\bar{Z})$  (respectively  $u_n \in C^\infty(\bar{Z})$ ). The sequence  $\{\frac{1}{\sqrt{\lambda_n}}u_n\}_{n \geq 1}$  is an orthonormal basis of  $H_0^1(Z)$ . For every integer  $m \geq 1$ , let  $E(\lambda_m)$  be the eigenspace corresponding to the eigenvalue  $\lambda_m$ . We define

$$\bar{H}_m \stackrel{df}{=} \bigoplus_{i=1}^{m-1} E(\lambda_i) \quad \text{and} \quad \hat{H}_m \stackrel{df}{=} \overline{\bigoplus_{i=m+1}^{\infty} E(\lambda_i)}.$$

We have the following orthogonal direct sum decomposition:

$$H_0^1(Z) = \bar{H}_m \oplus E(\lambda_m) \oplus \hat{H}_m.$$

The eigenspace  $E(\lambda_m) \subseteq H_0^1(Z) \cap C^\infty(Z)$  has the unique continuation property, namely if  $u \in E(\lambda_m)$  is such that  $u$  vanishes on a set of positive measure, then  $u(z) = 0$  for all  $z \in Z$ .

If we set

$$V_m \stackrel{df}{=} \bar{H}_m \oplus E(\lambda_m) \quad \text{and} \quad W_m \stackrel{df}{=} E(\lambda_m) \oplus \hat{H}_m,$$

then on these spaces we have variational characterizations of the eigenvalues (Rayleigh quotients), which can be found in [20].

Let us recall two modes of convergence of sets and functions, which will be used in the proof of our nonsmooth extension of the Castro–Lazer–Thews reduction method. So let  $(Y_1, \tau_1)$  and  $(Y_2, \tau_2)$  be two Hausdorff topological spaces ( $\tau_1$  and  $\tau_2$  being the respective topologies). Also let  $\{G_n\}_{n \geq 1}$  be a sequence of nonempty subsets of  $Y_1 \times Y_2$ . We define:

$$\begin{aligned} (\tau_1 \times \tau_2) - \liminf_{n \rightarrow +\infty} G_n &\stackrel{df}{=} \left\{ (u, v) \in Y_1 \times Y_2 : u = \tau_1 - \lim_{n \rightarrow +\infty} u_n, \right. \\ &\quad \left. v = \tau_2 - \lim_{n \rightarrow +\infty} v_n, (u_n, v_n) \in G_n, n \geq 1 \right\}, \\ (\tau_1 \times \tau_2) - \limsup_{n \rightarrow +\infty} G_n &\stackrel{df}{=} \left\{ (u, v) \in Y_1 \times Y_2 : u = \tau_1 - \lim_{n \rightarrow +\infty} u_{n_k}, \right. \\ &\quad \left. v = \tau_2 - \lim_{n \rightarrow +\infty} v_{n_k}, (u_{n_k}, v_{n_k}) \in G_{n_k}, \right. \\ &\quad \left. n_i < n_{i+1} \text{ for } i \geq 1 \right\}. \end{aligned}$$

If

$$G = (\tau_1 \times \tau_2) - \liminf_{n \rightarrow +\infty} G_n = (\tau_1 \times \tau_2) - \limsup_{n \rightarrow +\infty} G_n,$$

then we say that the sequence  $\{G_n\}_{n \geq 1}$  converges in the  $(\tau_1 \times \tau_2)$ -sequential Kuratowski sense to  $G$  and denote it by

$$G_n \xrightarrow{K_{\tau_1, \tau_2}} G \quad \text{as } n \rightarrow +\infty.$$

Now, let  $Y$  be a Banach space and  $\{\varphi_n\}_{n \geq 1} \subseteq \bar{\mathbb{R}}^Y$  and  $\varphi \in \bar{\mathbb{R}}^Y$  (with  $\bar{\mathbb{R}} \stackrel{df}{=} \mathbb{R} \cup \{+\infty\}$ ).

We say that the sequence  $\{\varphi_n\}_{n \geq 1}$  converges in the Mosco sense to  $\varphi$ , denoted by  $\varphi_n \xrightarrow{M} \varphi$  if and only if the following two conditions hold:

- (1) for every  $y \in Y$  and every sequence  $\{y_n\}_{n \geq 1} \subseteq Y$  such that  $y_n \xrightarrow{w} y$  in  $Y$ , we have that

$$\varphi(y) \leq \liminf_{n \rightarrow +\infty} \varphi_n(y_n);$$

- (2) for every  $y \in Y$ , there exists a sequence  $\{y_n\}_{n \geq 1} \subseteq Y$  such that  $y_n \rightarrow y$  in  $Y$  and  $\varphi_n(y_n) \rightarrow \varphi(y)$ .

Further analysis of these two notions can be found in [17].

### 3. The nonsmooth reduction method

In this section we extend the Castro, Lazer and Thews reduction method to the present nonsmooth setting. By  $2^*$  we denote the Sobolev critical exponent defined by

$$2^* \stackrel{df}{=} \begin{cases} \frac{2N}{N-2}, & \text{if } N > 2 \\ +\infty, & \text{if } N \leq 2. \end{cases}$$

Our hypotheses on the nonsmooth potential  $j$  are the following:

H(j):  $j: Z \times \mathbb{R} \rightarrow \mathbb{R}$  is a function, such that

- (i) for all  $\zeta \in \mathbb{R}$ , the function  $Z \ni z \mapsto j(z, \zeta) \in \mathbb{R}$  is measurable and for almost all  $z \in Z$ , we have that  $j(z, 0) = 0$ ;
- (ii) for almost all  $z \in Z$ , the function  $\mathbb{R} \ni \zeta \mapsto j(z, \zeta) \in \mathbb{R}$  is locally Lipschitz;
- (iii) for almost all  $z \in Z$ , all  $\zeta \in \mathbb{R}$  and all  $u \in \partial j(z, \zeta)$ , we have that  $|u| \leq a_1(z) + c_1 |\zeta|^{r-1}$ , with  $a_1 \in L^\infty(Z)$ ,  $c_1 > 0$  and  $1 \leq r < 2^*$ ;
- (iv)  $\lim_{|\zeta| \rightarrow +\infty} [u(\zeta)\zeta - 2j(z, \zeta)] = -\infty$  uniformly for almost all  $z \in Z$  and all  $u(\zeta) \in \partial j(z, \zeta)$ ;
- (v) there exists  $l \in L^\infty(Z)$ , such that for almost all  $z \in Z$ , we have  $l(z) \leq \lambda_{k+1} - \lambda_k$  with strict inequality on a set of positive measure and for almost all  $z \in Z$ , all  $\zeta_1, \zeta_2 \in \mathbb{R}$  with  $\zeta_1 \neq \zeta_2$  and all  $v_1 \in \partial j(z, \zeta_1)$ ,  $v_2 \in \partial j(z, \zeta_2)$ , we have  $\frac{v_1 - v_2}{\zeta_1 - \zeta_2} \leq l(z)$ ;
- (vi) there exist  $\beta \in L^\infty(Z)_-$  and  $\delta_0 > 0$  such that for some integer  $m \in [1, k]$  and for almost all  $z \in Z$ , we have  $\beta(z) \leq \lambda_m - \lambda_k$ , with strict inequality on a set of positive measure and for almost all  $z \in Z$  and all  $\zeta \in \mathbb{R}$ , such that  $|\zeta| \leq \delta_0$ , we have  $\lambda_{m-1} - \lambda_k \leq \frac{2j(z, \zeta)}{\zeta^2} \leq \beta(z)$ ;
- (vii)  $0 \leq \liminf_{|\zeta| \rightarrow +\infty} \frac{2j(z, \zeta)}{\zeta^2} \leq \limsup_{|\zeta| \rightarrow +\infty} \frac{2j(z, \zeta)}{\zeta^2} \leq \gamma(z)$  uniformly for almost all  $z \in Z$ , with  $\gamma \in L^\infty(Z)_+$  and  $\gamma(z) \leq \lambda_{k+1} - \lambda_k$  for almost all  $z \in Z$ , with strict inequality on a set of positive measure.

*Remark 3.1.* Hypothesis H(j)(vi) implies that we have double resonance at the origin. The resonance is complete from below and incomplete from above. The same double resonance situation at infinity is implied by hypothesis H(j)(vii). So hypotheses H(j)(vi) and H(j)(vii) together provide the double-double resonance character of our problem. Also note that hypothesis H(j)(v) permits only downward discontinuities of the derivative of the potential function  $j(z, \cdot)$ . Recall that for almost all  $z \in Z$ , the derivative of  $j(z, \cdot)$  exists almost everywhere (Rademacher theorem).

