

Enveloping σ - C^* -algebra of a smooth Frechet algebra crossed product by \mathbb{R} , K -theory and differential structure in C^* -algebras

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Abstract. Given an m -tempered strongly continuous action α of \mathbb{R} by continuous $*$ -automorphisms of a Frechet $*$ -algebra A , it is shown that the enveloping σ - C^* -algebra $E(S(\mathbb{R}, A^\infty, \alpha))$ of the smooth Schwartz crossed product $S(\mathbb{R}, A^\infty, \alpha)$ of the Frechet algebra A^∞ of C^∞ -elements of A is isomorphic to the σ - C^* -crossed product $C^*(\mathbb{R}, E(A), \alpha)$ of the enveloping σ - C^* -algebra $E(A)$ of A by the induced action. When A is a hermitian Q -algebra, one gets K -theory isomorphism $RK_*(S(\mathbb{R}, A^\infty, \alpha)) = K_*(C^*(\mathbb{R}, E(A), \alpha))$ for the representable K -theory of Frechet algebras. An application to the differential structure of a C^* -algebra defined by densely defined differential seminorms is given.

Keywords. Frechet $*$ -algebra; enveloping σ - C^* -algebra; smooth crossed product; m -tempered action; K -theory; differential structure in C^* -algebras.

1. Introduction

Given a strongly continuous action α of \mathbb{R} by continuous $*$ -automorphisms of a Frechet $*$ -algebra A , several crossed product Frechet algebras can be constructed [11,14]. They include the smooth Schwartz crossed product $S(\mathbb{R}, A, \alpha)$, the L^1 -crossed products $L^1(\mathbb{R}, A, \alpha)$ and $L^1_{|\cdot|}(\mathbb{R}, A, \alpha)$, and the σ - C^* -crossed product $C^*(\mathbb{R}, A, \alpha)$. Let $E(A)$ denote the enveloping σ - C^* -algebra of A [1,6]; and (A^∞, τ) denote the Frechet $*$ -algebra consisting of all C^∞ -elements of A with the C^∞ -topology τ ([14], Appendix I). The following theorem shows that for a smooth action, the enveloping algebra of smooth crossed product is the continuous crossed product of the enveloping algebra.

Theorem 1. *Let α be an m -tempered strongly continuous action of \mathbb{R} by continuous $*$ -automorphisms of a Frechet $*$ -algebra A . Let A admit a bounded approximate identity which is contained in A^∞ and which is a bounded approximate identity for the Frechet algebra A^∞ . Then $E(S(\mathbb{R}, A^\infty, \alpha)) \cong E(L^1_{|\cdot|}(\mathbb{R}, A^\infty, \alpha)) \cong C^*(\mathbb{R}, E(A), \alpha)$. Further, if α is isometric, then $E(L^1(\mathbb{R}, A, \alpha)) \cong C^*(\mathbb{R}, E(A), \alpha)$.*

Notice that neither $L^1(\mathbb{R}, A, \alpha)$ nor $S(\mathbb{R}, A^\infty, \alpha)$ need be a subalgebra of $C^*(\mathbb{R}, E(A), \alpha)$. A particular case of Theorem 1 when A is a dense subalgebra of C^* -algebra has been treated in [2]. Let RK_* (respectively K_*) denote the representable K -theory functor (respectively K -theory functor) on Frechet algebras [10]. We have the following isomorphism of K -theory, obtained without direct appeal to spectral invariance.

Theorem 2. *Let A be as in the statement of Theorem 1. Assume that A is hermitian and a Q -algebra. Then $RK_*(S(\mathbb{R}, A^\infty, \alpha)) \cong K_*(C^*(\mathbb{R}, E(A), \alpha))$. Further if the action α is isometric on A , then $RK_*(L^1(\mathbb{R}, A, \alpha)) \cong K_*(C^*(\mathbb{R}, E(A), \alpha))$.*

We apply this to the differential structure of a C^* -algebra. Let α be an action of \mathbb{R} on a C^* -algebra A leaving a dense $*$ -subalgebra \mathcal{U} invariant. Let $T \sim (T_k)_0^\infty$ be a differential $*$ -seminorm on \mathcal{U} in the sense of Blackadar and Cuntz [5] with $T_0(x) = \|\cdot\|$ the C^* -norm from A . Let T be α -invariant. Let $\mathcal{U}_{(k)}$ be the completion of \mathcal{U} in the submultiplicative $*$ -norm $p_k(x) = \sum_{i=0}^k T_i(x)$. The differential Frechet $*$ -algebra defined by T is $\mathcal{U}_\tau = \varprojlim \mathcal{U}_{(k)}$, the inverse limit of Banach $*$ -algebras $\mathcal{U}_{(k)}$.

