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Quotient normed cones

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Abstract. Given a normed cone (X, p) and a subcone Y, we construct and study the quotient normed cone $(X/Y, \tilde{p})$ generated by Y. In particular we characterize the bicompleteness of $(X/Y, \tilde{p})$ in terms of the bicompleteness of (X, p), and prove that the dual quotient cone $((X/Y)^*, \|\cdot\|_{\tilde{p},u})$ can be identified as a distinguished subcone of the dual cone $(X^*, \|\cdot\|_{p,u})$. Furthermore, some parts of the theory are presented in the general setting of the space $CL(X, Y)$ of all continuous linear mappings from a normed cone (X, p) to a normed cone (Y, q), extending several well-known results related to open continuous linear mappings between normed linear spaces.

Keywords. Normed cone; extended quasi-metric; continuous linear mapping; bicompleteness; quotient cone; dual.

1. Introduction and preliminaries

In recent years many works on functional analysis have been obtained in order to extend the well-known results of the classical theory of normed linear spaces to the framework of asymmetric normed linear spaces and quasi-normed cones. In particular, the dual of an asymmetric normed linear space has been constructed and studied in [10]. In the same reference an asymmetric version of the celebrated Alouglu theorem has been proved (see also [11]). Several appropiate generalizations of the structure of the dual of an asymmetric normed linear space can be found in [20] and [19]. Hahn–Banach type theorems in the frame of quasi-normed spaces have been given in [1,22] and [30]. In [9] and [17], the completion of asymmetric normed linear spaces and quasi-normed cones have been explored. An asymmetric version of the Riesz theorem for finite dimension linear spaces can be found in [7].

It seems interesting to point out that quasi-normed cones and other related 'nonsymmetric' structures from topological algebra and functional analysis, have been successfully applied, in the last few years, to several problems in theoretical computer science, approximation theory and physics, respectively (see §§ 11 and 12 of [16], and also [2,8,21,13,24–26,30,18]).

The purpose of this paper is to show that it is possible to generate in a natural way a quotient quasi-normed cone from a subcone of a given quasi-normed cone. Actually, we analyse when such quotient cones are bicomplete. We also construct and study the dual cone of a quasi-normed cone and we prove that it can be identified as the dual of a quotient cone. This is done with the help of an appropiate notion of a 'polar' cone.

Throughout this paper, $\mathbb{R}^+$, $\mathbb{N}$ and ω will denote the set of nonnegative real numbers, the set of natural numbers and the set of nonnegative integer numbers, respectively.

Recall that a *monoid* is a semigroup $(X, +)$ with neutral element 0.

According to [13], a *cone* (on $\mathbb{R}^+$) is a triple $(X, +, \cdot)$ such that $(X, +)$ is an abelian monoid, and $\cdot$ is a mapping from $\mathbb{R}^+ \times X$ to X such that for all $x, y \in X$ and $r, s \in \mathbb{R}^+$:

(i) $r \cdot (s \cdot x) = (rs) \cdot x$;
(ii) $r \cdot (x + y) = (r \cdot x) + (r \cdot y)$;
(iii) $(r + s) \cdot x = (r \cdot x) + (s \cdot x)$;
(iv) $1 \cdot x = x$;
(v) $0 \cdot x = 0$.

A cone $(X, +, \cdot)$ is called *cancellative* if for all $x, y, z \in X, z+x = z+y$ implies $x = y$.

Obviously, every linear space $(X, +, \cdot)$ can be considered as a cancellative cone when we restrict the operation $\cdot$ to $\mathbb{R}^+ \times X$.

Let us recall that a *linear mapping* from a cone $(X, +, \cdot)$ to a cone $(Y, +, \cdot)$ is a mapping $f: X \to Y$ such that $f(\alpha \cdot x + \beta \cdot y) = \alpha \cdot f(x) + \beta \cdot f(y)$.

A *subcone* of a cone $(X, +, \cdot)$ is a cone $(Y, +|_Y, \cdot|_Y)$ such that Y is a subset of X and $+|_Y$ and $\cdot|_Y$ are the restriction of $+$ and $\cdot$ to Y, respectively.

A *quasi-norm* on a cone $(X, +, \cdot)$ is a function $q: X \to \mathbb{R}^+$ such that for all $x, y \in X$ and $r \in \mathbb{R}^+$:

(i) $x = 0$ if and only if there is $-x \in X$ and $q(x) = q(-x) = 0$;
(ii) $q(r \cdot x) = rq(x)$;
(iii) $q(x + y) \leq q(x) + q(y)$.

If the quasi-norm q satisfies: (i′) $q(x) = 0$ if and only if $x = 0$, then q is called a *norm* on the cone $(X, +, \cdot)$.

A quasi-norm defined on a linear space is called an *asymmetric norm* in [8], [9] and [10].

Our main references for quasi-pseudo-metric spaces are [6] and [16].

Let us recall that a *quasi-pseudo-metric* on a set X is a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X$: (i) $d(x, x) = 0$; (ii) $d(x, z) \leq d(x, y) + d(y, z)$.

In our context, by a *quasi-metric on* X we mean a quasi-pseudo-metric d on X that satisfies the following condition: $d(x, y) = d(y, x) = 0$ if and only if $x = y$.

We will also consider *extended quasi-(pseudo-)metrics*. They satisfy the above three axioms, except that we allow $d(x, y) = +\infty$.

Each extended quasi-pseudo-metric d on a set X induces a topology $\mathcal{T}(d)$ on X which has as a base the family of open d-balls $\{B_d(x, r): x \in X, r > 0\}$, where $B_d(x, r) = \{y \in X: d(x, y) < r\}$ for all $x \in X$ and $r > 0$.

A (*n extended*) *quasi-*(*pseudo-*)*metric space* is a pair (X, d) such that X is a set and d is a (*n* extended) quasi-(pseudo-)metric on X.

Similarly to [15], an extended quasi-metric d on a cone $(X, +, \cdot)$ is said to be *invariant* if for each $x, y, z \in X$ and $r \in \mathbb{R}^+$, $d(x + z, y + z) = d(x, y)$ and $d(rx, ry) = rd(x, y)$.

If d is a (*n* extended) quasi-metric on a set X, then the function d^s defined on $X \times X$ by $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ is a (*n* extended) metric on X. An extended quasi-metric d on a set X is said to be *bicomplete* if d^s is a complete extended metric on X.

In [17] it was shown that it is possible to generate in a natural way extended quasi-metrics from quasi-norms on cancellative cones, extending to the well-known result that establishes that each norm on a linear space X induces a metric on X. In particular it was proved the following result:

PROPOSITION 1

Let p be a quasi-norm on a cancellative cone $(X, +, \cdot)$. Then the function d_p defined on $X \times X$ by

$$d_p(x, y) = \begin{cases} p(a), & \text{if } x \in X \text{ and } y \in x + X \text{ with } y = x + a \\ +\infty, & \text{otherwise} \end{cases}$$

is an invariant extended quasi-metric on X. Furthermore for each $x \in X$, $r \in \mathbb{R}^+\backslash\{0\}$ and each $\varepsilon > 0$, $rB_{e_p}(x, \varepsilon) = rx + \{y \in X\colon p(y) < r\varepsilon\}$, and the translations are $\mathcal{T}(d_p)$-open.

The (extended) Sorgenfrey topology is obtained as a particular case of the above construction.

The following well-known example will be useful later on. For each $x, y \in \mathbb{R}$, let $u(x) = x \vee 0$. Then u is clearly a quasi-norm on $\mathbb{R}$ whose induced extended quasi-metric is the so-called (extended) upper quasi-metric d_u on $\mathbb{R}$.