Let  $\varphi: H_0^1(Z) \rightarrow \mathbb{R}$  be the energy functional defined by

$$\varphi(x) \stackrel{df}{=} \frac{1}{2} \|\nabla x\|_2^2 - \frac{\lambda_k}{2} \|x\|_2^2 - \int_Z j(z, x(z)) dz \quad \forall x \in H_0^1(Z)$$

(by  $\|\cdot\|_p$  we denote the norm of  $L^p(Z)$ ). We know that  $\varphi$  is locally Lipschitz (see p. 313 of [18]). Since  $k \geq 1$  is fixed, for what follows we set

$$\bar{H} \stackrel{df}{=} \bar{H}_k = \bigoplus_{i=1}^{k-1} E(\lambda_i) \quad \text{and} \quad \hat{H} \stackrel{df}{=} \hat{H}_k = \overline{\bigoplus_{i=k+1}^{\infty} E(\lambda_i)}.$$

We have that

$$H_0^1(Z) = \bar{H} \oplus E(\lambda_k) \oplus \hat{H}.$$

Also we set

$$\bar{H}_0 \stackrel{df}{=} \bar{H} \oplus E(\lambda_k) = \bigoplus_{i=1}^k E(\lambda_i)$$

and for  $u \in \bar{H}_0$ , we consider the following minimization problem:

$$\inf_{v \in \hat{H}} \varphi(u + v). \tag{3.1}$$

Since we do not identify  $H_0^1(Z)$  with its dual, we have that

$$(H_0^1(Z))^* = H^{-1}(Z) = \bar{H}_0^* \oplus \hat{H}^*.$$

We start with a simple lemma which is needed in what follows.

*Lemma 3.2.* *If  $n \geq 1$  and  $\beta \in L^\infty(Z)_+$ , with*

$$\beta(z) \leq \lambda_{n+1} \text{ for a.a. } z \in Z$$

*and the inequality is strict on a set of positive measure, then there exists  $\xi_1 > 0$ , such that*

$$\|\nabla x\|_2^2 - \int_Z \beta(z) |x(z)|^2 dz \geq \xi_1 \|\nabla x\|_2^2 \quad \forall x \in \hat{H}_n.$$

*Proof.* Let

$$\theta(x) \stackrel{df}{=} \|\nabla x\|_2^2 - \int_Z \beta(z) |x(z)|^2 dz \quad \forall x \in \hat{H}_n.$$

By virtue of the variational characterization of the eigenvalues, we have that  $\theta \geq 0$ . Suppose that the lemma is not true. Because of the positive homogeneity of  $\theta$ , we can find a sequence  $\{x_m\}_{m \geq 1} \subseteq \hat{H}_n$ , such that  $\|\nabla x_m\|_2 = 1$  for  $m \geq 1$  and  $\theta(x_m) \searrow 0$ . By passing to a subsequence if necessary, we may assume that

$$\begin{aligned} x_m &\xrightarrow{w} x_0 \quad \text{in } H_0^1(Z), \\ x_m &\longrightarrow x_0 \quad \text{in } L^2(Z), \end{aligned}$$

with some  $x_0 \in \hat{H}_n$ . Since the norm in a Banach space is weakly lower semicontinuous, in the limit as  $m \rightarrow +\infty$ , we obtain

$$\theta(x_0) = \|\nabla x_0\|_2^2 - \int_Z \beta(z)|x_0(z)|^2 dz \leq 0,$$

so

$$\|\nabla x_0\|_2^2 \leq \int_Z \beta(z)|x_0(z)|^2 dz \leq \lambda_{n+1} \|x_0\|_2^2$$

and since  $x_0 \in \hat{H}_n$ , from the variational characterization of  $\lambda_{n+1}$ , we have that

$$\|\nabla x_0\|_2^2 = \lambda_{n+1} \|x_0\|_2^2. \tag{3.2}$$

If  $x_0 = 0$ , then taking into account that  $\theta(x_m) \rightarrow 0$  we would have that  $\|\nabla x_m\|_2 \rightarrow 0$ . Because  $\|\nabla x_m\|_2 = 1$  for  $m \geq 1$ , this is not possible, so  $x_0 \neq 0$ . From (3.2), it follows that  $x_0 \in E(\lambda_{n+1})$ . Then, from the hypothesis that  $\beta(z) < \lambda_{n+1}$  on a set of positive measure and from the unique continuity property of  $E(\lambda_{n+1})$ , we have that

$$\|\nabla x_0\|_2^2 \leq \int_Z \beta(z)|x_0(z)|^2 dz < \lambda_{n+1} \|x_0\|_2^2,$$

a contradiction to (3.2). □

The next proposition essentially extends the Castro–Lazer–Thews reduction method to a nonsmooth setting.

**PROPOSITION 3.3**

*If hypotheses H(j) hold, then there exists a continuous map  $\vartheta: \bar{H}_0 \rightarrow \hat{H}$ , such that for every  $u \in \bar{H}_0$ , we have*

$$\inf_{v \in \hat{H}} \varphi(u + v) = \varphi(u + \vartheta(u))$$

and  $\vartheta(u) \in \hat{H}$  is the unique solution of the operator inclusion

$$0 \in p_{\hat{H}^*} \partial \varphi(u + v),$$

with  $u \in \bar{H}_0$  fixed, where  $p_{\hat{H}^*}$  is the orthogonal projection on  $\hat{H}^* = [\bar{H}_0^*]^\perp$ .

*Proof.* For a fixed  $u \in \bar{H}_0$ , let  $\varphi_u: H_0^1(Z) \rightarrow \mathbb{R}$  be defined by

$$\varphi_u(w) \stackrel{df}{=} \varphi(u + w) \quad \forall w \in H_0^1(Z).$$

For every  $w, h \in H_0^1(Z)$ , we have that

$$\begin{aligned} \varphi_u^0(w; h) &= \limsup_{\substack{w' \rightarrow w \\ t \searrow 0}} \frac{\varphi_u(w' + th) - \varphi_u(w')}{t} \\ &= \limsup_{\substack{w' \rightarrow w \\ t \searrow 0}} \frac{\varphi(u + w' + th) - \varphi(u + w')}{t} = \varphi^0(u + w; h), \end{aligned}$$

so

$$\partial\varphi_u(w) = \partial\varphi(u + w) \quad \forall w \in H_0^1(Z). \quad (3.3)$$

Let  $\hat{i}: \hat{H} \rightarrow H_0^1(Z)$  be the inclusion map and let  $\hat{\varphi}_u: \hat{H} \rightarrow \mathbb{R}$  be defined by

$$\hat{\varphi}_u(v) = \varphi(u + v) \quad \forall v \in \hat{H}.$$

We have that  $\varphi_u \circ \hat{i} = \hat{\varphi}_u$  and so

$$\partial(\varphi_u \circ \hat{i})(v) = \partial\hat{\varphi}_u(v) \quad \forall v \in \hat{H}. \quad (3.4)$$

But from the chain rule of Clarke (p. 45–46 of [10]), we have

$$\partial(\varphi_u \circ \hat{i})(v) \subseteq p_{\hat{H}^*} \partial\varphi_u(\hat{i}(v)) \quad \forall v \in \hat{H},$$

since  $\hat{i}^* = p_{\hat{H}^*}$ . Hence, from (3.3) and (3.4), it follows that

$$\partial\hat{\varphi}_u(v) \subseteq p_{\hat{H}^*} \partial\varphi_u(\hat{i}(v)) = p_{\hat{H}^*} \partial\varphi(u + v) \quad \forall v \in \hat{H}. \quad (3.5)$$

Now we have

$$x^* = A(x) - \lambda_k x - h \quad \forall x \in H_0^1(Z), x^* \in \partial\varphi(x), \quad (3.6)$$

with  $A \in \mathcal{L}(H_0^1(Z), H^{-1}(Z))$  being defined by

$$\langle A(x), y \rangle_{H_0^1(Z)} \stackrel{df}{=} \int_Z (\nabla x(z), \nabla y(z))_{\mathbb{R}^N} dz \quad \forall x, y \in H_0^1(Z)$$

and  $h \in L^{r'}(Z)$  (where  $\frac{1}{r} + \frac{1}{r'} = 1$ ), such that  $h(z) \in \partial j(z, x(z))$  for almost all  $z \in Z$  (see p. 83 of [10]). So for any  $v_1, v_2 \in \hat{H}$  and  $x_1^* \in \partial\hat{\varphi}_u(v_1), x_2^* \in \partial\hat{\varphi}_u(v_2)$ , we have

$$x_i^* = p_{\hat{H}^*} A(u + v_i) - \lambda_k v_i - p_{\hat{H}^*} h_i \quad \text{for } i \in \{1, 2\},$$

where  $h_i \in L^{r'}(Z) \subseteq H^{-1}(Z)$  (recall  $r < 2^*$ ) is such that  $h_i(z) \in \partial j(z, (u + v_i)(z))$  for almost all  $z \in Z$  and  $i \in \{1, 2\}$ . Since  $p_{\hat{H}^*}^* = \hat{i}$ , we have that

$$\begin{aligned} & \langle p_{\hat{H}^*} (A(u + v_1) - A(u + v_2)), v_1 - v_2 \rangle_{\hat{H}} \\ &= \langle A(u + v_1) - A(u + v_2), \hat{i}(v_1) - \hat{i}(v_2) \rangle_{H_0^1(Z)} \\ &= \langle A(u + v_1) - A(u + v_2), v_1 - v_2 \rangle_{H_0^1(Z)} = \|\nabla v_1 - \nabla v_2\|_2^2. \end{aligned}$$

By hypothesis H(j)(v), we have that

$$\frac{h_1(z) - h_2(z)}{v_1(z) - v_2(z)} \leq l(z) \quad \text{a.e. on } \{v_1 \neq v_2\}. \quad (3.7)$$

