

On p -quermassintegral differences function

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Abstract. In this paper we establish Minkowski inequality and Brunn–Minkowski inequality for p -quermassintegral differences of convex bodies. Further, we give Minkowski inequality and Brunn–Minkowski inequality for quermassintegral differences of mixed projection bodies.

Keywords. Quermassintegral difference function; convex body; projection body; the Brunn–Minkowski inequality.

1. Introduction

The well-known classical Brunn–Minkowski inequality can be stated as follows:

If K and L are convex bodies in R^n , then

$$V(K + L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}, \quad (1)$$

with equality if and only if K and L are homothetic.

The Brunn–Minkowski inequality, has in recent decades, dramatically extended its influence in many areas of mathematics. Various applications have surfaced, for example, to probability and multivariate statistics, shape of crystals, geometric tomography, elliptic partial differential equations, and combinatorics (see [1,5,9,10,17]). Several remarkable analogs have been established in other areas, such as potential theory and algebraic geometry (see [3,4,6,7,12,16]). Reverse forms and similar forms of the inequality are important in the local theory of Banach space (see [17,18,20–22]). An elegant survey on this inequality is provided by Gardner (see [11]).

In fact, let K and L be convex bodies in R^n and let $0 \leq i \leq n-1$. The Brunn–Minkowski inequality for quermassintegral is the following inequality [11]:

$$W_i(K + L)^{1/(n-i)} \geq W_i(K)^{1/(n-i)} + W_i(L)^{1/(n-i)} \quad (2)$$

with equality if and only if K and L are homothetic.

Recently, i -quermassintegral difference function was defined by Leng [13] as follows:

$$Dw_i(K, D) = W_i(K) - W_i(D), \quad K, D \in \mathcal{K}^n, \quad D \subset K \text{ and } 0 \leq i \leq n-1.$$

Moreover, inequality (2) was extended to quermassintegral differences of convex bodies as follows [13]:

Theorem A. *If K, L , and D are convex bodies in R^n , $D \subset K$ and D' is a homothetic copy of D , then*

$$Dw_i(K + L, D + D')^{1/(n-i)} \geq Dw_i(K, D)^{1/(n-i)} + Dw_i(L, D')^{1/(n-i)}, \quad (3)$$

with equality for $0 \leq i < n - 1$ if and only if K and L are homothetic and $(W_i(K), W_i(D)) = \mu(W_i(L), W_i(D'))$, where μ is a constant.

In [8], Firey introduced, for each real $p \geq 1$, new linear combinations of convex bodies: For $K, L \in \mathcal{K}^n$, and $\lambda, \mu \geq 0$ (both are not zero), the Firey combination, $\lambda \cdot K +_p \mu \cdot L$, is a convex body. The main aim of this paper is to establish the Brunn–Minkowski inequality for quermassintegral differences about the Firey combination, which is an extension of the inequality (3).

Theorem 1. *If K, L , and D are convex bodies in R^n , $D \subset K$ and D' is a homothetic copy of D , then for $p \geq 1$,*

$$Dw_i(K +_p L, D +_p D')^{p/(n-i)} \geq Dw_i(K, D)^{p/(n-i)} + Dw_i(L, D')^{p/(n-i)}, \quad (4)$$

with equality for $0 \leq i < n - p$ if and only if K and L are homothetic ($p = 1$) (or are dilates ($p > 1$)) and $(W_i(K), W_i(D)) = \mu(W_i(L), W_i(D'))$, where μ is a constant.

For two convex bodies K and L , an important inequality of mixed volume is the well-known Minkowski inequality

$$V_1(K, L)^n \geq V(K)^{n-1}V(L),$$

with equality if and only if K and L are homothetic.

In 1984, the inequality was extended to compact domains by Zhang [19] as follows:

Theorem B. *If K is a compact domain with piecewise C^1 boundary ∂K , and L is a convex body in R^n , then*

$$V_1(K, L)^n \geq V(K)^{n-1}V(L), \quad (5)$$

with equality if and only if K and L are homothetic.