According to [17], a quasi-normed cone is a pair (X, p) where X is a cancellative cone and p is a quasi-norm on X.

2. The quotient cone of a quasi-normed cone

We can find several techniques to construct new normed linear spaces from a given one in the literature (see, for example, [27,12] and [14]). In particular one of them consists of deriving quotient linear spaces from a given linear space. Let us briefly recall this construction.

Let X be a linear space and let Y be a linear subspace of X. For each $x \in X$, let $[x] = x + Y$. The family of such subsets is called the quotient of X by Y and it is denoted by X/Y. The usual set-addition defines a sum on X/Y by $x + Y + z + Y = x + z + Y$. Moreover, the product by scalars is defined in X/Y by $\lambda \cdot (x + Y) = \lambda \cdot x + Y$. Under these operations X/Y forms a linear space. Furthemore, if $\|\cdot\|$ is a norm defined on X and Y is closed then $(E/Y, \|\cdot\|_{E/Y})$ admits a normed linear structure, with $\|[x]\|_{E/Y} = \inf\{\|x+y\|\colon y \in Y\}$.

Next we introduce a general method for generating quotient spaces from a quasi-normed cone (X, p), which preserves the quasi-normed cone structure, obtaining as a particular case of our construction the above classical technique.

Let X be a cancellative cone and let Y be a subcone of X. Denote by G_Y the set $\{y \in Y\colon -y \in Y\}$. Clearly G_Y is a nonempty set because of $0 \in G_Y$. It is easy to see that G_Y admits a subcone structure, in fact it is a linear space.

The following relation will allow us to construct a quasi-normed cone with quotient space structure.

A pair of elements $x, z \in X$ are $\mathcal{R}$-related if there exists $g \in G_Y$ such that $x + g = z$. In this case, we write $x\mathcal{R}y$.

This relation is obviously an equivalence relation, i.e., reflexive, symmetric and transitive. We will denote by $[x]$ the equivalence class of x with respect to $\mathcal{R}$. Thus, the classes are given by $[x] = \{x + y\colon y \in G_Y\}$ and the family of them by $X/Y = \{[x]\colon x \in X\}$.

PROPOSITION 2

The set X/Y endowed with the usual sum and product of equivalence classes $[x]+[z] := [x+z]$ and $\lambda \cdot [x] := [\lambda \cdot x]$ where $x, z \in X$ and $\lambda \in \mathbb{R}^+$, is a cancellative cone.

Proof. Let us show that the sum of equivalence classes is well-defined. Indeed, if $v \in [x]$ and $w \in [z]$ there are elements $g_1, g_2 \in G_Y$ such that $x = v + g_1$ and $z = w + g_2$. Then $x + z = v + w + g_1 + g_2$. Whence $[v + w] = [x + z]$. The natural product is also well-defined because $\lambda \cdot x = \lambda \cdot v + \lambda \cdot g_1$, so that $[\lambda \cdot x] = [\lambda \cdot v]$.

Direct calculations show that $(X/Y, +, \cdot)$ is a cone with neutral element $[0] = G_Y$. Finally, we show that X/Y has the cancellation law. Let $x, y, z \in X$ such that $[x] + [y] = [x]+[z]$. Then $[x+y] = [x+z]$. It follows that there exists $g \in G_Y$ with $x+y = x+z+g$. Since X is cancellative, $y = z + g$, so that $y \in [z]$. Consequently $[y] = [z]$. ∎

DEFINITION 3

Let X be a cancellative cone and Y be a subcone. Then X/Y is called *the quotient cone of X by Y*.

Remark 4. Note that if $[x] + [y] = [0]$ for any $x, y \in X$, then $[x + y] = [0]$ and, as a consequence there exists $g \in G_Y$ such that $x + y = 0 + g$. Hence $x + y + (-g) = 0$. Therefore $-x = y + (-g)$ and $-x \in [y]$.

Remark 5. In case X is a linear space and Y is a linear subspace of X we obtain as a particular case of our construction the classical technique for generating quotient linear spaces from linear spaces mentioned above, since $[0] = Y$ and $[x] = x + Y$.

Example 6. Let $(\mathbb{R}^2, +, \cdot)$ be endowed with the usual operations. Let Y be the subcone of $\mathbb{R}^2$ given by $Y = \{(x, y) : y \geq 0\}$. Obviously the subcone $A = \{(x, 0) : x \in \mathbb{R}\}$ satisfies that $A \subset Y$ and $G_Y = A$. Thus the (cancellative) quotient cone $\mathbb{R}^2/Y = \{[(x, y)] : (x, y) \in \mathbb{R}^2\}$, where $[(x, y)] = (x, y) + A = \{(z + x, y) : z \in \mathbb{R}\}$.

In the sequel, if A is a subset of a quasi-metric space (X, d), we will denote by $\bar{A}^d$ the closure of A with respect to $\mathcal{T}(d)$.

The following property, whose proof is well-known and we omit, will be useful later on.

PROPOSITION 7

Let (X, d) be a quasi-metric space and let $Y \subset X$. Then, $x \in \bar{Y}^d$ if and only if $d(x, Y) = 0$, where $d(x, Y) = \inf\{d(x, y) : y \in Y\}$.

Next we denote by $d(Y, \cdot)$ the function defined as $d(Y, x) = \inf\{d(y, x) : y \in Y\}$.

In Proposition 8, we construct a quasi-norm on X/Y from a quasi-normed cone (X, p). For this, it is necessary to assume that G_Y is a closed subcone of X with respect to $\mathcal{T}(d_p)$ (see Example 9 below).

PROPOSITION 8

Let (X, p) be a quasi-normed cone and let Y be a subcone of X such that G_Y is closed in $(X, \mathcal{T}(d_p))$. Then the pair $(X/Y, \hat{p})$ is a quasi-normed cone, where the function $\hat{p} : X/Y \to \mathbb{R}^+$ is defined by

$$\hat{p}([x]) = \inf\{p(x + y) : y \in G_Y\}.$$

Proof. It is clear that

$$\hat{p}([0]) = \inf\{p(y) \colon y \in G_Y\} = 0.$$

Let $x \in X$ such that $-x \in X$ and $\hat{p}([x]) = \hat{p}([-x]) = 0$. Then,

$$d_p(x, G_Y) \vee d_p(G_Y, x) \leq (d_p)^s(x, 0) = \max\{p(x), p(-x)\} < +\infty.$$

Hence, given $\varepsilon > 0$, there exist $a, b \in X$ and $g_1, g_2, h \in G_Y$ with

(i) $g_1 = x + b$ and $x = g_2 + a$,
(ii) $p(b) < d_p(x, G_Y) + \varepsilon$,
(iii) $p(-x + h) < \hat{p}([-x]) + \varepsilon$.

Since $d_p(x, g_1) = p(b)$ we deduce that

$$\hat{p}([-x]) \leq p(-x + g_1) = p(b) < d_p(x, G_Y) + \varepsilon$$

and

$$d_p(x, G_Y) \leq p(-x + h) < \hat{p}([-x]) + \varepsilon.$$

Then $\hat{p}([-x]) \leq d_p(x, G_Y) \leq \hat{p}([-x])$. Therefore $d_p(x, G_Y) = 0$. Similarly it was showed that $\hat{p}([x]) \leq d_p(-x, G_Y) \leq \hat{p}([x])$.

Since G_Y is closed in $(X, \mathcal{T}(d_p))$ and by Proposition 7, $x, -x \in G_Y$. Thus $[x] = [0]$.