So, from (3.6) and (3.7), it follows that

$$\begin{aligned} & \langle x_1^* - x_2^*, v_1 - v_2 \rangle_{\hat{H}} \\ &= \|\nabla v_1 - \nabla v_2\|_2^2 - \lambda_k \|v_1 - v_2\|_2^2 - \int_Z (h_1(z) - h_2(z))(v_1(z) - v_2(z)) dz \\ &\geq \|\nabla v_1 - \nabla v_2\|_2^2 - \lambda_k \|v_1 - v_2\|_2^2 - \int_Z l(z) |v_1(z) - v_2(z)|^2 dz. \end{aligned}$$



By hypothesis H(j)(v), we know that

$$l(z) \leq \lambda_{k+1} - \lambda_k \quad \text{for a.a. } z \in Z,$$

with strict inequality on a set of positive measure. So we can apply Lemma 3.2 (with  $\beta(z) = l(z) + \lambda_k$ ) and obtain  $\xi_1 > 0$ , such that

$$\langle x_1^* - x_2^*, v_1 - v_2 \rangle_{\hat{H}} \geq \xi_1 \|\nabla v_1 - \nabla v_2\|_2^2.$$

So the multifunction  $v \mapsto \partial \hat{\varphi}_u(v)$  is strongly monotone in the dual pair  $(\hat{H}, \hat{H}^*)$ . Hence the function  $\hat{H} \ni v \mapsto \hat{\varphi}_u(v) \in \mathbb{R}$  is strongly convex, i.e. the function  $\hat{H} \ni v \mapsto \hat{\varphi}_u(v) - \frac{\xi_1}{2} \|v\|_{H_0^1(Z)}^2 \in \mathbb{R}$  is convex (see p. 37 of [10]).

Let  $v \in \hat{H}$ ,  $x^* \in \partial \hat{\varphi}_u(v)$  and  $y^* \in \partial \hat{\varphi}_u(0)$ . From the previous considerations, we have

$$\begin{aligned} \langle x^*, v \rangle_{\hat{H}} &= \langle x^* - y^*, v \rangle_{\hat{H}} + \langle y^*, v \rangle_{\hat{H}} \\ &\geq \xi_1 \|\nabla v\|_2^2 - \xi_2 \|y^*\|_{H^{-1}(Z)} \|\nabla v\|_2, \end{aligned}$$

for some  $\xi_2 > 0$ , so the multifunction  $v \mapsto \partial \varphi_u(v)$  is coercive.

The multifunction  $v \mapsto \partial \hat{\varphi}_u(v)$  is maximal monotone (since  $\hat{\varphi}_u$  is convex). But a maximal monotone, coercive operator is surjective (see p. 322 of [17]). Thus, we can find  $v_0 \in \hat{H}$ , such that

$$0 \in \partial \hat{\varphi}_u(v_0) \quad \text{and} \quad \inf_{v \in \hat{H}} \varphi(u + v) = \varphi(u + v_0).$$

Because of the strong convexity of  $\hat{\varphi}_u$ , we infer that the minimizer  $v_0 \in \hat{H}$  is unique.

Therefore we can define a map  $\vartheta: \bar{H}_0 \rightarrow \hat{H}$  which to each fixed  $u \in \bar{H}_0$  assigns the unique solution  $v_0 \in \hat{H}$  of the minimization problem (3.1). Then from (3.5), we have

$$0 \in \partial \hat{\varphi}_u(\vartheta(u)) \subseteq p_{\hat{H}^*} \partial \varphi(u + \vartheta(u))$$

and

$$\inf_{v \in \hat{H}} \varphi(u + v) = \varphi(u + \vartheta(u)).$$

Finally, we have to show that  $\vartheta$  is continuous. To this end suppose that  $u_n \rightarrow u$  in  $\bar{H}_0$ . If  $v_n \rightarrow v$  in  $\hat{H}$ , we have  $\hat{\varphi}_{u_n}(v_n) \rightarrow \hat{\varphi}_u(v)$  (in fact it is easy to see that  $\hat{\varphi}_{u_n} \rightarrow \hat{\varphi}_u$  in  $C(\hat{H})$ ). On the other hand, if  $v_n \xrightarrow{w} v$  in  $\hat{H}$ , by virtue of the weak lower semicontinuity of the norm in a Banach space and the compactness of the embedding  $H_0^1(Z) \subseteq L^2(Z)$ , we have

$$\hat{\varphi}_u(v) \leq \liminf_{n \rightarrow +\infty} \hat{\varphi}_{u_n}(v_n)$$

(recall the definition of  $\varphi$ ). It follows that  $\hat{\varphi}_{u_n} \xrightarrow{M} \hat{\varphi}_u$  and so by virtue of Theorem 5.6.9 of p. 766 of [17], we have that

$$\text{Gr } \partial \hat{\varphi}_{u_n} \xrightarrow{K_{s,s}} \text{Gr } \partial \hat{\varphi}_u \quad \text{as } n \rightarrow +\infty.$$

Because  $0 \in \partial \hat{\varphi}_u(\vartheta(u))$ , we can find  $v_n^* \in \partial \hat{\varphi}_{v_n}(v_n)$ , for  $n \geq 1$ , such that  $v_n \rightarrow \vartheta(u)$  in  $\hat{H}$  and  $v_n^* \rightarrow 0$  in  $\hat{H}^*$ . Recall that  $0 \in \partial \hat{\varphi}_{u_n}(\vartheta(u_n))$ , from the strong monotonicity of  $\partial \hat{\varphi}_{u_n}$ , we have that

$$\langle v_n^*, v_n - \vartheta(u_n) \rangle_{\hat{H}} \geq \xi_1 \|v_n - \vartheta(u_n)\|_{H_0^1(Z)}^2,$$

so

$$\begin{aligned} & \| \vartheta(u) - \vartheta(u_n) \|_{H_0^1(Z)} \\ & \leq \frac{1}{\xi_1} \| v_n^* \|_{H^{-1}(Z)} + \| v_n - \vartheta(u) \|_{H_0^1(Z)} \longrightarrow 0 \quad \text{as } n \rightarrow +\infty \end{aligned}$$

and thus we have proved that  $\vartheta$  is continuous.  $\square$

Using Proposition 3.3, we can define the map  $\bar{\varphi}: \bar{H}_0 \rightarrow \mathbb{R}$ , by

$$\bar{\varphi}(u) = \varphi(u + \vartheta(u)) \quad \forall u \in \bar{H}_0. \quad (3.8)$$

Note that from the definition of  $\vartheta$ , for all  $u, h \in \bar{H}_0$ , we have that

$$\begin{aligned} \bar{\varphi}(u+h) - \bar{\varphi}(u) &= \varphi(u+h + \vartheta(u+h)) - \varphi(u + \vartheta(u)) \\ &\leq \varphi(u+h + \vartheta(u)) - \varphi(u + \vartheta(u)). \end{aligned}$$

Similarly from the definition of  $\vartheta$ , for all  $u, h \in \bar{H}_0$ , we obtain

$$\begin{aligned} \bar{\varphi}(u) - \bar{\varphi}(u+h) &= \varphi(u + \vartheta(u)) - \varphi(u+h + \vartheta(u+h)) \\ &\leq \varphi(u + \vartheta(u+h)) - \varphi(u+h + \vartheta(u+h)). \end{aligned}$$

It follows that  $\bar{\varphi}$  is locally Lipschitz (since  $\varphi$  is).

Now we will show that

$$\partial \bar{\varphi}(u) \subseteq p_{\bar{H}_0^*} \partial \varphi(u + \vartheta(u)) \quad \forall u \in \bar{H}_0. \quad (3.9)$$

First for all  $u, h \in \bar{H}_0$ , we have

$$\begin{aligned} \bar{\varphi}^0(u; h) &= \limsup_{\substack{u' \rightarrow u \\ t \searrow 0}} \frac{\bar{\varphi}(u' + th) - \bar{\varphi}(u')}{t} \\ &= \limsup_{\substack{u' \rightarrow u \\ t \searrow 0}} \frac{\varphi(u' + th + \vartheta(u' + th)) - \varphi(u' + \vartheta(u'))}{t} \\ &\leq \limsup_{\substack{u' \rightarrow u \\ t \searrow 0}} \frac{\varphi(u' + th + \vartheta(u')) - \varphi(u' + \vartheta(u'))}{t} \\ &\leq \varphi^0(u + \vartheta(u); h). \end{aligned}$$

Denoting by  $\bar{i}_0: \bar{H}_0 \rightarrow H_0^1(Z)$  the inclusion map and noting that  $\bar{i}_0^* = p_{\bar{H}_0^*}$ , for all  $u, h \in \bar{H}_0$ , we have that

$$\begin{aligned} \bar{\varphi}^0(u; h) &\leq \varphi^0(u + \vartheta(u); \bar{i}_0(h)) \\ &= \sup_{u^* \in \partial\varphi(u + \vartheta(u))} \langle u^*, \bar{i}_0(h) \rangle_{H_0^1(Z)} = \sup_{u^* \in \partial\varphi(u + \vartheta(u))} \langle p_{\bar{H}_0^*}(u^*), h \rangle_{\bar{H}_0}. \end{aligned}$$

Suppose that  $u_0^* \in \partial\bar{\varphi}(u)$ . From the definition of the Clarke directional derivative, we have

$$\langle u_0^*, h \rangle_{\bar{H}_0} \leq \bar{\varphi}^0(u; h) \quad \forall h \in \bar{H}_0,$$

so

$$\langle u_0^*, h \rangle_{\bar{H}_0} \leq \sup_{u^* \in \partial\varphi(u + \vartheta(u))} \langle p_{\bar{H}_0^*}(u^*), h \rangle_{\bar{H}_0} \quad \forall h \in \bar{H}_0$$

and thus

$$u_0^* \in p_{\bar{H}_0^*} \partial\varphi(u + \vartheta(u)).$$

Therefore, we obtain (3.9).