Recently, inequality (5) was extended to volume differences by Leng [13] as follows:

Theorem C. *Suppose that K and D are compact domains, L is a convex body, and $D \subset K$, $D' \subset L$ and D' is a homothetic copy of D . Then*

$$(V_1(K, L) - V_1(D, D'))^n \geq Dv(K, D)^{n-1}Dv(L, D'), \quad (6)$$

with equality if and only if K and L are homothetic and $(V(K), V(D)) = \mu(V(L), V(D'))$, where μ is a constant.

In [14], Lutwak introduced the *mixed p -quermassintegrals* $W_{p,0}(K, L), W_{p,1}(K, L), \dots, W_{p,n-1}(K, L)$, for $K, L \in \mathcal{K}^n$, and real number $p \geq 1$. The next aim of this paper is to establish the Minkowski inequality for mixed p -quermassintegral differences of convex bodies. A new generalization of the classical Minkowski inequality is presented as follows:

Theorem 2. *Let K, L , and D be convex bodies in R^n and $D \subset K, D' \subset L$ and D' be a homothetic copy of D . Then for $p \geq 1$,*

$$(W_{p,i}(K, L) - W_{p,i}(D, D'))^{n-i} \geq Dw_i(K, D)^{n-i-p} Dw_i(L, D)^p, \quad (7)$$

with equality for $0 \leq i < n - p$ if and only if K and L are homothetic ($p = 1$) (or are dilates ($p > 1$)) and $(W_i(K), W_i(D)) = \mu(W_i(L), W_i(D'))$, where μ is a constant.

On the other hand, we establish Minkowski inequality and Brunn–Minkowski inequality for quermassintegral differences of mixed projection bodies, respectively. which are extensions of Lutwak’s results [15].

Theorem 3. *Let K, L , and D be convex bodies in $R^n, D \subset K$ and D' a homothetic copy of D . Then for $0 \leq j < n - 2$,*

$$\begin{aligned} &Dw_i(\Pi_j(K + L), \Pi_j(D + D'))^{1/(n-i)(n-j-1)} \\ &\geq Dw_i(\Pi_j K, \Pi_j D)^{1/(n-i)(n-j-1)} + Dw_i(\Pi_j L, \Pi_j D')^{1/(n-i)(n-j-1)}, \end{aligned} \quad (8)$$

with equality for $0 \leq i < n - 1$ if and only if K and L are homothetic and $(W_i(K), W_i(D)) = \mu(W_i(L), W_i(D'))$, where μ is a constant.

Theorem 4. *Let K, L , and D be convex bodies in $R^n, D \subset K, D'$ be a homothetic copy of D , and $0 \leq j < n - 1$. Then*

$$Dw_i(\Pi_j(K, L), \Pi_j(D, D'))^{n-1} \geq Dw_i(\Pi K, \Pi D)^{n-j-1} Dw_i(\Pi L, \Pi D')^j, \quad (9)$$

with equality for $0 \leq i < n - 1$ if and only if K and L are homothetic.

The above interrelated notations, definitions and background materials are given in §2.

2. Definitions and preliminaries

The setting for this paper is the n -dimensional Euclidean space \mathbb{R}^n ($n > 2$). Let \mathcal{K}^n denote the set of convex bodies (compact, convex subsets with non-empty interiors) in \mathbb{R}^n .

Firey [8] introduced, for each real $p \geq 1$, new linear combinations of convex bodies: For $K, L \in \mathcal{K}^n$, and $\lambda, \mu \geq 0$ (both are not zero), the *Firey combination*, $\lambda \cdot K +_p \mu \cdot L$, is a convex body whose support function is defined by

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot)^p = \lambda h(K, \cdot)^p + \mu h(L, \cdot)^p.$$

Obviously, $\alpha \cdot K = \alpha^{1/p} K$.