Next we prove that $\hat{p}$ is homogeneous with respect to nonnegative real numbers. We distinguish two cases:

Case 1. $\lambda = 0$. Then the homogeneousness is immediately obtained by definition of $\hat{p}$ and the fact that X/Y is a cone.

Case 2. $\lambda > 0$. Then

$$\begin{aligned}\hat{p}(\lambda \cdot [x]) &= \inf\{p(\lambda x + g) \colon g \in G_Y\} = \inf\left\{\lambda p\left(x + \frac{g}{\lambda}\right) \colon g \in G_Y\right\} \\ &= \inf\{\lambda p(x + h) \colon h \in G_Y\} = \lambda \inf\{p(x + h) \colon h \in G_Y\} \\ &= \lambda \hat{p}([x]).\end{aligned}$$

It remains to show the triangular inequality. Let $x, z \in X$ and $\varepsilon > 0$. Then there exist $g, h \in G_Y$ such that

$$p(x + g) < \hat{p}([x]) + \varepsilon/2 \quad \text{and} \quad p(z + h) < \hat{p}([z]) + \varepsilon/2.$$

It follows that

$$\begin{aligned}\hat{p}([x] + [z]) = \hat{p}([x + z]) &\leq p((x + z) + (g + h)) \\ &\leq p(x + g) + p(z + h) < \hat{p}([x]) + \hat{p}([z]) + \varepsilon.\end{aligned}$$

We conclude that $\hat{p}([x] + [z]) \leq \hat{p}([x]) + \hat{p}([z])$. ■

The following example shows that the condition that the subcone G_Y is closed in $(X, \mathcal{T}(d_p))$ can not be omitted in the statement of the preceding theorem.

Example 9. Consider $X = (\mathbb{R}^2, +, \cdot)$ endowed with the usual operations $+$ and $\cdot$. Let A be the subcone of $\mathbb{R}^2$ given by $A = \{(x, x) : x \in \mathbb{R}\}$. Thus, it is clear that $G_A = A$. On the other hand, the function $p(x, y) = u(x) + u(y)$ is a quasi-norm on $\mathbb{R}^2$, so that $(\mathbb{R}^2, p)$ is a quasi-normed cone. Furthermore, $(2, 3) \in \bar{A}^{d_p}$ because of $d_p\big((2, 3), \big(2 - \frac{1}{n}, 2 - \frac{1}{n}\big)\big) = p\big(-\frac{1}{n}, -1 - \frac{1}{n}\big) = u\big(-\frac{1}{n}\big) + u\big(-1 - \frac{1}{n}\big) = 0$ for all $n \in \mathbb{N}$ and, as a consequence, G_A is not closed in $(\mathbb{R}^2, \mathcal{T}(d_p))$. Finally, the function $\hat{p}$ satisfies that $\hat{p}([(2, -3)]) = \hat{p}([(-2, 3)]) = 0$. Whence $\hat{p}$ is a prenorm on $\mathbb{R}^2/A$ but is not a quasi-norm.

3. Bicomplete quasi-normed quotient cones

Let us recall that a (n extended) quasi-metric space (Y, q) is said to be a bicompletion of the (extended) quasi-metric space (X, d) if (Y, q) is a bicomplete (extended) quasi-metric space such that (X, d) is isometric to a dense subspace of the (extended) metric space (Y, q^s). It is well-known that each (extended) quasi-metric space (X, d) has an (up to isometry) unique bicompletion $(\tilde{X}, \tilde{d})$ (see [3,28]).

In [17] it was introduced that the notion of bicomplete quasi-normed cone and the construction of the bicompleteness was given. Following [17], a quasi-normed cone (X, p) is said to be *bicomplete* if the induced extended quasi-metric d_p is bicomplete. In connection with bicompleteness of quasi-normed structures, some results for quasi-normed monoids and asymmetric normed linear spaces may be found in [23] and [9].

In this section we characterize the bicompleteness of the quasi-normed quotient cone in terms of the bicompleteness of the original quasi-normed cone.

Lemma 10. *Let (X, p) be a quasi-normed cone. Then $d_{\hat{p}}([x], [y]) \leq d_p(x, y)$ for all $x, y \in X$.*

Proof. We distinguish two cases:

Case 1. $d_p(x, y) = +\infty$. Then $d_{\hat{p}}([x], [y]) = +\infty$, because otherwise we have that there exists $[z] \in X/Y$ such that $[y] = [x] + [z]$. It follows that $y = x + z + g$ for any $g \in G_Y$. Hence we obtain that $d_p(x, y) \leq p(z + g)$, a contradiction with the hypothesis.

Case 2. Now we suppose that $d_p(x, y) < +\infty$. Then, given $\varepsilon > 0$, there exists $c_\varepsilon \in X$ such that $y = x + c_\varepsilon$ and $p(c_\varepsilon) < d_p(x, y) + \varepsilon$. Therefore $[y] = [x] + [c_\varepsilon]$ with $d_{\hat{p}}([x], [y]) \leq \hat{p}([c_\varepsilon]) < p(c_\varepsilon) < d_p(x, y) + \varepsilon$. ∎

Remark 11. Note that if X is a cancellative cone, then $x = y + z$ and $y = x + w$ imply $w = -z$ for all $x, y, z, w \in X$.

The next theorem provides necessary and sufficient conditions for bicompleteness of the quasi-normed quotient cone.

Theorem 12. *Let (X, p) be a quasi-normed cone and let Y be a subcone of X such that G_Y is closed in $(X, \mathcal{T}(d_p))$. Then (X, p) is bicomplete if and only if $(G_Y, p|_{G_Y})$ and $(X/Y, \hat{p})$ are bicomplete.*

Proof. First we assume that (X, p) is bicomplete. Let $([x_n])_{n\in\mathbb{N}}$ be a Cauchy sequence in the extended metric space $(X/Y, (d_{\hat{p}})^s)$. Then there exists a subsequence $([x_{n_i}])_{i\in\mathbb{N}}$ such that, given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ with $(d_{\hat{p}})^s([x_{n_i}], [x_{n_{i+1}}]) < \varepsilon$ whenever $n_i \geq n_0$. It follows that there exist $[a_{n_{i,i+1}}], [b_{n_{i,i+1}}] \in X/Y$, such that

$$[x_{n_i}] = [x_{n_{i+1}}] + [a_{n_{i,i+1}}], \quad [x_{n_{i+1}}] = [x_{n_i}] + [b_{n_{i+1,i}}]$$

and $\max\{\hat{p}([a_{n_{i,i+1}}]), \hat{p}([ba_{n_{i+1,i}}])\} < \varepsilon$ for all $n_i \geq n_0$. By Remark 11, $[a_{n_{i,i+1}}] = -[b_{n_{i,i+1}}]$ and by Remark 4, $-[a_{n_{i,i+1}}] = [-a_{n_{i,i+1}}]$. Consequently, there is $g_{n_{i,i+1}} \in G_Y$ which satisfies

$$x_{n_i} = x_{n_{i+1}} + a_{n_{i,i+1}} + g_{n_{i,i+1}}, \quad x_{n_{i+1}} = x_{n_i} + (-(a_{n_{i+1,i}} + g_{n_{i,i+1}}))$$

with $\max\{p(a_{n_{i,i+1}} + g_{n_{i,i+1}}), p(-(a_{n_{i+1,i}} + g_{n_{i,i+1}}))\} < \varepsilon$ whenever $n_i \geq n_0$. Hence $(x_{n_i})_{i\in\mathbb{N}}$ is a Cauchy sequence in the extended metric space $(X, (d_p)^s)$. Since (X, p) is a bicomplete quasi-normed cone there exists $x \in X$ such that $\lim_{i\to+\infty}(d_p)^s(x, x_{n_i}) = 0$. Thus, by Lemma 10, $\lim_{i\to+\infty}(d_{\hat{p}})^s([x], [x_{n_i}]) \leq \lim_{i\to+\infty}(d_p)^s(x, x_{n_i}) = 0$. Therefore $([x_n])_{n\in\mathbb{N}}$ is a convergent sequence so that $(X/Y, \hat{p})$ is a bicomplete quasi-normed cone. Furthermore, if (X, p) is bicomplete we deduce that all Cauchy sequence in $(G_Y, (d_{p|_{G_Y}})^s)$ are convergent. Since the convergence with respect to $\mathcal{T}(d_{p|_{G_Y}})^s)$ implies convergence with respect to $\mathcal{T}(d_{p|_{G_Y}})$ and G_Y is closed in $(X, \mathcal{T}(d_p))$ we obtain that $(G_Y, p|_{G_Y})$ is bicomplete.