Next let  $\psi = -\bar{\varphi}$ . Then  $\psi$  is locally Lipschitz on the finite dimensional space  $\bar{H}_0$ . In the next section working with  $\psi$  and using Theorem 2.1, we prove a multiplicity theorem for problem (HVI).

#### 4. Existence of multiple solutions

As  $m \leq k$  are fixed (see hypothesis H(j)(vi)), let us put

$$Y \stackrel{df}{=} \bigoplus_{i=1}^{m-1} E(\lambda_i) \quad \text{and} \quad V \stackrel{df}{=} \bigoplus_{i=m}^k E(\lambda_i).$$

We have

$$\bar{H}_0 = \bar{H} \oplus E(\lambda_k) = Y \oplus V.$$

The next proposition shows that  $\psi = -\bar{\varphi}$  satisfies the local linking condition (see Theorem 2.1).

##### PROPOSITION 4.1

If hypotheses H(j) hold, then there exists  $\delta > 0$  such that

$$\begin{cases} \psi(u) \leq 0 & \text{if } u \in V, \|u\|_{H_0^1(Z)} \leq \delta \\ \psi(u) \geq 0 & \text{if } u \in Y, \|u\|_{H_0^1(Z)} \leq \delta \end{cases}.$$

*Proof.* Because  $Y$  is finite dimensional, all norms are equivalent and so we can find  $M_1 > 0$ , such that

$$\sup_{z \in Z} |u(z)| \leq M_1 \|u\|_{H_0^1(Z)} \quad \forall u \in Y.$$

Let  $\beta \in L^\infty(Z)$  and  $\delta_0 > 0$  as in hypothesis H(j)(vi). Thus, if  $\delta' \stackrel{df}{=} \frac{\delta_0}{M_1}$ , we have

$$\sup_{z \in Z} |u(z)| \leq \delta_0 \quad \forall u \in Y, \|u\|_{H_0^1(Z)} \leq \delta'.$$

By virtue of hypothesis H(j)(vi) and the definition of  $\vartheta$ , for all  $u \in Y$  with  $\|u\|_{H_0^1(Z)} \leq \delta'$ , we have

$$\begin{aligned} \psi(u) &= -\bar{\varphi}(u) \geq -\frac{1}{2}\|\nabla u\|_2^2 + \frac{\lambda_k}{2}\|u\|_2^2 + \int_Z j(z, u(z))dz \\ &\geq -\frac{1}{2}\|\nabla u\|_2^2 + \frac{\lambda_k}{2}\|u\|_2^2 + \frac{\lambda_{m-1} - \lambda_k}{2}\|u\|_2^2 \\ &= -\frac{1}{2}\|\nabla u\|_2^2 + \frac{\lambda_{m-1}}{2}\|u\|_2^2 \geq 0. \end{aligned}$$

Also for all  $u \in V$ , we have

$$\begin{aligned} \psi(u) &= -\varphi(u + \vartheta(u)) \tag{4.1} \\ &= -\frac{1}{2}\|\nabla(u + \vartheta(u))\|_2^2 + \frac{\lambda_k}{2}\|u + \vartheta(u)\|_2^2 + \int_Z j(z, (u + \vartheta(u))(z))dz. \end{aligned}$$

From hypothesis H(j)(vi), we have that

$$j(z, \zeta) \leq \frac{1}{2}\beta(z)\zeta^2 \quad \text{for a.a. } z \in Z \text{ and all } |\zeta| \leq \delta_0,$$

while by virtue of hypothesis H(j)(iii) and the Lebourg mean value theorem (see p. 41 of [10] or [24]), we have that

$$j(z, \zeta) \leq c_2|\zeta|^\eta \quad \text{for a.a. } z \in Z, \text{ all } |\zeta| > \delta_0,$$

with some  $2 < \eta \leq 2^*$  and  $c_2 > 0$ . So finally we can say that

$$j(z, \zeta) \leq \frac{1}{2}\beta(z)\zeta^2 + c_3|\zeta|^\eta \quad \text{for a.a. } z \in Z, \text{ all } \zeta \in \mathbb{R}$$

with  $c_3 \stackrel{df}{=} c_2 + \frac{1}{2}\|\beta\|_\infty > 0$ . Using this in (4.1), we obtain

$$\begin{aligned} \psi(u) &\leq -\frac{1}{2}\|\nabla(u + \vartheta(u))\|_2^2 + \frac{\lambda_k}{2}\|u + \vartheta(u)\|_2^2 \tag{4.2} \\ &\quad + \frac{1}{2} \int_Z \beta(z)|(u + \vartheta(u))(z)|^2 dz + c_3\|u + \vartheta(u)\|_\eta^\eta \quad \forall u \in V. \end{aligned}$$

Note that

$$\lambda_{m-1} \leq \beta(z) + \lambda_k \leq \lambda_m \quad \text{for almost all } z \in Z,$$

with the second inequality strict on a set of positive measure. Since  $u \in V$  and  $\vartheta(u) \in \hat{H}$ ,  $u + \vartheta(u) \in \hat{H}_{m-1}$  and we can apply Lemma 3.2 with  $\beta + \lambda_k$  to obtain  $\xi_1 > 0$  such that

$$\begin{aligned} &\|\nabla(u + \vartheta(u))\|_2^2 - \int_Z (\beta(z) + \lambda_k)|(u + \vartheta(u))(z)|^2 dz \\ &\geq \xi_1\|\nabla(u + \vartheta(u))\|_2^2 \quad \forall v \in V. \end{aligned}$$

Using this inequality in (4.2), we have

$$\psi(u) \leq -\frac{1}{2}\xi_1 \|\nabla(u + \vartheta(u))\|_2^2 + c_4 \|\nabla(u + \vartheta(u))\|_2^\eta \quad \forall u \in V,$$

for some  $c_4 > 0$ . Here we have used the Sobolev embedding theorem since  $\eta \leq 2^*$  and the Poincaré inequality. Because  $2 < \eta$ ,  $\vartheta(0) = 0$ , we can find  $\delta'' > 0$ , such that

$$\psi(u) \leq 0 \quad \forall u \in V, \|u\|_{H_0^1(Z)} \leq \delta''.$$

Finally let  $\delta \stackrel{df}{=} \min\{\delta', \delta''\}$  to finish the proof of the proposition. □

Since we aim in applying Theorem 2.1, we need to show that  $\psi$  satisfies the nonsmooth C-condition. To establish this for  $\psi$ , we show first that  $\varphi$  satisfies the nonsmooth C-condition.

**PROPOSITION 4.2**

*If hypotheses H(j) hold, then  $\varphi$  satisfies the nonsmooth C-condition.*

*Proof.* Let  $\{x_n\}_{n \geq 1} \subseteq H_0^1(Z)$  be a sequence, such that

$$\varphi(x_n) \longrightarrow c \quad \text{and} \quad (1 + \|x_n\|_{H_0^1(Z)})m_\varphi(x_n) \longrightarrow 0 \text{ as } n \rightarrow +\infty.$$

Since the norm functional is weakly lower semicontinuous and  $\partial\varphi(x_n) \subseteq H^{-1}(Z)$  is weakly compact, from the Weierstrass theorem, we know that there exists  $x_n^* \in \partial\varphi(x_n)$ , such that  $\|x_n^*\|_{H^{-1}(Z)} = m_\varphi(x_n)$  for all  $n \geq 1$ . We know that

$$x_n^* = A(x_n) - \lambda_k x_n - h_n \quad \forall n \geq 1,$$

with  $A \in \mathcal{L}(H_0^1(Z), H^{-1}(Z))$  being the maximal monotone operator defined by

$$\langle A(x), y \rangle_{H_0^1(Z)} \stackrel{df}{=} \int_Z (\nabla x(z), \nabla y(z))_{\mathbb{R}^N} dz \quad \forall x, y \in H_0^1(Z)$$

and  $h_n \in L^{r'}(Z)$  (with  $\frac{1}{r} + \frac{1}{r'} = 1$ ) is such that  $h_n(z) \in \partial j(z, x_n(z))$  for almost all  $z \in Z$  (see p. 80 of [10]). From the choice of the sequence  $\{x_n\}_{n \geq 1} \subseteq H_0^1(Z)$ , we have

$$\begin{aligned} & |\langle x_n^*, x_n \rangle_{H_0^1(Z)} - 2\varphi(x_n) + 2c| \\ & \leq \|x_n\|_{H_0^1(Z)} \|x_n^*\|_{H^{-1}(Z)} + 2|\varphi(x_n) - c| \\ & \leq (1 + \|x_n\|_{H_0^1(Z)})m_\varphi(x_n) + 2|\varphi(x_n) - c| \longrightarrow 0 \text{ as } n \rightarrow +\infty. \end{aligned} \quad (4.3)$$