The mixed quermassintegral $W_0(K, L), W_1(K, L), \dots, W_{n-1}(K, L)$ of $K, L \in \mathcal{K}^n$ are defined by

$$(n - i)W_i(K, L) = \lim_{\varepsilon \rightarrow 0} \frac{W_i(K + \varepsilon L) - W_i(K)}{\varepsilon}, \tag{10}$$

where

$$W_i(K, L) = V(\underbrace{K, \dots, K}_{n-i-1}, \underbrace{B, \dots, B}_i, L).$$

The mixed p -quermassintegrals $W_{p,0}(K, L), W_{p,1}(K, L), \dots, W_{p,n-1}(K, L)$, for $K, L \in \mathcal{K}^n$, and real $p \geq 1$, are defined by [14]

$$\frac{n - i}{p}W_{p,i}(K, L) = \lim_{\varepsilon \rightarrow 0} \frac{W_i(K +_p \varepsilon \cdot L) - W_i(K)}{\varepsilon}. \tag{11}$$

Of course for $p = 1$, the mixed p -quermassintegral $W_{p,i}(K, L)$ is just $W_i(K, L)$. Obviously, $W_{p,i}(K, K) = W_i(K)$, for all $p \geq 1$.

If $K_1, \dots, K_r \in \mathcal{K}^n$ and $\lambda_1, \dots, \lambda_r \geq 0$, then the projection body of the Minkowski linear combination $\lambda_1 K_1 + \dots + \lambda_r K_r \in \mathcal{K}^n$ can be written as a symmetric homogeneous polynomial of degree $(n - 1)$ in λ_i [15]:

$$\Pi(\lambda_1 K_1 + \dots + \lambda_r K_r) = \sum \lambda_{i_1} \dots \lambda_{i_{n-1}} \Pi_{i_1 \dots i_{n-1}}, \tag{12}$$

where the sum is a Minkowski sum taken over all $(n - 1)$ -tuples (i_1, \dots, i_{n-1}) of positive integers not exceeding r . The body $\Pi_{i_1 \dots i_{n-1}}$ depends only on the bodies $K_{i_1}, \dots, K_{i_{n-1}}$, and is uniquely determined by (12). It is called *the mixed projection bodies* of $K_{i_1}, \dots, K_{i_{n-1}}$, and is written as $\Pi(K_{i_1}, \dots, K_{i_{n-1}})$. If $K_1 = \dots = K_{n-1-i} = K$ and $K_{n-i} = \dots = K_{n-1} = L$, then $\Pi(K_{i_1}, \dots, K_{i_{n-1}})$ will be written as $\Pi_i(K, L)$. If $L = B$, then $\Pi_i(K, L)$ is denoted by $\Pi_i K$ and when $i = 0$, $\Pi_i K$ is denoted by ΠK .

3. Some lemmas

The following results will be required to prove our main theorems.

Lemma 1 [14]. *If $p \geq 1$, and $K, L \in \mathcal{K}^n$, when $0 \leq i < n$, then*

$$W_i(K +_p L)^{p/(n-i)} \geq W_i(K)^{p/(n-i)} + W_i(L)^{p/(n-i)}, \tag{13}$$

for $p > 1$ with equality if and only if K and L are dilates; for $p = 1$ with equality if and only if K and L are homothetic.

$$W_{p,i}(K, L)^{n-i} \geq W_i(K)^{n-i-p} W_i(L)^p \tag{14}$$

for $p > 1$ with equality if and only if K and L are dilates; for $p = 1$ with equality if and only if K and L are homothetic.

Lemma 2 [15]. *If $K, L \in \mathcal{K}^n$, and $0 \leq i < n$, then*

$$\begin{aligned} W_i(\Pi_j(K + L))^{1/(n-i)(n-j-i)} \\ \geq W_i(\Pi_j K)^{1/(n-i)(n-j-1)} + W_i(\Pi_j L)^{1/(n-i)(n-j-1)}, \end{aligned} \tag{15}$$

with equality if and only if K and L are homothetic.

$$W_i(\Pi_j(K, L))^{n-1} \geq W_i(\Pi K)^{n-j-1} W_i(\Pi L)^j, \tag{16}$$

with equality if and only if K and L are homothetic.