Conversely, let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(X, (d_p)^s)$. Given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d_p(x_n, x_m) < \varepsilon$ whenever $n, m \geq n_0$. Whence there exists $a_{nm} \in X$ such that

$$x_n = x_m + a_{nm} \quad \text{and} \quad x_m = x_n + (-a_{nm})$$

with $\max\{p(a_{nm}), p(-a_{nm})\} < \varepsilon$ whenever $n, m \geq n_0$. By Lemma 10 we obtain that $(d_{\hat{p}})^s([x_n], [x_m]) \leq (d_p)^s(x_n, x_m) = \max\{p(a_{nm}), p(-a_{nm})\} < \varepsilon$ whenever $n, m \geq n_0$. It follows that $([x_n])_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X/Y, (d_{\hat{p}})^s)$. Since $(X/Y, \hat{p})$ is bicomplete, $\lim_{n\to\infty}(d_{\hat{p}})^s([x], [x_n]) = 0$ for any $[x] \in X/Y$. Hence there exist $n_1 \in \mathbb{N}$ and $[a_n] \in X/Y$ such that $[x] = [x_n] + [a_n]$, $[x_n] = [x] + [-a_n]$ and $\max\{\hat{p}([a_n]), \hat{p}([-a_n])\} < \varepsilon$ whenever $n \geq n_1$. Put $n_2 := \max\{n_0, n_1\}$. Then there exists $w_n \in G_Y$ such that $x_n = x + (-a_n) + w_n, x_m = x + (-a_m) + w_m$ and $\max\{p((-a_n) + w_n), p((-a_m) + w_m)\} < \varepsilon$ whenever $n, m \geq n_2$. Thus we deduce that $w_n - a_n = w_m - a_m + a_{nm}$, $w_m - a_m = w_n - a_n + (-a_{nm})$ and $\max\{p(a_{nm}), p(-a_{nm})\} < \varepsilon$ whenever $m, n \geq n_2$. Whence we deduce that $d_p(w_n - a_n, G_Y) \leq \hat{p}([a_n - w_n] = \hat{p}([a_n]) < \varepsilon$ and $d_p(G_Y, w_n - a_n) \leq d_p(w_n - g, w_n - a_n) = p(-a_n + g) < \hat{p}([-a_n]) + \varepsilon < 2\varepsilon$ for all $n \geq n_2$. Since

$$\begin{aligned}\inf_{g\in G_Y}(d_p)^s(g, w_n - a_n) &\leq \inf_{g\in G_Y}\{d_p(w_n - a_n, G_Y) + d_p(G_Y, w_n - a_n)\} \\ &\leq \inf_{g\in G_Y} d_p(w_n - a_n, G_Y) + \inf_{g\in G_Y} d_p(G_Y, w_n - a_n) \\ &= d_p(w_n - a_n, G_Y) + d_p(G_Y, w_n - a_n) < 3\varepsilon,\end{aligned}$$

there exist $g_n \in G_Y$ and $t_n \in X$ such that $g_n = w_n - a_n + t_n$, $w_n - a_n = g_n - t_n$ and $\max\{p(t_n), p(-t_n)\} < 3\varepsilon$ whenever $n \geq n_2$. Thus $g_n = w_n - a_n + t_n = w_m - a_m +$

$a_{nm} + t_n = g_m - t_m + a_{nm} + t_n$ and $g_m = g_n - t_n - a_{nm} + t_m$ with $\max\{p(-t_m + a_{nm} + t_n), p(-t_n - a_{nm} + t_m)\} < 7\varepsilon$ for all $m, n \geq n_2$, so

$$(d_p)^s(g_n, g_m) = \max\{p(-t_m + a_{nm} + t_n), p(-t_n - a_{nm} + t_m)\} < 5\varepsilon$$

whenever $n, m \geq n_2$. It follows that $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the extended metric space $(G_Y, (d_p)^s|_{G_Y})$. Hence $\lim_{n\to\infty} d_p(g, g_n) = 0$ for any $g \in G_Y$ because $(G_Y, p|_{G_Y})$ is bicomplete. Therefore

$$\lim_{n\to\infty} d_p(g, w_n - a_n) = \lim_{n\to\infty} d_p(g, g_n - t_n) = 0$$

and

$$\lim_{n\to\infty} d_p(x + g, x_n) = \lim_{n\to\infty} d_p(x + g, x + w_n - a_n) = 0.$$

This concludes the proof. ■

As a consequence of Theorem 12 we have the following well-known result for normed linear spaces (see [4]).

COROLLARY 13

Let $(X, \|\cdot\|)$ be a normed linear space and let Y be a closed subspace of X. Then $(X, \|\cdot\|)$ is a Banach space if and only if $(Y, \|\cdot\|)$ and $(X/Y, \|\cdot\|_{X/Y})$ are Banach spaces.

4. Continuous linear mappings on quasi-normed cones

Throughout this section we give several results concerning continuous linear mappings, which will be useful in the next section. A mapping from a quasi-normed cone (X, p) to a quasi-normed cone (Y, q) will be called *continuous* if it is continuous from (X, d_p) to (Y, d_q).

In the sequel the set of all continuous linear mappings between quasi-normed cones (X, p) and (Y, q) will be denoted by $CL(X, Y)$. Obviously $CL(X, Y)$ is a cancellative cone for the usual point-wise operations. In particular, when the pair (Y, q) is the quasi-normed cone $(\mathbb{R}, u)$ we will denote $CL(X, \mathbb{R})$ by X^*, and X^* will be called the *dual cone* of X. Note that $f \in X^*$ if and only if it is a linear and upper semicontinuous real-valued function on (X, p). Some results concerning metrization and quasi-metrization for several weak * topologies defined on the dual of a quasi-normed cone and a generalization of Alouglu's theorem to quasi-normed cone context were obtained in [11].

A version of Theorem 15 and Proposition 19 for asymmetric normed linear spaces can be found in [5] and [10].

We omit the easy proof of the following lemma.

Lemma 14. *Let (X, p) and (Y, q) be two quasi-normed cones and let $f: X \to Y$ be a linear mapping. Then $\frac{s}{r} f(B_{d_p}(0, r)) = f(B_{d_p}(0, s))$ for all $r, s > 0$.*

The next result characterizes the continuity of linear mappings defined between quasi-normed cones. In order to help the reader we include the complete proof of the below theorem although the equivalence (1)–(3) has been proved in [11].