Note that

$$\langle x_n^*, x_n \rangle_{H_0^1(Z)} = \langle A(x_n), x_n \rangle_{H_0^1(Z)} - \lambda_k \|x_n\|_2^2 - \int_Z h_n(z)x_n(z) dz.$$

Then using (4.3), we obtain

$$\begin{aligned} & \int_Z (h_n(z)x_n(z) - 2j(z, x_n(z))) dz \\ & = 2\varphi(x_n) - \langle x_n^*, x_n \rangle_{H_0^1(Z)} \longrightarrow 2c \text{ as } n \rightarrow +\infty. \end{aligned} \quad (4.4)$$

We claim that  $\{x_n\}_{n \geq 1} \subseteq H_0^1(Z)$  is bounded. Suppose that this is not the case. By passing to a subsequence if necessary, we may assume that  $\|x_n\|_{H_0^1(Z)} \rightarrow +\infty$ . Let  $y_n \stackrel{df}{=} \frac{x_n}{\|x_n\|_{H_0^1(Z)}}$  for  $n \geq 1$ . Since  $\|y_n\|_{H_0^1(Z)} = 1$ , passing to a subsequence if necessary, we may assume that

$$\begin{aligned} y_n &\xrightarrow{w} y \quad \text{in } H_0^1(Z), \\ y_n &\rightarrow y \quad \text{in } L^2(Z), \\ y_n(z) &\rightarrow y(z) \quad \text{for a.a. } z \in Z \end{aligned}$$

and

$$|y_n(z)| \leq k(z) \quad \text{for a.a. } z \in Z \text{ and all } n \geq 1,$$

with some  $k \in L^2(Z)$ . Because of hypothesis H(j)(vii), for a given  $\varepsilon > 0$ , we can find  $M_2 = M_2(\varepsilon) > 0$  such that for almost all  $z \in Z$  and all  $|\zeta| \geq M_2$ , we have

$$j(z, \zeta) \leq \frac{1}{2}(\gamma(z) + \varepsilon)\zeta^2.$$

Also from hypothesis H(j)(iii) and the Lebourg mean value theorem, for almost all  $z \in Z$  and all  $|\zeta| < M_2$ , we have  $|j(z, \zeta)| \leq \xi_2$  for some  $\xi_2 > 0$ . So we can say that for almost all  $z \in Z$  and all  $\zeta \in \mathbb{R}$ , we have

$$j(z, \zeta) \leq \frac{1}{2}(\gamma(z) + \varepsilon)\zeta^2 + \xi_2.$$

Then for every  $n \geq 1$ , we have

$$\begin{aligned} \frac{\varphi(x_n)}{\|x_n\|_{H_0^1(Z)}^2} &= \frac{1}{2}\|\nabla y_n\|_2^2 - \frac{\lambda_k}{2}\|y_n\|_2^2 - \int_Z \frac{j(z, x_n(z))}{\|x_n\|_{H_0^1(Z)}^2} dz \\ &\geq \frac{c_4}{2} - \frac{\lambda_k}{2}\|y_n\|_2^2 - \frac{1}{2} \int_Z \gamma(z)y_n(z)^2 dz - \frac{\varepsilon}{2}\|y_n\|_2^2 - \frac{\xi_2|Z|}{\|x_n\|_{H_0^1(Z)}^2}, \end{aligned}$$

for some  $c_4 > 0$  (by the Poincaré inequality). Passing to the limit as  $n \rightarrow +\infty$ , we obtain

$$0 \geq \frac{1}{2}(c_4 - (\lambda_k + \|\gamma\|_\infty + \varepsilon)\|y\|_2^2)$$

and thus  $y \neq 0$ .

Because of hypothesis H(j)(iv), we can find  $M_3 > 0$  such that for almost all  $z \in Z$ , all  $|\zeta| \geq M_3$  and all  $u \in \partial j(z, \zeta)$ , we have

$$u\zeta - 2j(z, \zeta) \leq -1.$$

On the other hand, as above, for almost all  $z \in Z$  and all  $|\zeta| < M_3$ , we have  $|j(z, \zeta)| \leq \xi_3$  for some  $\xi_3 > 0$ . Thus using hypothesis H(j)(iii), we see that for almost all  $z \in Z$ , all  $|\zeta| < M_3$  and all  $u \in \partial j(z, \zeta)$ , we have

$$|u\zeta - 2j(z, \zeta)| \leq \xi_4,$$

for some  $\xi_4 > 0$ . Thus finally, for almost all  $z \in Z$ , all  $\zeta \in \mathbb{R}$  and all  $u \in \partial j(z, \zeta)$ , we have

$$u\zeta - 2j(z, \zeta) \leq \xi_4. \tag{4.5}$$

Let  $C \stackrel{df}{=} \{z \in Z: y(z) \neq 0\}$ . Evidently  $|C|_N > 0$  (with  $|\cdot|_N$  being the Lebesgue measure on  $\mathbb{R}^N$ ) and for all  $z \in C$ , we have that

$$|x_n(z)| \longrightarrow +\infty \quad \text{as } n \rightarrow +\infty. \tag{4.6}$$

From (4.6) and by virtue of Lemma 1 of [35], for a given  $\delta \in (0, |C|_N)$ , we can find a measurable subset  $C_1 \subseteq C$ , such that  $|C \setminus C_1|_N < \delta$  and  $|x_n(z)| \longrightarrow +\infty$  uniformly for all  $z \in C_1$ . Using hypothesis H(j)(iv), we infer that

$$\int_{C_1} (h_n(z)x_n(z) - 2j(z, x_n(z)))dz \longrightarrow -\infty \quad \text{as } n \rightarrow +\infty.$$

From (4.5), we have

$$h_n(z)x_n(z) - 2j(z, x_n(z)) \leq \xi_4 \quad \text{for a.a. } z \in Z \setminus C_1,$$

and so

$$\begin{aligned} & \int_Z (h_n(z)x_n(z) - 2j(z, x_n(z)))dz \\ &= \int_{C_1} (h_n(z)x_n(z) - 2j(z, x_n(z)))dz + \xi_4|(Z \setminus C_1)_c|_N \longrightarrow -\infty, \end{aligned}$$

as  $n \rightarrow +\infty$ , which contradicts (4.4). This proves the boundedness of  $\{x_n\} \subseteq H_0^1(Z)$ . So, passing to a subsequence if necessary, we may assume that

$$\begin{aligned} x_n &\xrightarrow{w} x_0 \quad \text{in } H_0^1(Z) \\ x_n &\longrightarrow x_0 \quad \text{in } L^2(Z), \end{aligned}$$

for some  $x_0 \in H_0^1(Z)$ . From the choice of the sequence  $\{x_n\}_{n \geq 1} \subseteq H_0^1(Z)$ , we have

$$\begin{aligned} & | \langle x_n^*, x_n - x_0 \rangle_{H_0^1(Z)} | \\ &= | \langle A(x_n), x_n - x_0 \rangle_{H_0^1(Z)} - \lambda_k \langle x_n^*, x_n - x_0 \rangle_{H_0^1(Z)} \\ &\quad - \int_Z h_n(z)(x_n(z) - x_0(z))dz | \\ &\leq m_\varphi(x_n) \|x_n - x_0\|_{H_0^1(Z)} \longrightarrow 0 \quad \text{as } n \rightarrow +\infty. \end{aligned}$$

But

$$\lambda_k \langle x_n^*, x_n - x_0 \rangle_{H_0^1(Z)} + \int_Z h_n(z)(x_n(z) - x_0(z))dz \longrightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

So it follows that

$$\lim_{n \rightarrow +\infty} \langle A(x_n), x_n - x_0 \rangle_{H_0^1(Z)} = 0.$$

From the maximal monotonicity of  $A$ , we have

$$\langle A(x_n), x_n \rangle_{H_0^1(Z)} \longrightarrow \langle A(x_0), x_0 \rangle_{H_0^1(Z)}$$

and so

$$\|\nabla x_n\|_2 \longrightarrow \|\nabla x_0\|_2.$$

Because  $\nabla x_n \xrightarrow{w} \nabla x_0$  in  $L^2(Z; \mathbb{R}^N)$ , from the Kadec–Klee property of Hilbert spaces, we conclude that  $\nabla x_n \longrightarrow \nabla x_0$  in  $L^2(Z; \mathbb{R}^N)$ , hence  $x_n \longrightarrow x_0$  in  $H_0^1(Z)$  as  $n \rightarrow +\infty$ .  $\square$

Using this proposition, we can establish that  $\bar{\varphi}$  satisfies the nonsmooth C-condition.