Lemma 3 [2] (the Bellman's inequality). Let $a = \{a_1, \dots, a_n\}$ and $b = \{b_1, \dots, b_n\}$ be two series of positive real numbers and $p > 1$ such that $a_1^p - \sum_{i=2}^n a_i^p > 0$ and $b_1^p - \sum_{i=2}^n b_i^p > 0$. Then

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{1/p} + \left(b_1^p - \sum_{i=2}^n b_i^p\right)^{1/p} \leq \left((a_1 + b_1)^p - \sum_{i=2}^n (a_i + b_i)^p\right)^{1/p} \tag{17}$$

with equality if and only if $a = vb$ where v is a constant.

Lemma 4. If $a, b, c, d > 0, 0 < \alpha < 1, 0 < \beta < 1$ and $\alpha + \beta = 1$. Let $a > b$ and $c > d$, then

$$a^\alpha c^\beta - b^\alpha d^\beta \geq (a - b)^\alpha (c - d)^\beta, \tag{18}$$

with equality if and only if $a/b = c/d$.

Proof. Consider the following function

$$f(x) = x^\alpha c^\beta - (x - b)^\alpha (c - d)^\beta, \quad x > 0.$$

Let

$$f'(x) = \alpha c^\beta x^{\alpha-1} - \alpha (c - d)^\beta (x - b)^{\alpha-1} = 0.$$

We get $x = bc/d$.

On the other hand, if $x \in (0, \frac{bc}{d})$, then $f'(x) < 0$; if $x \in (\frac{bc}{d}, +\infty)$, then $f'(x) > 0$, and it follows that

$$\min_{x>0}\{f(x)\} = f\left(\frac{bc}{d}\right) = b^\alpha d^\beta.$$

This completes the proof. □

4. Inequalities for mixed p -quermassintegral differences of convex bodies

Theorem 1. If K, L , and D are convex bodies in R^n , $D \subset K$ and D' is a homothetic copy of D , then for $p \geq 1$,

$$Dw_i(K +_p L, D +_p D')^{p/(n-i)} \geq Dw_i(K, D)^{p/(n-i)} + Dw_i(L, D')^{p/(n-i)}, \tag{19}$$

with equality for $0 \leq i < n - p$ if and only if K and L are homothetic ($p = 1$) (or are dilates ($p > 1$)) and $(W_i(K), W_i(D)) = \mu(W_i(L), W_i(D'))$, where μ is a constant.

Using inequality (13) and in view of the Bellman's inequality, we get the above Brunn–Minkowski inequality for quermassintegral differences of the Firey combination.

Taking $p = 1$ in inequality (19), the inequality (19) changes to inequality (3).

In the following, we will prove the Minkowski inequality for mixed p -quermassintegral differences of convex bodies.

Theorem 2. *If K, L and D are convex bodies in R^n , $D \subset K, D' \subset L$ and D' is a homothetic copy of D , then for $p \geq 1$,*

$$(W_{p,i}(K, L) - W_{p,i}(D, D'))^{n-i} \geq Dw_i(K, D)^{n-i-p} Dw_i(L, D)^p, \quad (20)$$

with equality for $0 \leq i < n - p$ if and only if K and L are homothetic ($p = 1$) (or are dilates ($p > 1$)) and $(W_i(K), W_i(D)) = \mu(W_i(L), W_i(D'))$, where μ is a constant.