Theorem 15. *Let (X, p) and (Y, q) be two quasi-normed cones and let $f: X \to Y$ be a linear mapping. Then the following statements are equivalent*:

(1) *f is continuous.*
(2) *f is bounded in $B_{d_p}(0, r)$ for every $r > 0$.*
(3) *There exists $c > 0$ such that $q(f(x)) \le cp(x)$ for all $x \in X$.*

Proof. First, we prove that (1) implies (2). Given $\varepsilon = 1$, there exists $r > 0$ such that $d_q(0, f(x)) = q(f(x)) < 1$ whenever $d_p(0, x) = p(x) < r$. Thus, $q(f(x)) < 1$ for all $x \in B_{d_p}(0, r)$. Consider $r_0 > 0$. Next we show that f is bounded in $B_{d_p}(0, r_0)$. To this end we only consider that $r_0 > r$, otherwise there is nothing to prove. By Lemma 14, $\frac{r_0}{r} f(B_{d_p}(0, r)) = f(B_{d_p}(0, r_0))$. It follows that $q(f(y)) \le \frac{r_0}{r}$ for all $y \in B_{d_p}(0, r_0)$, and so f is bounded. Since f is bounded in $B_{d_p}(0, r)$ with $r > 0$ and $B_{d_q}(0, 0) \subseteq B_{d_q}(0, r)$, f is bounded in $B_{d_q}(0, 0)$.

Next we show that (2) implies (3). Let $M, r \in \mathbb{R}^+ \backslash \{0\}$ satisfying $q(f(x)) \le M$ for all $x \in X$ such that $p(x) < r$. Let $y \in X$. Now we show that $q(f(y)) \le cp(y)$. We distinguish two cases.

Case 1. $p(y) = 0$. Suppose that $q(f(y)) \neq 0$. Then $0 < q(f(y)) < M$. Hence there exists $\delta > 0$ such that $\delta \cdot q(f(y)) > M$. It follows that $q(f(\delta \cdot y)) > M$ with $p(\delta \cdot y) = \delta p(y) = 0$, which contradicts (2).

Case 2. $p(x) > 0$. Then $p\big(\frac{r}{2} \frac{y}{p(y)}\big) = \frac{r}{2} \frac{p(y)}{p(y)} = \frac{r}{2} < r$. Furthermore, by (2), $q\big(f\big(\frac{r}{2} \frac{y}{p(y)}\big)\big) \le M$. Immediatelly we deduce that $\frac{r}{2} \frac{1}{p(y)} q(f(y)) \le M$. Therefore $q(f(y)) \le \frac{2}{r} M\, p(y)$. Put $c = \frac{2}{r} M$. Then $q(f(y)) \le cp(y)$ for all $y \in X$.

Finally, we prove that (3) implies (1). Indeed, let $\varepsilon > 0$, $\delta = \frac{\varepsilon}{c}$ and $x, y \in X$ such that $d_p(x, y) < \delta$, then there exists $b \in X$ with $y = x + b$ and $p(b) < \delta$. Thus $d_q(f(x), f(y)) \le q(f(b)) \le cp(b) < c\delta = \varepsilon$. This concludes the proof. ∎

Remark 16. Let $(X, +, \cdot)$ be a cancellative cone. Then, $x = 0$ if and only if $\lambda \cdot x = 0$ for any $\lambda > 0$.

Proof. Suppose that $\lambda \cdot x = 0$ for any $\lambda > 0$. Since $\lambda \cdot x + \lambda \cdot y = \lambda \cdot y$ for all $y \in X$, $\frac{1}{\lambda}(\lambda \cdot x) + \frac{1}{\lambda}(\lambda \cdot y) = \frac{1}{\lambda}(\lambda \cdot y)$. Whence $x + y = y$ for all y, and so $x = 0$.

Conversely, consider that $x = 0$. Then, for every $\lambda > 0$, $0 + \frac{1}{\lambda} \cdot z = \frac{1}{\lambda} \cdot z$ for all $z \in X$. Thus $\lambda \cdot 0 + z = z$ for all $z \in X$. Whence $\lambda \cdot 0 = 0$ for all $\lambda > 0$. ∎

Observe that $\overline{B_{d_p}}(x, r) \neq \overline{B_{d_p}(x, r)}^{d_p}$, where $\overline{B_{d_p}}(x, r) = \{y \in X: d_p(x, y) \le r\}$ and $r \ge 0$.

Example 17. Let $(\mathbb{R}, +, \cdot)$ be the real numbers endowed with the usual operations. It is clear that $(\mathbb{R}, u)$ is a quasi-normed cone such that $\overline{B_{d_u}}(0, 1) = (-\infty, 1]$ and $\mathbb{R} \backslash \overline{B_{d_u}}(0, 1) = (1, +\infty)$. So $\overline{B_{d_u}}(0, 1)$ is not a closed set in $(\mathbb{R}, \mathcal{T}(d_u))$.

Lemma 18. *Let (X, p) and (Y, q) be two quasi-normed cones and let $f \in CL(X, Y)$. If $f|_{\bar{B}_{d_p}(0,1)} \equiv 0$ then $f \equiv 0$ on X.*

Proof. Let $x \in X$. We distinguish two cases.

Case 1. $p(x) = 0$. Then $d_p(0, x) = p(x) = 0$ and $x \in \bar{B}_{d_p}(0, 1)$, so $f(x) = 0$.

Case 2. $p(x) \neq 0$. Then $\frac{x}{p(x)} \in \bar{B}_{d_p}(0, 1)$ and $f\left(\frac{x}{p(x)}\right) = 0$. Consequently $\frac{1}{p(x)} f(x) = 0$ and by Remark 16, $f(x) = 0$. ■

PROPOSITION 19

Let (X, p) and (Y, q) be two quasi-normed cones. Then $(CL(X, Y), \|\cdot\|_{p,q})$ is a quasi-normed cone, where $\|f\|_{p,q} := \sup_{x\in\overline{B_{d_p}}(x,1)} q(f(x))$.

Proof. Obviously, the function $\|\cdot\|_{p,q}$ is well-defined by Theorem 15. Next, we only have to show that $\|\cdot\|_{p,q}$ is a quasi-norm on X. Indeed, $\|0\|_{p,q} = 0$. Let $f \in CL(X, Y)$ satisfy $-f \in CL(X, Y)$ such that $\|f\|_{p,q} = \|-f\|_{p,q} = 0$. Then

$$\sup_{x\in\overline{B_{d_p}}(x,1)} q(f(x)) = \sup_{x\in\overline{B_p}(x,1)} q(-f(x)) = 0.$$

Hence, we deduce that $f(x) = -f(x) = 0$ for all $x \in \overline{B_{d_p}}(0, 1)$ and by Lemma 18, $f = 0$.
On the other hand, for each $\lambda \in \mathbb{R}^+$ and $f \in CL(X, Y)$ we obtain

$$\begin{aligned}\|\lambda \cdot f\|_{p,q} &= \sup_{x\in\overline{B_{d_p}}(x,1)} q(\lambda f(x)) = \sup_{x\in\overline{B_{d_p}}(x,1)} \lambda q(f(x)) \\ &= \lambda \sup_{x\in\overline{B_{d_p}}(x,1)} q(f(x)) = \lambda\|f\|_{p,q}.\end{aligned}$$

Finally, we show the triangle inequality. Let $f, g \in CL(X, Y)$. Then, given $\varepsilon > 0$, there exists $x_\varepsilon \in X$ such that $p(x_\varepsilon) \leq 1$ and

$$\begin{aligned}\|f + g\|_{p,q} &= \sup_{x\in\overline{B_{d_p}}(x,1)} q(f(x) + g(x)) < q(f(x_\varepsilon)) + q(g(x_\varepsilon)) + \varepsilon \\ &\leq \|f\|_{p,q} + \|g\|_{p,q} + \varepsilon.\end{aligned}$$

Therefore $\|f + g\|_{p,q} \leq \|f\|_{p,q} + \|g\|_{p,q}$. ■

We omit the easy proof of the following useful results.