#### PROPOSITION 4.3

*If hypotheses H(j) hold, then  $\bar{\varphi}$  satisfies the nonsmooth C-condition.*

*Proof.* Let  $c \in \mathbb{R}$  and let  $\{u_n\}_{n \geq 1} \subseteq \bar{H}_0$  be a sequence, such that

$$\bar{\varphi}(u_n) \longrightarrow c \quad \text{and} \quad (1 + \|u_n\|_{H_0^1(Z)})m_{\bar{\varphi}}(u_n) \longrightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

As before we can find  $\bar{v}_n^* \in \partial \bar{\varphi}(u_n)$ , such that  $m_{\bar{\varphi}}(u_n) = \|\bar{v}_n^*\|_{H^{-1}(Z)}$ . By virtue of (3.9), we can find  $v_n^* \in \partial \varphi(u_n + \vartheta(u_n))$ , such that

$$\bar{v}_n^* = p_{\bar{H}_0} v_n^* \quad \forall n \geq 1.$$

Recall that, by Proposition 3.3,  $0 \in p_{\hat{H}^*} \partial \varphi(u_n + \vartheta(u_n))$ , for  $n \geq 1$ . Then using hypothesis H(j)(v) we have  $m_{\varphi}(u_n + \vartheta(u_n)) \leq m_{\bar{\varphi}}(u_n)$ . Therefore,

$$(1 + \|u_n\|_{H_0^1(Z)})\|v_n^*\|_{H_0^1(Z)} \longrightarrow 0 \quad \text{with } v_n^* \in \partial \varphi(u_n + \vartheta(u_n)).$$

But from Proposition 4.2, we know that  $\varphi$  satisfies the nonsmooth C-condition. So we can extract a subsequence of  $\{u_n\}_{n \geq 1}$ , which is strongly convergent. This proves the proposition.  $\square$

#### PROPOSITION 4.4

*If hypotheses H(j) hold, then  $\psi$  is bounded below.*

*Proof.* We show that  $-(\varphi|_{\bar{H}_0})$  is bounded below. Then because  $-\psi = \bar{\varphi} \leq (\varphi|_{\bar{H}_0})$ , we can conclude that  $\psi$  is bounded below. To this end we proceed by contradiction. Suppose that  $-(\varphi|_{\bar{H}_0})$  is not bounded below. Then we can find  $x_n \in \bar{H}_0$ , such that

$$\varphi(x_n) \geq n \quad \forall n \geq 1 \tag{4.7}$$

and

$$\|x_n\|_{H_0^1(Z)} \longrightarrow +\infty.$$



By virtue of hypothesis H(j)(vii), for a given  $\varepsilon > 0$ , we can find  $M_4 = M_4(\varepsilon) > 0$ , such that for almost all  $z \in Z$  and all  $|\zeta| \geq M_4$ , we have

$$-\frac{\varepsilon}{2}\zeta^2 \leq j(x, \zeta).$$

On the other hand, as before via Lebourg mean value theorem, we can find  $\xi_5 > 0$ , such that for almost all  $z \in Z$  and all  $|\zeta| \leq M_4$ , we have

$$|j(z, \zeta)| \leq \xi_5.$$

So finally we see that

$$-\frac{\varepsilon}{2}\zeta^2 - \xi_5 \leq j(z, \zeta) \quad \text{for a.a. } z \in Z, \text{ all } \zeta \in \mathbb{R}. \tag{4.8}$$

Let

$$x_n = \bar{x}_n + \bar{\bar{x}}_n, \quad \text{with } \bar{x}_n \in \bar{H}, \bar{\bar{x}}_n \in E(\lambda_k), \quad n \geq 1.$$

First assume that

$$\frac{\|\nabla \bar{x}_n\|_2}{\|\nabla x_n\|_2} \longrightarrow \mu \neq 0 \quad \text{as } n \rightarrow +\infty. \tag{4.9}$$

Exploiting the orthogonality relations, the fact that  $\|\nabla \bar{x}_n\|_2^2 = \lambda_k \|\bar{x}_n\|_2^2$  and estimate (4.8), we have

$$\begin{aligned} \varphi(x_n) &= \frac{1}{2} \|\nabla \bar{x}_n\|_2^2 - \frac{\lambda_k}{2} \|\bar{x}_n\|_2^2 - \int_Z j(z, x_n(z)) dz \\ &\leq \frac{1}{2} \|\nabla \bar{x}_n\|_2^2 - \frac{\lambda_k}{2} \|\bar{x}_n\|_2^2 + \frac{\varepsilon}{2} \|x_n\|_2^2 + \xi_5 |Z|_N. \end{aligned}$$

Thus from the variational characterization of the eigenvalues we get

$$\begin{aligned} \varphi(x_n) &\leq \frac{1}{2} \left( 1 - \frac{\lambda_k}{\lambda_{k-1}} \right) \|\nabla \bar{x}_n\|_2^2 + \frac{\varepsilon}{2} \|x_n\|_2^2 + \xi_5 |Z|_N \\ &\leq \frac{1}{2} \|\nabla x_n\|_2^2 \left( \left( 1 - \frac{\lambda_k}{\lambda_{k-1}} \right) \frac{\|\nabla \bar{x}_n\|_2^2}{\|\nabla x_n\|_2^2} + \frac{\varepsilon}{\lambda_k} \right) + \xi_5 |Z|_N. \end{aligned} \tag{4.10}$$

Since by hypothesis, we have that  $\|\nabla x_n\|_2 \longrightarrow +\infty$ , so from (4.9) and recalling that  $\lambda_{k-1} < \lambda_k$ , by passing to the limit as  $n \rightarrow +\infty$  in (4.10), we see that  $\varphi(x_n) \longrightarrow -\infty$  as  $n \rightarrow +\infty$ , a contradiction to (4.7).

Next assume that

$$\frac{\|\nabla \bar{x}_n\|_2}{\|\nabla x_n\|_2} \longrightarrow 0 \quad \text{as } n \rightarrow +\infty. \tag{4.11}$$

By virtue of hypothesis H(j)(iv), for a given  $\eta > 0$ , we can find  $M_5 = M_5(\eta) > 0$ , such that for almost all  $z \in Z$ , all  $|\zeta| \geq M_5$  and all  $u \in \partial j(z, \zeta)$ , we have

$$u\zeta - 2j(z, \zeta) \leq -\eta. \tag{4.12}$$

From p. 48 of [10], we know that for almost all  $z \in Z$  and all  $\zeta > 0$ , the function  $\zeta \mapsto \frac{j(z, \zeta)}{\zeta^2}$  is locally Lipschitz and we have that

$$\begin{aligned} \partial \left( \frac{j(z, \zeta)}{\zeta^2} \right) &= \frac{\zeta^2 \partial j(z, \zeta) - 2\zeta j(z, \zeta)}{\zeta^4} \\ &= \frac{\zeta \partial j(z, \zeta) - 2j(z, \zeta)}{\zeta^3}. \end{aligned}$$

Using (4.12), we see that for almost all  $z \in Z$ , all  $\zeta \geq M_5$  and all  $v \in \partial \left( \frac{j(z, \zeta)}{\zeta^2} \right)$ , we have

$$v \leq -\frac{\eta}{\zeta^3}.$$

Since for all  $z \in Z \setminus E$ , with  $|E|_N = 0$ , the function  $\zeta \mapsto \frac{j(z, \zeta)}{\zeta^2}$  is locally Lipschitz on  $[M_5, +\infty)$ , it is differentiable at every  $x \in [M_5, +\infty) \setminus L(z)$  with some  $|L(z)|_1 = 0$ . We set

$$\xi_0(z, \zeta) \stackrel{df}{=} \begin{cases} \frac{d}{dz} \left( \frac{j(z, \zeta)}{\zeta^2} \right), & \text{if } x \in [M_5, +\infty) \setminus L(z), \\ 0, & \text{if } x \in L(z). \end{cases}$$

For all  $z \in Z \setminus E$  and all  $\zeta \in [M_5, +\infty) \setminus L(z)$ , we have  $\xi_0(z, \zeta) \in \partial \left( \frac{j(z, \zeta)}{\zeta^2} \right)$  and so

$$\xi_0(z, \zeta) \leq -\frac{\eta}{\zeta^3}.$$

Let  $\tau, \bar{\tau} \in [M_5, +\infty)$ , with  $\tau \leq \bar{\tau}$ . Integrating the above inequality with respect to  $\zeta \in \mathbb{R}$  on the interval  $[\tau, \bar{\tau}]$ , we obtain

$$\frac{j(z, \bar{\tau})}{\bar{\tau}^2} - \frac{j(z, \tau)}{\tau^2} \leq -\frac{\eta}{2} \left( \frac{1}{\tau^2} - \frac{1}{\bar{\tau}^2} \right).$$

Let  $\bar{\tau} \rightarrow +\infty$  and using hypothesis H(j)(vii), we obtain

$$\frac{j(z, \tau)}{\tau^2} \geq \frac{\eta}{2\tau^2}$$

and thus, for almost all  $z \in Z$  and all  $\tau \geq M_5$ , we have that

$$j(z, \tau) \geq \frac{\eta}{2}.$$

Since  $\eta > 0$  was arbitrary, it follows that  $j(z, \tau) \rightarrow +\infty$  as  $\tau \rightarrow +\infty$  uniformly for almost all  $z \in Z$ . Similarly we show that  $j(z, \tau) \rightarrow +\infty$ ,  $\tau \rightarrow -\infty$  uniformly for almost all  $z \in Z$ . Therefore

$$j(z, \zeta) \rightarrow +\infty \quad \text{as } |\zeta| \rightarrow +\infty \text{ uniformly for a.a. } z \in Z. \quad (4.13)$$

From [1] (see the proof of Lemma 3.2) or [33], we have that for a given  $\delta > 0$  we can find  $\xi_\delta > 0$ , such that

$$\{|z \in Z: |v(z)| < \xi_\delta \|v\|_{H_0^1(Z)}\}_N < \delta \quad \forall v \in E(\lambda_k).$$

Let us set

$$C_n \stackrel{df}{=} \{z \in Z: |\bar{x}_n(z)| \geq \xi_\delta \|\bar{x}_n\|_{H_0^1(Z)}\} \quad \forall n \geq 1.$$

We have that  $|Z \setminus C_n|_N < \delta$ .