Proof. From inequality (14), we have

$$W_{p,i}(K, L)^{n-i} \geq W_i(K)^{n-i-p} W_i(L)^p, \quad (21)$$

for $p > 1$ with equality if and only if K and L are dilates; for $p = 1$ with equality if and only if K and L are homothetic and

$$W_{p,i}(D, D')^{n-i} = W_i(D)^{n-i-p} W_i(D')^p.$$

Hence,

$$\begin{aligned} W_{p,i}(K, L) - W_{p,i}(D, D') &\geq W_i(K)^{(n-i-p)/(n-i)} W_i(L)^{p/(n-i)} \\ &\quad - W_i(D)^{(n-i-p)/(n-i)} W_i(D')^{p/(n-i)}. \end{aligned}$$

On the other hand, in view of inequality (18), we have

$$\begin{aligned} &W_i(K)^{(n-i-p)/(n-i)} W_i(L)^{p/(n-i)} - W_i(D)^{(n-i-p)/(n-i)} W_i(D')^{p/(n-i)} \\ &\geq (W_i(K) - W_i(D))^{(n-i-p)/(n-i)} (W_i(L) - W_i(D'))^{p/(n-i)}, \end{aligned} \quad (21')$$

with equality if and only if $W_i(K)/W_i(D) = W_i(L)/W_i(D')$.

Thus,

$$W_{p,i}(K, L) - W_{p,i}(D, D') \geq Dw_i(K, D)^{(n-i-p)/(n-i)} Dw_i(L, D)^{p/(n-i)}.$$

Combining equality conditions of inequalities (21) and (21'), it shows that the equality holds for $0 \leq i < n - p$ if and only if K and L are homothetic ($p = 1$) (or are dilates ($p > 1$)) and $(W_i(K), W_i(D)) = \mu(W_i(L), W_i(D'))$, where μ is a constant.

The proof is complete. \square

Taking $p = 1$ in inequality (20), we get the following result.

COROLLARY 1

Suppose that K, L and D are convex bodies, and $D \subset K, D' \subset L, D'$ is a homothetic copy of D . Then

$$(W_i(K, L) - W_i(D, D'))^{n-i} \geq Dw_i(K, D)^{n-i-1} Dw_i(L, D), \quad (22)$$

with equality for $0 \leq i < n - 1$ if and only if K and L are homothetic and $(W_i(K), W_i(D)) = \mu(W_i(L), W_i(D'))$, where μ is a constant.

Taking $i = 0$ in (22), it changes to inequality (6).

Remark 1. Let $p = 1$, $i = 0$, and D, D' be a single point in (20). Then (20) reduces to the classical Minkowski inequality. Hence, (20) is a generalization of the classical Minkowski inequality.

5. Inequalities for quermassintegral differences of mixed projection bodies

In this section, we first establish the Brunn–Minkowski inequality for quermassintegral differences of mixed projection bodies as follows:

Theorem 3. *Let K, L , and D be convex bodies in R^n , $D \subset K$ and D' a homothetic copy of D . Then for $0 \leq j < n - 2$,*

$$\begin{aligned} Dw_i(\Pi_j(K + L), \Pi_j(D + D'))^{1/(n-i)(n-j-1)} \\ \geq Dw_i(\Pi_j K, \Pi_j D)^{1/(n-i)(n-j-1)} + Dw_i(\Pi_j L, \Pi_j D')^{1/(n-i)(n-j-1)}, \end{aligned} \tag{23}$$

with equality for $0 \leq i < n - 1$ if and only if K and L are homothetic and $(W_i(K), W_i(D)) = \mu(W_i(L), W_i(D'))$, where μ is a constant.

Proof. Applying inequality (15), we have

$$\begin{aligned} W_i(\Pi_j(K + L))^{1/(n-i)(n-j-1)} \\ \geq W_i(\Pi_j K)^{1/(n-i)(n-j-1)} + W_i(\Pi_j L)^{1/(n-i)(n-j-1)}, \end{aligned} \tag{24}$$

with equality if and only if K and L are homothetic.

$$\begin{aligned} W_i(\Pi_j(D + D'))^{1/(n-i)(n-j-1)} \\ = W_i(\Pi_j D)^{1/(n-i)(n-j-1)} + W_i(\Pi_j D')^{1/(n-i)(n-j-1)}. \end{aligned} \tag{25}$$