Now consider $\tilde{\mathcal{U}}$ to be the α -invariant smooth envelope of \mathcal{U} defined to be the completion of \mathcal{U} in the collection of all α -invariant differential $*$ -seminorms. Notice that neither \mathcal{U}_τ nor $\tilde{\mathcal{U}}$ is a subalgebra of A , though each admits a continuous surjective $*$ -homomorphism onto A induced by the inclusion $\mathcal{U} \rightarrow A$. There exists actions of \mathbb{R} on each of \mathcal{U}_τ and $\tilde{\mathcal{U}}$ induced by α . The following is a smooth Frechet analogue of Connes' analogue of Thom isomorphism [7]. It supplements an analogues result in [11].

Theorem 3.

- (a) $RK_*(S(\mathbb{R}, \mathcal{U}_\tau^\infty, \alpha)) = K_{*+1}(A)$.
- (b) Assume that $\tilde{\mathcal{U}}$ is metrizable. Then $RK_*(S(\mathbb{R}, \tilde{\mathcal{U}}, \alpha)) = K_{*+1}(A)$.

2. Preliminaries and notations

A Frechet $*$ -algebra (A, t) is a complete topological involutive algebra A whose topology t is defined by a separating sequence $\{\|\cdot\|_n : n \in \mathbb{N}\}$ of seminorms satisfying $\|xy\|_n \leq \|x\|_n \|y\|_n$, $\|x^*\|_n = \|x\|_n$, $\|x\|_n \leq \|x\|_{n+1}$ for all x, y in A and all n in \mathbb{N} . If each $\|\cdot\|_n$ satisfies $\|x^*x\|_n = \|x\|_n^2$ for all x in A , then A is a σ - C^* -algebra [9]. A is called a Q -algebra if the set of all quasi-regular elements of A is an open set. For each n in \mathbb{N} , let A_n be the Hausdorff completion of $(A, \|\cdot\|_n)$. There exists norm decreasing surjective $*$ -homomorphisms $\pi_n: A_{n+1} \rightarrow A_n$, $\pi_n(x + \ker \|\cdot\|_{n+1}) = x + \ker \|\cdot\|_n$ for all $x \in A$. Then the sequence

$$A_1 \xleftarrow{\pi_1} A_2 \xleftarrow{\pi_2} A_3 \xleftarrow{\pi_3} \dots \xleftarrow{\pi_{n-1}} A_n \xleftarrow{\pi_n} A_{n+1} \xleftarrow{\dots}$$

is an inverse limit sequence of Banach $*$ -algebras and $A = \varprojlim A_n$, the inverse limit of Banach $*$ -algebras. Let $\text{Rep}(A)$ be the set of all $*$ -homomorphisms $\pi: A \rightarrow B(H_\pi)$ of A into the C^* -algebras $B(H_\pi)$ of all bounded linear operators on Hilbert spaces H_π . Let

$$\text{Rep}_n(A) := \{\pi \in \text{Rep}(A) : \text{there exists } k > 0 \text{ such that}$$

$$\|\pi(x)\| \leq k\|x\|_n \text{ for all } x\}.$$

Then $|x|_n := \sup\{\|\pi(x)\| : \pi \in \text{Rep}_n(A)\}$ defines a C^* -seminorm on A . The star radical of A is

$$\text{srad}(A) = \{x \in A : |x|_n = 0 \text{ for all } n \text{ in } \mathbb{N}\}.$$

The enveloping σ - C^* -algebra $(E(A), \tau)$ of A is the completion of $A/\text{srad}(A)$ in the topology τ defined by the C^* -seminorms $\{\|\cdot\|_n : n \in \mathbb{N}\}$, $|x + \text{srad}(A)|_n = |x|_n$ for x in A .

Let α be a strongly continuous action of \mathbb{R} by continuous $*$ -automorphisms of A . The C^∞ -elements of A for the action α are

$$A^\infty := \{x \in A : t \rightarrow \alpha_t(x) \text{ is a } C^\infty\text{-function}\}.$$

It is a dense $*$ -subalgebra of A which is a Frechet algebra with the topology defined by the submultiplicative $*$ -seminorms

$$\|x\|_{k,n} = \|x\|_n + \sum_{j=0}^k (1/j!) \|\delta^j x\|_n, \quad n \in \mathbb{N}, \quad k \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\}$$

where δ is the derivation $\delta(x) = (d/dt)\alpha_t(x)|_{t=0}$. By Theorem A.2 of [14], α leaves A^∞ invariant and each α_t restricted to A^∞ gives a continuous $*$ -automorphism of the Frechet algebra A^∞ . The action α is *smooth* if $A^\infty = A$.