Let us establish that the sequence  $\{\bar{x}_n\}_{n \geq 1}$  is bounded in  $H_0^1(Z)$ . Arguing by contradiction, assume that along a relabeled subsequence we have  $\|\bar{x}_n\|_{H_0^1(Z)} \rightarrow +\infty$ . Recall that

$$\varphi(x_n) \leq \frac{1}{2} \left(1 - \frac{\lambda_k}{\lambda_{k-1}}\right) \|\nabla \bar{x}_n\|_2^2 - \int_Z j(z, x_n(z)) dz \quad \forall n \geq 1.$$

Using (4.13) we find a constant  $\alpha > 0$  such that  $j(z, \zeta) \geq 0$  a.a.  $z \in Z, \forall |\zeta| > \alpha$ . We see that

$$\begin{aligned} - \int_Z j(z, x_n(z)) dz &= - \int_{\{z \in Z: |x_n(z)| > \alpha\}} j(z, x_n(z)) dz \\ &\quad - \int_{\{z \in Z: |x_n(z)| \leq \alpha\}} j(z, x_n(z)) dz \\ &\leq - \int_{\{z \in Z: |x_n(z)| \leq \alpha\}} j(z, x_n(z)) dz \\ &\leq k_0 |Z|_N, \end{aligned}$$

where  $k_0 > 0$  is a constant. In writing the last inequality above we made use of assumption H(j)(iii) and Lebourg's mean value theorem. Then we obtain the estimate

$$\varphi(x_n) \leq \frac{1}{2} \left(1 - \frac{\lambda_k}{\lambda_{k-1}}\right) \|\nabla \bar{x}_n\|_2^2 + k_0 |Z|_N.$$

Since  $\lambda_{k-1} < \lambda_k$  and we supposed that  $\|\bar{x}_n\|_{H_0^1(Z)} \rightarrow +\infty$ , we arrive at the conclusion that  $\varphi(x_n) \rightarrow -\infty$  as  $n \rightarrow +\infty$ . This contradicts relation (4.7), and thus the sequence  $\{\bar{x}_n\}_{n \geq 1}$  is bounded in  $H_0^1(Z)$ .

Because  $\bar{x}_n \in \bar{H}$  and the space  $\bar{H}$  is finite dimensional, it follows from the boundedness of the sequence  $\{\bar{x}_n\}_{n \geq 1} \subseteq H_0^1(Z)$  that we can find  $c_5 > 0$  such that

$$|\bar{x}_n(z)| \leq c_5 \quad \forall z \in Z, n \geq 1.$$

From (4.13), we know that for a given  $\eta_1 > 0$  we can find  $M_6 = M_6(\eta_1) > 0$  such that

$$j(z, \zeta) \geq \eta_1 \quad \text{for a.a. } z \in Z, \text{ all } |\zeta| \geq M_6.$$

Let

$$D_n \stackrel{df}{=} \{z \in Z: |x_n(z)| \geq M_6\} \quad \forall n \geq 1.$$

If  $z_0 \in C_n$ , then

$$|x_n(z_0)| \geq |\bar{x}_n(z_0)| - |\bar{x}_n(z_0)| \geq \xi_\delta \|\bar{x}_n\|_{H_0^1(Z)} - c_5.$$

Because of (4.11), we must have that  $\|\bar{x}_n\|_{H_0^1(Z)} \rightarrow +\infty$  as  $n \rightarrow +\infty$ . So there exists  $n_0 \geq 1$  large enough such that

$$\xi_\delta \|\bar{x}_n\|_{H_0^1(Z)} - c_5 \geq M_6 \quad \forall n \geq n_0$$

and so  $z_0 \in D_n$  for  $n \geq n_0$ , i.e.

$$C_n \subseteq D_n \quad \forall n \geq n_0.$$

Then using (4.8) and the fact that  $|Z \setminus D_n| < \delta$  (as  $Z \setminus D_n \subseteq Z \setminus C_n$ ), for  $n \geq n_0$ , we have that

$$\begin{aligned} \int_Z j(z, x_n(z)) dz &= \int_{D_n} j(z, x_n(z)) dz + \int_{Z \setminus D_n} j(z, x_n(z)) dz \\ &\geq \eta_1 |D_n|_N - \left( \frac{\varepsilon}{2} M_6^2 + \xi_5 \right) |Z \setminus D_n|_N \\ &\geq \eta_1 |D_n|_N - \left( \frac{\varepsilon}{2} M_6^2 + \xi_5 \right) \delta \\ &\geq \eta_1 (|Z|_N - \delta) - \left( \frac{\varepsilon}{2} M_6^2 + \xi_5 \right) \delta, \end{aligned}$$

so

$$\liminf_{n \rightarrow +\infty} \int_Z j(z, x_n(z)) dz \geq \eta_1 (|Z|_N - \delta) - \left( \frac{\varepsilon}{2} M_6^2 + \xi_5 \right) \delta.$$

Since  $\delta > 0$  is arbitrary, we let  $\delta \searrow 0$ . We obtain

$$\liminf_{n \rightarrow +\infty} \int_Z j(z, x_n(z)) dz \geq \eta_1 |Z|_N.$$

Because  $\eta_1 > 0$  is arbitrary, we conclude that

$$\int_Z j(z, x_n(z)) dz \rightarrow +\infty \quad \text{as } n \rightarrow +\infty.$$

From the choice of the sequence  $\{x_n\}_{n \geq 1} \subseteq \bar{H}_0$ , we have

$$n \leq \varphi(x_n) \leq \frac{1}{2} \left( 1 - \frac{\lambda_k}{\lambda_{k-1}} \right) \|\nabla \bar{x}_n\|_2^2 - \int_Z j(z, x_n(z)) dz \rightarrow -\infty,$$

a contradiction. Therefore  $-(\varphi|_{\bar{H}_0})$  is bounded below and so  $\psi$  is bounded below too.  $\square$

Now we are ready for our multiplicity result.

**Theorem 4.5.** *If hypotheses H(j) hold, then problem (HVI) has at least two nontrivial solutions.*

*Proof.* If  $\inf_{\bar{H}_0} \psi = 0$  (remark that  $\psi(0) = 0$ ), then by virtue of Proposition 4.1, all  $x \in V$  with  $\|x\|_V \leq \delta$  are critical points of  $\psi$ .

So assume that  $\inf_{\bar{H}_0} \psi < 0$ . By virtue of Propositions 4.1, 4.3 and 4.4 and since  $\psi(0) = 0$ , we can apply Theorem 2.1 and obtain two nontrivial critical points  $x_1, x_2 \in \bar{H}_0$  of  $\psi$ , i.e.

$$0 \in \partial \psi(x_i) \quad \text{for } i = 1, 2.$$

Hence  $0 \in \partial\bar{\varphi}(x_i)$  for  $i = 1, 2$  and finally from (3.9), we have

$$0 \in p_{\bar{H}_0^*} \partial\varphi(x_i + \vartheta(x_i)) \quad \text{for } i = 1, 2$$

(see the definition of  $\vartheta$  in Proposition 3.3). Recall that  $0 \in p_{\hat{H}^*} \partial\varphi(x_i + \vartheta(x_i))$  for  $i = 1, 2$ , where  $\hat{H} = \bar{H}_0^\perp$ . Therefore we have  $0 \in \partial\varphi(x_i + \vartheta(x_i))$  for  $i = 1, 2$ . Set  $u_i = x_i + \vartheta(x_i)$  for  $i = 1, 2$ . Then  $u_1$  and  $u_2$  are two nontrivial critical points of  $\varphi$ , and thus they are two nontrivial solutions of (HVI).  $\square$

*Remark 4.6.* An example of a nonsmooth locally Lipschitz potential satisfying hypotheses H(j) is the following. For simplicity we drop the  $z$ -dependence (see figure 1 for  $j$  and figure 2 for the Clarke subdifferential  $\partial j$ ).

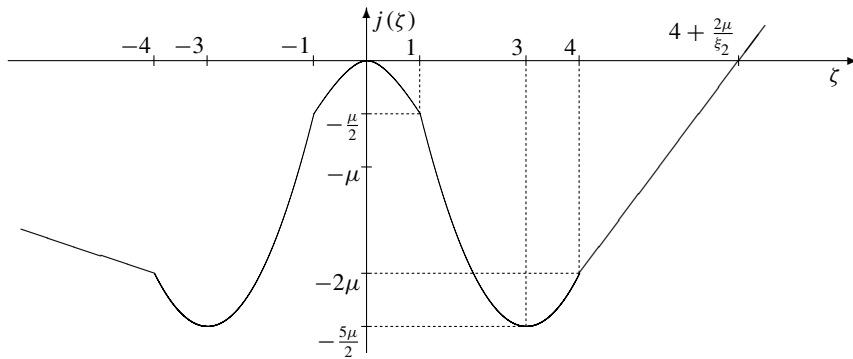


Figure 1. Potential function.