From (24) and (25), we obtain that

$$\begin{aligned} Dw_i(\Pi_j(K + L), \Pi_j(D + D')) \\ \geq [W_i(\Pi_j K)^{1/(n-i)(n-j-1)} + W_i(\Pi_j L)^{1/(n-i)(n-j-1)}]^{(n-i)(n-j-1)} \\ - [W_i(\Pi_j D)^{1/(n-i)(n-j-1)} + W_i(\Pi_j D')^{1/(n-i)(n-j-1)}]^{(n-i)(n-j-1)}. \end{aligned} \tag{26}$$

From (26) and in view of the Bellman’s inequality,

$$\begin{aligned} Dw_i(\Pi_j(K + L), \Pi_j(D + D'))^{1/(n-i)(n-j-1)} \\ \geq (W_i(\Pi_j K) - W_i(\Pi_j D))^{1/(n-i)(n-j-1)} \\ + (W_i(\Pi_j L) - W_i(\Pi_j D'))^{1/(n-i)(n-j-1)}. \end{aligned}$$

This completes the proof. □

Taking $i = 0, j = 0$ in inequality (23), we obtain the following result.

COROLLARY 2

Let K, L , and D be convex bodies in R^n , $D \subset K$ and D' is a homothetic copy of D . Then

$$\begin{aligned} Dv(\Pi(K+L), \Pi(D+D'))^{1/n(n-1)} \\ \geq Dv(\Pi K, \Pi D)^{1/n(n-1)} + Dv(\Pi L, \Pi D')^{1/n(n-1)} \end{aligned} \quad (27)$$

with equality if and only if K and L are homothetic and $V((K), V(D)) = \mu(V(L), V(D'))$, where μ is a constant.

This is just a projection form of ‘Theorem 1’ which was given by Leng [13].

Remark 2. Let D and D' be a single point in (23). Then (23) changes to (16). This shows that (23) is a generalization of the Brunn–Minkowski inequality for mixed projection bodies.

In the following, we establish the Minkowski inequality for quermassintegral differences of mixed projection bodies.

Theorem 4. Let K, L , and D be convex bodies in R^n , $D \subset K$, D' is a homothetic copy of D , and $0 \leq j < n - 1$. Then

$$Dw_i(\Pi_j(K, L), \Pi_j(D, D'))^{n-1} \geq Dw_i(\Pi K, \Pi D)^{n-j-1} Dw_i(\Pi L, \Pi D')^j, \quad (28)$$

with equality for $0 \leq i < n - 1$ if and only if K and L are homothetic and $(W_i(K), W_i(D)) = \mu(W_i(L), W_i(D'))$ where μ is a constant.

Proof. Applying inequality (16), we have

$$W_i(\Pi_j(K, L)) \geq W_i(\Pi K)^{(n-j-1)/(n-1)} W_i(\Pi L)^{j/(n-1)},$$

with equality if and only if K and L are homothetic.

$$W_i(\Pi_j(D, D')) = W_i(\Pi D)^{(n-j-1)/(n-1)} W_i(\Pi D')^{j/(n-1)}.$$

Hence, from inequality (18) in Lemma 4, we obtain that

$$\begin{aligned} Dw_i(\Pi_j(K, L), \Pi_j(D, D')) \\ \geq W_i(\Pi K)^{(n-j-1)/(n-1)} W_i(\Pi L)^{j/(n-1)} \\ - W_i(\Pi D)^{(n-j-1)/(n-1)} W_i(\Pi D')^{j/(n-1)} \\ \geq (W_i(\Pi K) - W_i(\Pi D))^{(n-j-1)/(n-1)} (W_i(\Pi L) - W_i(\Pi D'))^{j/(n-1)}. \end{aligned}$$

The proof is complete. \square

Taking $i = 0$, $j = 1$ in inequality (28), inequality (28) changes to the following result.