2.1 Smooth Schwartz crossed product [14]

Assume that α is *m-tempered* in the sense that for each $n \in \mathbb{N}$, there exists a polynomial P_n such that $\|\alpha_r(x)\|_n \leq P_n(r)\|x\|_n$ for all $r \in \mathbb{R}$ and all $x \in A$. Let $S(\mathbb{R})$ be the Schwartz space. The completed (projective) tensor product $S(\mathbb{R}) \otimes A = S(\mathbb{R}, A)$ consisting of A -valued Schwartz functions on \mathbb{R} is a Frechet algebra with the twisted convolution

$$(f * g)(r) = \int_{\mathbb{R}} f(s)\alpha_s(g(r - s))ds$$

called the *smooth Schwartz crossed product* by \mathbb{R} denoted by $S(\mathbb{R}, A, \alpha)$. The algebra $S(\mathbb{R}, A^\infty, \alpha)$ is a Frechet $*$ -algebra with the involution $f^*(r) = \alpha_r(f(-r)^*)$ (Corollary 4.9 of [14]) whose topology τ_s is defined by the seminorms

$$\|f\|_{n,l,m} = \sum_{i+j=n} \int_{\mathbb{R}} (1 + |r|)^i \|f^{(j)}(r)\|_{l,m} dr, \quad n \in \mathbb{Z}^+, \quad l \in \mathbb{Z}^+, \quad m \in \mathbb{N}$$

where

$$\|f^{(j)}(r)\|_{l,m} = \sum_{k=0}^l (1/k!) \|\delta^k(\alpha_s((d^j/dr^j)f(r)))|_{s=0}\|_m$$

(Theorem 3.1.7 of [14], [11]). These seminorms are submultiplicative if α is isometric on A in the sense that $\|\alpha_r(x)\|_n = \|x\|_n$ for all $n \in \mathbb{N}$ and all $x \in A$.

2.2 L^1 -crossed products [11,14]

Let F_d be the set of all functions $f: \mathbb{R} \rightarrow A$ for which

$$\|f\|_{d,m} := \int_{\mathbb{R}} (1 + |r|)^d \|f(r)\|_m dr < \infty$$

for all m in \mathbb{N} . Here \int denotes the upper integral. Let \mathbb{L}_d be the closure in F_d of the set of all measurable simple functions $f: \mathbb{R} \rightarrow A$ in the topology on F_d given by the seminorms $\{\|\cdot\|_{d,m} : m \in \mathbb{N}\}$. Let $N_d = \cap\{\ker \|\cdot\|_{d,m} : m \in \mathbb{N}\}$. Then $N_d = N_{d+1}$; $L_d := \mathbb{L}_d/N_d$

is complete in $\{\|\cdot\|_{d,m}: m \in \mathbb{N}\}$ and $L_{d+1} \rightarrow L_d$ continuously. The space of $|\cdot|$ -rapidly vanishing L^1 -functions from \mathbb{R} to A is $L^1_{|\cdot|}(\mathbb{R}, A, \alpha) := \cap\{L_d: d \in \mathbb{Z}^+\}$, a Frechet algebra with the topology given by the seminorms $\{\|\cdot\|_{d,m}: m \in \mathbb{N}, d \in \mathbb{Z}^+\}$ and with twisted convolution. Assume that α is isometric on $(A, \{\|\cdot\|_n\})$. Then the completed projective tensor product $L^1(\mathbb{R}) \otimes A = L^1(\mathbb{R}, A)$ is a Frechet $*$ -algebra with twisted convolution and the involution $f \rightarrow f^*$. This L^1 -crossed product is denoted by $L^1(\mathbb{R}, A, \alpha)$. Notice that α is isometric on $(A^\infty, \{\|\cdot\|_{n,m}\})$ also, so that the Frechet $*$ -algebra $L^1(\mathbb{R}, A^\infty, \alpha)$ is defined; and then the induced actions $(\alpha_r f)(s) = \alpha_r(f(s))$ on $L^1(\mathbb{R}, A^\infty, \alpha)$ and on $L^1(\mathbb{R}, A, \alpha)$ are also isometric.

2.3 σ - C^* -crossed product

Assume that α is isometric. We define the σ - C^* -crossed product $C^*(\mathbb{R}, A, \alpha)$ of A by \mathbb{R} to be the enveloping σ - C^* -algebra $E(L^1(\mathbb{R}, A, \alpha))$ of $L^1(\mathbb{R}, A, \alpha)$.