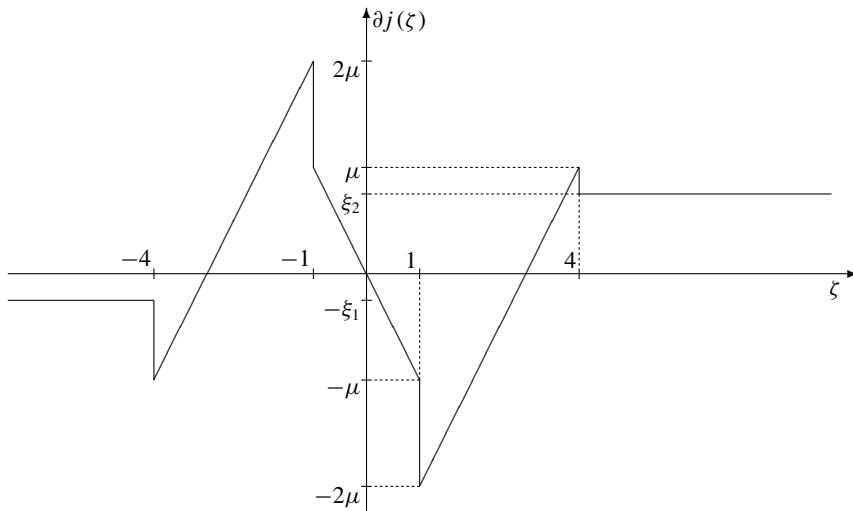


Figure 2. Subdifferential of the potential.

$$j(\zeta) \stackrel{df}{=} \begin{cases} -\xi_1\zeta - 2\mu - 4\xi_1, & \text{if } \zeta < -4 \\ \frac{\mu}{2}\zeta^2 + 3\mu\zeta + 2\mu, & \text{if } -4 \leq \zeta < -1 \\ -\frac{\mu}{2}\zeta^2, & \text{if } -1 \leq \zeta < 1 \\ \frac{\mu}{2}\zeta^2 - 3\mu\zeta + 2\mu, & \text{if } 1 \leq \zeta < 4 \\ \xi_2\zeta - 2\mu - 4\xi_2, & \text{if } 4 \leq \zeta. \end{cases}$$

Here  $\lambda_k - \lambda_m < \mu < \min\{\lambda_k - \lambda_{m-1}, \lambda_{k+1} - \lambda_k\}$ ,  $0 < \xi_1, \xi_2 < \mu$ . All the assumptions (i)–(vii) in H(j) are verified. For instance, assumption H(j)(v) holds with  $l(z) \equiv \mu$ . In this case we have resonance at  $\pm\infty$  since  $\frac{j(\zeta)}{\zeta^2} \rightarrow 0$  as  $|\zeta| \rightarrow +\infty$ . Another possibility is the function  $j(x) = \max\left\{\frac{\xi}{2}x^2 + c|x|, \frac{\xi}{2}|x|\right\}$  with  $\xi < \lambda_{k+1} - \lambda_k$ ,  $c \leq \frac{\xi}{2}$ .

### Acknowledgement

The authors wish to thank the referee for his constructive remarks. The first author (LG) is an award holder of the NATO Science Fellowship Programme, which was spent in the National Technical University of Athens.

### References

- [1] Bartolo P, Benci V and Fortunato D, Abstract critical point theorems and applications to some nonlinear problems with ‘strong’ resonance at infinity, *Nonlin. Anal.* **7** (1983) 981–1012
- [2] Berestycki H and de Figueiredo D, Double resonance in semilinear elliptic problems, *Comm. Partial Diff. Eqns* **6** (1981) 91–120
- [3] Brezis H and Nirenberg L, Remarks on finding critical points, *Comm. Pure Appl. Math.* **44** (1991) 939–963
- [4] Cac N P, On an elliptic boundary value problem at double resonance, *J. Math. Anal. Appl.* **132** (1988) 473–483
- [5] Castro A and Cossio J, Multiple solutions for a nonlinear Dirichlet problem, *SIAM J. Math. Anal.* **25** (1994) 1554–1561
- [6] Castro A, Cossio J and Neuberger J, On multiple solutions of a nonlinear Dirichlet problem, *Nonlin. Anal.* **30** (1997) 3657–3662
- [7] Castro A and Lazer A, Critical point theory and the number of solutions of a nonlinear Dirichlet problem, *Annali di Mat. Pura ed Appl.* **70** (1979) 113–137
- [8] Cerami G, Un criterio di esistenza per i punti critici su varietà illimitate, *Istit. Lombardo Accad. Sci. Lett. Rend. A* **112** (1978) 332–336
- [9] Chang K C, Variational methods for nondifferentiable functionals and their applications to partial differential equations, *J. Math. Anal. Appl.* **80** (1981) 102–129
- [10] Clarke F H, Optimization and nonsmooth analysis (New York: Wiley) (1983)
- [11] Costa D and Silva E A, On a class of resonant problems at higher eigenvalues, *Diff. Int. Eqns* **8** (1995) 663–671
- [12] Gasiński L and Papageorgiou N S, Solutions and multiple solutions for quasilinear hemivariational inequalities at resonance, *Proc. R. Soc. Edinburgh* **131A** (2001) 1091–1111
- [13] Gasiński L and Papageorgiou N S, An existence theorem for nonlinear hemivariational inequalities at resonance, *Bull. Australian Math. Soc.* **63** (2001) 1–14

- [14] Gasiński L and Papageorgiou N S, Multiple solutions for semilinear hemivariational inequalities at resonance, *Publ. Math. Debrecen* **59** (2001) 121–146
- [15] Goeleven D, Motreanu D and Panagiotopoulos P D, Eigenvalue problems for variational–hemivariational inequalities at resonance, *Nonlin. Anal.* **33** (1998) 161–180
- [16] Hirano N and Nishimura T, Multiplicity results for semilinear elliptic problems at resonance with jumping nonlinearities, *J. Math. Anal. Appl.* **180** (1993) 566–586
- [17] Hu S and Papageorgiou N S, Handbook of multivalued analysis, volume I: Theory (The Netherlands: Kluwer, Dordrecht) (1997)
- [18] Hu S and Papageorgiou N S, Handbook of multivalued analysis, volume II: Applications (The Netherlands: Kluwer, Dordrecht) (2000)
- [19] Iannacci R and Nkashama M N, Nonlinear elliptic partial differential equations at resonance: Higher eigenvalues, *Nonlin. Anal.* **25** (1995) 455–471
- [20] Jost J, Post-modern analysis (Berlin: Springer-Verlag) (1998)
- [21] Kandilakis D, Kourogenis N and Papageorgiou N S, Two nontrivial critical points for nonsmooth functionals via local linking and applications, *J. Global Optimization* **34** (2006) 219–244
- [22] Kourogenis N and Papageorgiou N S, Nonsmooth critical point theory and nonlinear elliptic equations at resonance, *J. Austr. Math. Soc.* **69A** (2000) 245–271
- [23] Landesman E, Robinson S and Rumbos A, Multiple solutions of semilinear elliptic problems at resonance, *Nonlin. Anal.* **24** (1995) 1049–1059
- [24] Lebourg G, Valeur moyenne pour gradient généralisé, *CRAS Paris* **281** (1975) 795–797
- [25] Motreanu D and Panagiotopoulos P D, A minimax approach to the eigenvalue problem of hemivariational inequalities and applications, *Applicable Anal.* **58** (1995) 53–76
- [26] Motreanu D and Panagiotopoulos P D, On the eigenvalue problem for hemivariational inequalities: Existence and multiplicity of solutions, *J. Math. Anal. Appl.* **197** (1996) 75–89
- [27] Naniewicz Z and Panagiotopoulos P D, Mathematical theory of hemivariational inequalities and applications (New York: Marcel-Dekker) (1994)
- [28] Niculescu C and Radulescu V, A saddle point theorem for nonsmooth functionals and problems at resonance, *Annales Acad. Sci. Fennicae* **21** (1996) 117–131
- [29] Radulescu V, Mountain pass theorems for nondifferentiable functions and applications, *Proc. Japan Acad. Sci.* **A69** (1993) 193–198
- [30] Radulescu V and Panagiotopoulos P, Perturbations of hemivariational inequalities with constraints and applications, *J. Global Optim.* **12** (1998) 285–297
- [31] Robinson S, Double resonance in semilinear elliptic boundary value problem over bounded and unbounded domains, *Nonlin. Anal.* **21** (1993) 407–424
- [32] Su J, Semilinear elliptic boundary value problems with double resonance between two consecutive eigenvalues, *Nonlin. Anal.* **48** (2002) 881–895
- [33] Su J and Tang C, Multiplicity results for semilinear elliptic equations with resonance at higher eigenvalues, *Nonlin. Anal.* **44** (2001) 311–321
- [34] Tang C and Wu X P, Existence and multiplicity of semilinear elliptic equations, *J. Math. Anal. Appl.* **256** (2001) 1–12
- [35] Tang C and Wu X P, Periodic solutions for second order systems with not uniformly coercive potential, *J. Math. Anal. Appl.* **259** (2001) 386–397
- [36] Thews K, A reduction method for some nonlinear Dirichlet problems, *Nonlin. Anal.* **3** (1979) 795–813