PROPOSITION 20

Let (X, p) and (Y, q) be two quasi-normed cones and let $f \in CL(X, Y)$. Then $q(f(x)) \leq \|f\|_{p,q}\, p(x)$ for all $x \in X$.

PROPOSITION 21

Let (X, p), (Y, q) and (Z, w) be three quasi-normed cones. If $f \in CL(X, Y)$ and $g \in CL(Y, Z)$ then $f \circ g \in CL(X, Z)$ and $\|f \circ g\|_{p,w} \leq \|f\|_{p,q}\|g\|_{q,w}$.

PROPOSITION 22

Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be two asymmetric normed linear spaces and let $T: X_1 \to X_2$ be a linear mapping. Then the following assertions are equivalent:

(1) *T is injective and T^{-1} is continuous.*
(2) *There exists a positive constant k such that $k\|x\|_1 \leq \|T(x)\|_2$.*

The key to the proof of Proposition 22 is based on the fact that a linear mapping is injective if and only if $\ker T = \{x \in X\colon T(x) = 0\} = \{0\}$.

As a particular case of Proposition 22 we obtain the well-known result for the case of normed linear spaces (see, for example [12]). However, Example 23 shows that it is possible to construct a noninjective mapping on the quasi-normed cones which satisfies the condition (2) of the preceding proposition. This is due to the fact that there are functions whose ker is exactly the neutral element that are not injective, so that the above proposition does not hold for quasi-normed cones.

Example 23. Motivated by the applications to the analysis of complexity of programs and algorithms given in [29], it is introduced and studied in [26] that the so-called dual complexity space, which consists of the pair $(\mathcal{C}^*, d_{\mathcal{C}^*})$, where

$$\mathcal{C}^* = \left\{ f \in (\mathbb{R}^+)^\omega\colon \sum_{n=0}^{\infty} 2^{-n} f(n) < +\infty \right\},$$

and $d_{\mathcal{C}^*}$ is the quasi-metric on $\mathcal{C}^*$ given by $d_{\mathcal{C}^*}(f, g) = \sum_{n=0}^{\infty} 2^{-n}[(g(n) - f(n)) \vee 0]$.

Several properties of $d_{\mathcal{C}^*}$ are discussed in [26]. In particular, observe that the topology induced by $d_{\mathcal{C}^*}$ is not T_1.

On the other hand, $(\mathcal{C}^*, +, \cdot)$ is clearly a cancellative cone, with neutral element $f_0 \in \mathcal{C}^*$ given by $f_0(n) = 0$ for all $n \in \omega$, where $+$ is the usual pointwise addition and $\cdot$ is the operation defined by $(\lambda \cdot f)(n) = \lambda f(n)$ for all $n \in \omega$.

Let $p\colon \mathcal{C}^* \to \mathbb{R}^+$ defined by $p(f) = \sum_{n=0}^{\infty} 2^{-n} f(n)$. It is routine to see that p is a quasi-norm on $\mathcal{C}^*$. Then the induced extended quasi-metric e_p on $\mathcal{C}^*$ is given by

$$e_p(f, g) = \begin{cases} \sum_{n=0}^{\infty} 2^{-n}(g(n) - f(n)), & \text{if } f \leq g \\ +\infty, & \text{otherwise} \end{cases}.$$

Several properties of the extended quasi-metric e_p have been studied in [21] and [23].

Let $X = \{f \in \mathcal{C}^*\colon f(0) > 0\} \cup \{f_0\}$. It is routine to see that X is a subcone of $\mathcal{C}^*$.

Define $q\colon X \to \mathbb{R}^+$ by $q(f) = f(0)$. Clearly q is a quasi-norm on X.

Let $F\colon X \to \mathcal{C}^*$ defined by $F(f)(0) = f(0)$ and $F(f)(n) = 0$ for all $f \in X$ and $n \in \mathbb{N}$. Obviously F is linear from $(X, +, \cdot)$ to $(\mathcal{C}^*, +, \cdot)$.

Moreover $p(F(f)) = \sum_{n=0}^{\infty} 2^{-n} F(f(n)) = f(0) = q(f)$ for all $f \in X$. So F satisfies the condition (2) of Proposition 22 and $\ker F = \{f_0\}$. However, if $f, g \in X$ satisfy $f(0) = g(0)$ and $f(1) \neq g(1)$, we obtain $F(f) = F(g)$, and thus F is not injective.

In order to obtain a quasi-normed cone version of Proposition 22 we introduce the next result.

PROPOSITION 24

Let (X, p) and (Y, q) be two quasi-normed cones and let $f\colon X \to Y$ be a linear mapping. If f is injective then the following statements are equivalent:

(1) *There exists $c > 0$ such that $p(f^{-1}(y)) \leq cq(y)$ for all $y \in f(X)$.*
(2) *There exists $k > 0$ such that $kp(x) \leq q(f(x))$ for all $x \in X$.*

Proof. First, we assume that the statement (1) holds. Then for every $x \in X$ we have $p(f^{-1}(f(x))) \leq c\,q(f(x))$ and $kp(f^{-1}(f(x))) \leq q(f(x))$, where $k = \frac{1}{c}$.

Now, assume that the statement (2) holds and let $y \in f(X)$. Then there exists $x \in X$ such that $f(x) = y$ and $x = f^{-1}(y)$. Thus $kp(f^{-1}(y)) \leq q(y)$, so that $p(f^{-1}(y)) \leq cq(y)$, with $c = \frac{1}{k}$. ■

We finish the section extending some results related to open continuous linear mappings between normed linear spaces to our context. To this end, we introduce the *p-injectivity*.

DEFINITION 25

Let (X, p) be a quasi-normed cone and let Y be a nonempty subset of X. We will say that $f\colon X \to Y$ is p-injective if $f(x) = f(y)$ implies $p(x) = p(y)$ whenever $x, y \in X$.

Note that from the above definition one gathers that $\ker f \subseteq \ker p$ and that all injective mappings are p-injective. Furthermore, if (X, p) is an asymmetric normed linear space the linearity joint with the p-injectivity implies the injectivity. However, in the next example it is shown that the equivalence is not true for quasi-normed cones.

Example 26. Consider again, the quasi-normed cone (X, q) of Example 23. It is easy to see that the mapping $F\colon X \to \mathcal{C}^*$ defined as in the mentioned example is linear and q-injective, but it is not injective.