3. Technical lemmas

Lemma 3.1. *Let α be m -tempered on A . Then α extends as a strongly continuous isometric action of \mathbb{R} by continuous $*$ -automorphisms of the σ - C^* -algebra $E(A)$.*

Proof. By the m -temperedness of α , for each $n \in \mathbb{N}$, there exists a polynomial P_n such that for all $x \in A$ and all $r \in \mathbb{R}$, $\|\alpha_r(x)\|_n \leq P_n(r)\|x\|_n$. Let $r \in \mathbb{R}$. Let $x \in \text{srad}(A)$. Then for all $\pi \in \text{Rep}(A)$, $\pi(x) = 0$, so that $\sigma(\alpha_r(x)) = 0$ for all $\sigma \in \text{Rep}(A)$, hence $\alpha_r(x) \in \text{srad}(A)$. Thus $\alpha_r(\text{srad}(A)) \subseteq \text{srad}(A)$, and the map

$$\tilde{\alpha}_r: A/\text{srad}(A) \rightarrow A/\text{srad}(A), \quad \tilde{\alpha}_r([x]) = [\alpha_r(x)],$$

where $[x] = x + \text{srad}(A)$, is a well-defined $*$ -homomorphism. Further, let $\tilde{\alpha}_r[x] = 0$. Then $\alpha_r(x) \in \text{srad}(A)$. Hence $x = \alpha_{-r}(\alpha_r(x)) \in \text{srad}(A)$, $[x] = 0$. Thus $\tilde{\alpha}_r$ is one-to-one, which is clearly surjective. Now, for each $n \in \mathbb{N}$, and for all $x \in A$,

$$|\tilde{\alpha}_r[x]|_n = |[\alpha_r(x)]|_n \leq \|\alpha_r(x)\|_n \leq P_n(r)\|x\|_n.$$

Since, by definition, $|\cdot|_n$ is the greatest C^* -seminorm on $A/\text{srad}(A)$ satisfying that for some $k_n > 0$, $||[z]|_n \leq k_n \|z\|_n$ for all $z \in A$, it follows that $|\tilde{\alpha}_r[x]|_n \leq |[x]|_n$ for all x in A . Hence

$$|[x]|_n \leq |\tilde{\alpha}_{-r}(\tilde{\alpha}_r[x])|_n = |\tilde{\alpha}_{-r}[\alpha_r(x)]|_n \leq |[\alpha_r(x)]|_n = |\tilde{\alpha}_r[x]|_n$$

showing that $|\tilde{\alpha}_r[x]|_n = |[x]|_n$ for all $x \in A$, $r \in \mathbb{R}$, $n \in \mathbb{N}$. It follows that $\tilde{\alpha}_r$ extends as a $*$ -automorphism $\tilde{\alpha}_r: E(A) \rightarrow E(A)$ satisfying $|\tilde{\alpha}_r(z)|_n = |z|_n$ for all $z \in A$ and all $n \in \mathbb{N}$; and $\tilde{\alpha}: \mathbb{R} \rightarrow \text{Aut}^*(E(A))$, $r \rightarrow \tilde{\alpha}_r$ defines an isometric action of \mathbb{R} on $E(A)$. We verify that $\tilde{\alpha}$ is strongly continuous. Let $z \in E(A)$. It is sufficient to prove that the map $f: \mathbb{R} \rightarrow E(A)$, $f(r) = \alpha_r(z)$ is continuous at $r = 0$. Choose $z_n = [x_n]$ in $A/\text{srad}(A)$ such that $z_n \rightarrow z$ in $E(A)$. Fix $k \in \mathbb{N}$, $\varepsilon > 0$. Choose n_0 in \mathbb{N} such that $|z_{n_0} - z|_k < \varepsilon/3$ with $z_{n_0} = [x_{n_0}]$. Then for all $r \in \mathbb{R}$, $|\tilde{\alpha}_r(z) - \tilde{\alpha}_r(z_{n_0})|_k = |z - z_{n_0}|_k < \varepsilon/3$. Since α is strongly continuous, there exists a $\delta > 0$ such that $|r| < \delta$ implies that $\|\alpha_r(x_0) - x_0\|_k < \varepsilon/3$. Then for all such r , $|\tilde{\alpha}_r(z) - z|_k < \varepsilon$ showing the desired continuity of f . This completes the proof. \square

Notation. Henceforth we denote the action $\tilde{\alpha}$ by α .