COROLLARY 3

Let K, L , and D be convex bodies in R^n , $D \subset K$ and D' a homothetic copy of D . Then

$$Dv(\Pi_1(K, L), \Pi_1(D, D'))^{n-1} \geq Dv(\Pi K, \Pi D)^{n-2} Dv(\Pi L, \Pi D'), \quad (29)$$

with equality for $0 \leq i < n - 1$ if and only if K and L are homothetic and $(V(K), V(D)) = \mu(V(L), V(D'))$ where μ is a constant.

This is just a projection form of inequality (6).

Remark 3. Let D, D' be a single point in (29). Then (29) changes to the following inequality:

If $K, L \in \mathcal{K}^n$, then

$$V(\Pi_1(K, L))^{n-1} \geq V(\Pi K)^{n-2}V(\Pi L)$$

with equality if and only if K and L are homothetic.

This is just *the Minkowski inequality for mixed projection bodies* which was given by Lutwak [15].

6. Two open problems

In the following, we pose two open problems:

Problem 1. Let $K_i, i = 1, 2, \dots, n$ and $D_i, i = 1, 2, \dots, n$ be convex bodies in $R^n, D_i \subset K_i$ and D'_i a homothetic copy of $D_i, i = 1, 2, \dots, n$, respectively. Then for $0 \leq r \leq n$,

$$\begin{aligned} & (V(K_1, \dots, K_n) - V(D_1, \dots, D_n))^r \\ & \geq \prod_{j=1}^r (V(\underbrace{K_j, \dots, K_j}_r, K_{r+1}, \dots, K_n) - V(\underbrace{D_j, \dots, D_j}_r, D_{r+1}, \dots, D_n)). \end{aligned} \tag{30}$$

Remark 4. In (30), taking $r = n$, we obtain that

$$(V(K_1, \dots, K_n) - V(D_1, \dots, D_n))^n \geq \prod_{j=1}^n (V(K_j) - V(D_j)). \tag{31}$$

Taking $K_1 = \dots = K_{n-1} = K, K_n = L, D_1 = \dots = D_{n-1} = D, D_n = D'$ in (31), inequality (31) changes to

$$(V_1(K, L) - V_1(D, D'))^n \geq Dv(K, D)^{n-1}Dv(L, D').$$

This is just inequality (6).

On the other hand, let D and D' be a single point in (30). Then (30) changes to the well-known Aleksandrov–Fenchel inequality.

Problem 2. Let $K_i, i = 1, \dots, n - 1$ and $D_i, i = 1, \dots, n - 1$ be convex bodies in $R^n, D_i \subset K_i$ and $D'_i, i = 1, \dots, n - 1$ a homothetic copy of D_i , respectively. Then for $0 \leq r \leq n - 1$,

$$\begin{aligned} & Dv(\Pi(K_1, \dots, K_{n-1}), \Pi(D_1, \dots, D_{n-1}))^r \\ & \geq \prod_{j=1}^r Dv(\Pi(\underbrace{K_j, \dots, K_j}_r, K_{r+1}, \dots, K_n), \Pi(\underbrace{D_j, \dots, D_j}_r, D_{r+1}, \dots, D_n)). \end{aligned} \tag{32}$$

Remark 5. In (32), taking $r = n - 1$, we obtain that

$$Dv(\Pi(K_1, \dots, K_{n-1}), \Pi(D_1, \dots, D_{n-1}))^{n-1} \geq \prod_{j=1}^{n-1} Dv(\Pi K_j, \Pi D_j). \quad (33)$$

Taking $K_1 = \dots = K_{n-2} = K$, $K_{n-1} = L$, $D_1 = \dots = D_{n-2} = D$, $D_{n-1} = D'$ in (33), inequality (33) changes to

$$Dv(\Pi_1(K, L), \Pi_1(D, D'))^n \geq Dv(\Pi(K, D))^{n-1} Dv(\Pi(L, D')). \quad (34)$$

This is just inequality (29) which is proved in this paper.

On the other hand, let D and D' be a single point in (32). Then (32) changes to the well-known Aleksandrov–Fenchel inequality for mixed projection bodies which was given by Lutwak [15].

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