Theorem 27. *Let (X, p) and (Y, q) be two quasi-normed cones and let $f\colon X \to Y$ be a p-injective linear mapping. Then the following assertions are equivalent*:

(i) *f is open and onto.*
(ii) *There exists $\delta > 0$ such that $f(\overline{B_{d_p}}(0, 1)) \supseteq \delta \overline{B_{d_q}}(0, 1)$.*
(iii) *There exists $M > 0$ such that, given $y \in Y$, there is $x \in X$ with $f(x) = y$ and $p(x) \leq Mq(y)$.*

Proof. (i) clearly implies (ii). Next, suppose that (ii) holds and let $y \in Y$. We distinguish two cases:

Case 1. $q(y) = 0$. By hypothesis there exists $x \in \overline{B_{d_p}}(0, 1)$ such that $f(x) = \delta y$. Suppose that $p(x) > 0$, then there exists $r > 0$ with $rp(x) > 1$ and

$$f(r \cdot x) = r \cdot f(x) = r \cdot (\delta \cdot y),$$

since $q(r \cdot (\delta \cdot y)) = r\delta q(y) = 0$, $r \cdot (\delta \cdot y) \in \delta\overline{B_{d_q}}(0, 1)$. On the other hand, by (ii) we obtain that there exists $z \in \overline{B_{d_p}}(0, 1)$ such that $f(z) = \delta \cdot (r \cdot y) = f(r \cdot x)$ with $r \cdot y \in \overline{B_{d_q}}(0, 1)$. However $1 < p(rx)$ and $p(z) \leq 1$ which contradicts the fact that f is p-injective. Therefore $p(x) = 0$.

Case 2. $q(y) \neq 0$. Then, let $y' = \frac{\delta \cdot y}{q(y)}$. By hypothesis, there exists $x' \in \overline{B_{d_p}}(0, 1)$ such that $f(x') = y'$. Put $x = \left(\frac{q(y)}{\delta}\right) \cdot x'$. Thus $p(x) \leq q(y)/\delta$.

Next we prove that (iii) implies (i). Let $x \in X$ and let V be a neighborhood of x. Then, by Proposition 1, there exists $\varepsilon > 0$ such that $x + z \in V$ for all z with $p(z) \leq \varepsilon$. Moreover, if $y \in Y$ satisfies $q(y) \leq \varepsilon/M$ we have that there exists $z \in X$ such that $f(z) = y$ and $p(z) \leq \varepsilon$. Hence $f(x) + y = f(x + z) \in f(V)$, and $f(V)$ is a neighborhood of $f(x)$. This concludes the proof. ■

COROLLARY 28

Let (X, p) and (X, q) be two quasi-normed cones. If $f: X \to Y$ is an open, bijective, continuous linear mapping then it is an isomorphism.

The condition of p-injectivity of Theorem 27 cannot be omitted as seen in the following example.

Example 29. Consider the cancellative cones $(\mathbb{R}^2, +, \cdot)$ and $(\mathbb{R}, \pm, \odot)$ endowed with the usual operations. Define $f: \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = x$. It is clear that f is linear and onto. Consider the quasi-norms $p: \mathbb{R}^2 \to \mathbb{R}^+$ and $u: \mathbb{R} \to \mathbb{R}^+$ defined by $p(x, y) = |y|$ and $u(x) = x \vee 0$. Obviously $f(2, 3) = f(2, 4)$ and $p(2, 3) \neq p(2, 4)$, and as a consequence f is not p-injective. Next we show that f holds in (ii) in Theorem 27, but not in (iii). Indeed, we have that $(-\infty, 1] = \overline{B_{d_u}}(0, 1) \subset f(\overline{B_{d_p}}(0, 1)) = \mathbb{R}$ and $|y| = p(0, y) > Mu(f(0, y)) = 0$, for all $M > 0$ and $y \neq 0$.

5. Quasi-normed cones as image of quotient cones and duality

Given a normed linear space $(X, \|\cdot\|)$ and a linear subspace $Y \subset X$, it is defined the *polar of* Y by $Y^0 = \{f \in X^*: f|_Y \equiv 0\}$, where X^* is the dual of X. It is well-known that the polar inherits the linear structure of X^* and it is isometrically isomorphic to the dual of the quotient linear space X/Y whenever Y is closed (see [12]). Next, we introduce a suitable notion of polar which allows us to extend the mentioned property to our context.

Define the mapping $\varphi: X \to X/Y$ by $\varphi(x) = [x]$. Obviously φ is linear and onto. Such a mapping will be called the *quotient mapping of X induced by Y*. Next we show that the mapping φ is, in addition, continuous.

PROPOSITION 30

Let (X, p) be a quasi-normed cone and let Y be a subcone of X such that G_Y is closed in $(X, \mathcal{T}(d_p))$. Then the quotient mapping $\varphi: X \to X/Y$ is continuous.

Proof. Let $x \in X$. Then $\hat{p}(\varphi(x)) = \hat{p}([x]) = \inf_{y \in G_Y} p(x+y) \leq \inf_{y \in G_Y} p(x) + p(y) = p(x)$. ∎

COROLLARY 31

$\|\varphi\|_{p,\hat{p}} \leq 1$.

DEFINITION 32

Let (X, p) be a quasi-normed cone. We will call *polar* of the subcone Y of X to the set $Y^0 = \{f \in X^*: f|_{G_Y} \equiv 0\}$.

Note that as a particular case of our definition we have the polar of a normed linear space, because of $G_Y = Y$ whenever Y is a linear subspace of a normed linear space.

Let (X, p) and (Y, q) be two quasi-normed cones. A linear mapping $T: X \to Y$ is an *isometry* if $q(T(x)) = p(x)$ for all $x \in X$. If, in addition, T is an isomorphism, i.e., T is a continuous linear bijective mapping with T^{-1} continuous, then T is called an *isometric isomorphism.*

Observe that, contrary to the asymmetric linear case, there are isometries on (quasi-) normed cones which are not injective (see Example 23 above). Two quasi-normed cones are called *isometrically isomorphic* (*isomorphic*) if there exists an isometric isomorphism (isomorphism) between them.

PROPOSITION 33

Let (X, p) be a quasi-normed cone and let Y be a subcone of X such that $\overline{G_Y}^{d_p} = G_Y$. Then Y^0 is a subcone of X^* isometrically isomorphic to $(X/Y)^*$.

Proof. We only show that Y^0 is isometrically isomorphic to $(X/Y)^*$. Define the mapping $T\colon (X/Y)^* \to Y^0$ by $(Tf)(x) = f(\varphi(x)) = f([x])$. It is clear that T is a well-defined mapping because:

(1) $(Tf)(g) = f([0]) = 0$ whenever $g \in G_Y$, since $f \in (X/Y)^*$,
(2) Tf is linear on Y for all $f \in (X/Y)^*$, and so is φ,
(3) Tf is continuous for all $f \in (X/Y)^*$, because applying Proposition 20 for each $x \in X$ we have

$$u((Tf)(x)) = u(f(\varphi(x))) \leq \|f\|_{\hat{p},u}\hat{p}(\varphi(x)) \leq \|f\|_{\hat{p},u}\|\varphi\|_{\hat{p},p}p(x).$$

From the definition of T we immediately deduce that it is linear.

On the other hand, we obtain from (3) and Corollary 31 that

$$\|Tf\|_{p,u} = \sup\{u((Tf)(x))\colon p(x) \leq 1\} \leq \|f\|_{\hat{p},u}\|\varphi\|_{\hat{p},p} \leq \|f\|_{\hat{p},u}.$$

So T is continuous.

Since $\tilde{p}([x]) \leq p(x)$ for all $x \in X$, $\|f\|_{\hat{p},u} \leq \|Tf\|_{p,u}$. Hence $\|f\|_{\hat{p},u} = \|Tf\|_{p,u}$ and T is an isometry.

Next we show that T is onto. For each $h \in Y^0$, define $f_h\colon X/Y \to \mathbb{R}$ as $f_h([x]) = h(x)$. Thus f_h is a well-defined function. Indeed,

(1) given $z \in [x]$ such that $z = x + g$ for any $g \in G_Y$ we have

$$f_h([z]) = h(z) = h(x + g) = h(x) + h(g) = h(x) = f_h([x]).$$

(2) f_h is linear, and so is h.
(3) Since for each $x \in X$ and $g \in G_Y$,

$$u(f_h([x])) = u(h(x + g)) \leq \|h\|_{p,u}p(x),$$

we follow that

$$u(f_h([x])) \leq \|h\|_{p,u} \inf\{p(z)\colon z \in [x]\} = \|h\|_{p,u}\hat{p}([x]),$$

and $f_h \in (X/Y)^*$.

It is easy to see that $T(f_h) = h$.

On the other hand, T is injective. Suppose that $(Th) = (Tf)$ with $f, h \in (X/Y)^*$. Then $(Th)(x) = (Tf)(x)$ for all $x \in X$. Whence $h([x]) = f([x])$ for all $x \in X$, and $f = h$.

Finally, we prove that $T^{-1}\colon Y^0 \to (X/Y)^*$ is continuous. Let $h \in Y^0$, then $T^{-1}(h) = f_h$. It follows that

$$u((T^{-1}h)([x])) = u(f_h([x])) \leq \|h\|_{p,u}\hat{p}([x]).$$

So $\|T^{-1}h\|_{\hat{p},u} \leq \|h\|_{p,u}$, and from Theorem 15, T^{-1} is continuous. ■

In the classical context it is showed that, under open continuous linear mappings, every normed linear space is isomorphic to a distinguished quotient linear space (see [12]). To end this section we obtain an asymmetric version of such a result.

Let (X, p) and (Y, q) be two quasi-normed cones and let $T\colon X \to Y$ be an onto linear mapping. Define $\tilde{T}\colon X/\ker T \to Y$ by $\tilde{T}([x]) = T(x)$. It is obvious that $\tilde{T}$ is well-defined, since if $z \in [x]$, then $z = x + g$ for any $g \in G_{\ker T}$ and $\tilde{T}([z]) = T(z) = T(x) + T(g) = T(x)$.

Since T is linear and onto, so is $\tilde{T}$. Furthemore, if $\varphi_{\ker T}$ is the quotient mapping induced by $\ker T$, it is a routine to check that $T = \tilde{T} \circ \varphi_{\ker T}$.

PROPOSITION 34

Let (X, p) and (Y, q) be two quasi-normed cones. Then, $\tilde{T}$ is injective if and only if $T(x) = T(y)$ implies $y \in [x]$.

Proof. First, suppose that $\tilde{T}$ is injective. Then, if $T(x) = T(y)$ for any $x, y \in X$ we follow that $\tilde{T}([x]) = \tilde{T}([y])$. Thus $[x] = [y]$ and there exists $g \in G_{\ker T}$ such that $y = x + g$. So $T(y) = T(x)$.

Now assume that $T(x) = T(y)$ implies $y \in [x]$. Then, if $\tilde{T}([x]) = \tilde{T}([y])$ for any $x, y \in X$ we have that $y \in [x]$. Consequently $[y] = [x]$ and $\tilde{T}$ is injective. ■

DEFINITION 35

Let (X, p) and (Y, q) be two quasi-normed cones. A mapping $T\colon X \to Y$ is G-injective if $T(x) = T(y)$ implies $y \in [x]$.

Note that injectivity implies G-injectivity.

The below examples show that contrary to the case of normed linear spaces [12], $G_{\ker T}$ is not closed in general in $(X, \mathcal{T}(d_p))$.

Example 36. Consider $(\mathbb{R}^2, +, \cdot)$ with the usual operations $+$ and $\cdot$. Let A be the subcone of $\mathbb{R}^2$ given by $A = \{(x, 0)\colon x \in \mathbb{R}\}$. Thus, it is clear that $G_A = A$. On the other hand, the function $p(x, y) = u(x) + u(y)$ is a quasi-norm on $\mathbb{R}^2$, so that $(\mathbb{R}^2, p)$ is a quasi-normed cone. Furthermore, $(2, 3) \in \bar{A}^{d_p}$ because of $d_p((2, 3), (-3n + 2, 0)) = p((-3n, -3)) = u(-3n) + u(-3) = 0$ for all $n \in \mathbb{N}$ and, as a consequence, G_A is not closed in $(\mathbb{R}^2, \mathcal{T}(d_p))$. Define the linear mapping $f(x, y) = y$. Consequently $\ker f = A$, $G_{\ker f} = \ker f$ and $G_{\ker f}$ is not closed in $(X, \mathcal{T}(d_p))$.

Example 37. Let $Y = \{(x, y)\colon x \in \mathbb{R}, y \geq 0\}$ be endowed with the usual operations of $\mathbb{R}^2$. Thus (Y, p) is a quasi-normed cone, where $p(x, y) = y$. Define the linear mapping $f\colon Y \to \mathbb{R}$ by $f(x, y) = y$. Hence $\ker f = \{(x, 0)\colon x \in \mathbb{R}\}$ and $G_{\ker f} = \ker f$. Suppose that the sequence $\{(x_n, 0)\}_{n\in\mathbb{N}} \subset G_{\ker f}$ is convergent to (x, y) in $(Y, \mathcal{T}(d_p))$, then $(x_n, 0) = (x, y) + (z_n, s_n)$ eventually. It follows that $y = s_n = 0$ eventually. Therefore $(x, y) \in G_{\ker f}$, and $G_{\ker f}$ is closed in $(Y, \mathcal{T}(d_p))$.

In the light of the preceding examples we assume in the following desired result that $\overline{G_N}^{d_p} = G_N$.

PROPOSITION 38

Let (X, p) and (Y, q) be two quasi-normed cones. Let $T\colon X \to Y$ be a onto, continuous linear mapping such that $G_{\ker T}$ is closed in $(X, \mathcal{T}(d_p))$. Then the following holds:

(1) $\tilde{T}$ *is continuous.*
(2) $\|\tilde{T}\|_{\hat{p},q} = \|T\|_{p,q}$.
(3) *If in addition* T *is* p*-injective, open and* G*-injective, then* $\tilde{T}$ *is an isomorphism.*

Proof. Let $[x] \in X/\ker f$. By Proposition 20, $q(\tilde{T}([x])) = q(T(z)) \le \|T\|_{p,q}p(z)$ for all $z \in [x]$. Then $q(\tilde{T}([x])) \le \|T\|_{p,q}\hat{p}([x])$. Thus $\tilde{T}$ is continuous and $\|\tilde{T}\|_{\hat{p},q} \le \|T\|_{p,q}$. Since $T = \tilde{T} \circ \varphi_{\ker f}$ we deduce that $\|T\|_{p,q} = \|\tilde{T} \circ \varphi_{\ker f}\|_{p,q}$ and so, by Proposition 21, $\|T\|_{p,q} \le \|\tilde{T}\|_{\hat{p},q}\|\varphi_{\ker f}\|_{p,\hat{p}}$. By Corollary 31, $\|T\|_{p,q} \le \|\tilde{T}\|_{\hat{p},q}$. We conclude that $\|T\|_{p,q} = \|\tilde{T}\|_{\hat{p},q}$. Thus we have proved (1) and (2).

Item (3) remains. Assume that T is, in addition, open and p-injective. Then, by Theorem 27, there exists $M > 0$ such that every element $y \in Y$ is expressible as $T(x)$ for any $x \in X$, holding $p(x) \le Mq(y)$. Since T is G-injective, by Proposition 34, $\tilde{T}$ is injective and $\tilde{T}^{-1}(y) = [x]$. Furthermore, $\hat{p}([x]) \le p(x) \le Mq(y)$ and, by Proposition 24, $\tilde{T}^{-1}$ is continuous with $\|\tilde{T}\|_{\hat{p},q} \le M$. ■

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