A covariant representation of the Frechet algebra dynamical system (\mathbb{R}, A, α) is a triple (π, U, H) such that

- (a) $\pi: A \rightarrow B(H)$ is a $*$ -homomorphism;
- (b) $U: \mathbb{R} \rightarrow \mathcal{U}(H)$ is a strongly continuous unitary representation of \mathbb{R} on H ; and
- (c) $\pi(\alpha_t(x)) = U_t \pi(x) U_t^*$ for all $x \in A$ and all $t \in \mathbb{R}$.

The following is an analogue of Proposition 7.6.4, p. 257 of [12] which can be proved along the same lines. Let $C_c^\infty(\mathbb{R}, A^\infty) = C_c^\infty(\mathbb{R}) \otimes A^\infty$ (completed projective tensor product) be the space of all A^∞ -valued C^∞ -functions on \mathbb{R} with compact supports.

Lemma 3.2. Let A have a bounded approximate identity (e_i) contained in A^∞ which is also a bounded approximate identity for the Frechet algebra A^∞ . (In particular, let A be unital.)

- (a) If (π, U, H) is a covariant representation of $(\mathbb{R}, A^\infty, \alpha)$, then there exists a non-degenerate $*$ -representation $(\pi \times U, H)$ of $S(\mathbb{R}, A^\infty, \alpha)$ such that

$$(\pi \times U)y = \int_{\mathbb{R}} \pi(y(t))U_t dt$$

for every y in $C_c^\infty(\mathbb{R}, A^\infty)$. The correspondence $(\pi, U, H) \rightarrow (\pi \times U, H)$ is bijective onto the set of all non-degenerate $*$ -representations of $S(\mathbb{R}, A^\infty, \alpha)$.

- (b) Let α be isometric. Then the above gives a one-to-one correspondence between the covariant representations of (\mathbb{R}, A, α) and non-degenerate $*$ -representations of each of $L^1(\mathbb{R}, A^\infty, \alpha)$ and $L^1(\mathbb{R}, A, \alpha)$.

Lemma 3.3. $E(A^\infty) = E(A)$; and for all k in \mathbb{Z}^+ , n in \mathbb{N} , $\|_{n,k} = \|_n$.

Proof. Consider the inverse limit $A = \varprojlim A_n$ as in the Introduction. Since α satisfies $\|\alpha_r(x)\|_n \leq P_n(r)\|x\|_n$ for all $x \in \mathbb{R}$, all $r \in A$ and all $n \in \mathbb{N}$, it follows that for each n , α ‘extends’ uniquely as a strongly continuous action $\alpha^{(n)}$ of \mathbb{R} by continuous $*$ -automorphisms of the Banach $*$ -algebra A_n . Let $(A_{n,m}, \|\cdot\|_{n,m})$ be the Banach algebra consisting of all C^m -elements y of A_n with the norm $\|\cdot\|_{n,m} = \|y\|_n + \sum_{i=1}^m (1/i!) \|\delta^i(x)\|_n$. Let $(A_n^\infty, \{\|\cdot\|_{m,n}: m \in \mathbb{Z}^+\})$ be the Frechet algebra consisting of all C^∞ -elements of A_n for the action $\alpha^{(n)}$. Then

$$A^\infty = \varprojlim A_n^\infty = \varprojlim \varprojlim A_{m,n} = \varprojlim A_{n,n}.$$

By Theorem 2.2 of [15], each $A_{m,n}$ is dense and spectrally invariant in A_n . Hence each $A_{n,m}$ is a Q -normed algebra in the norm $\|\cdot\|_{n,m}$ of A_n .

Let $\pi: A^\infty \rightarrow B(H)$ be a $*$ -representation of A on a Hilbert space H . Since the topology of A^∞ is determined by the seminorms

$$\|x\|_{n,n} = \|x\|_n + \sum_{j=1}^n (1/j!) \|\delta^j(x)\|_n, \quad n \in \mathbb{N}$$

it follows that for some $k > 0$, $\|\pi(x)\| \leq k\|x\|_{n,n}$ for all $a \in A^\infty$. Hence π defines a $*$ -homomorphism $\pi: (A_{n,n}, \|\cdot\|_{n,n}) \rightarrow B(H)$ satisfying $\|\pi(x)\| \leq k\|x\|_{n,n}$ for all x in

$A_{n,n}$. Since $(A_{n,n}, \| \cdot \|_n)$ is a Q -normed $*$ -algebra, this map π is continuous in the norm $\| \cdot \|_n$ on $A_{n,n}$. In fact, for all x in A^∞ ,

$$\begin{aligned} \|\pi(x)\|^2 &= \|\pi(x^*x)\| = r_{B(H)}(\pi(x^*x)) \leq r_{A_{n,n}}(\pi(x^*x + \ker \| \cdot \|_n)) \\ &\leq \|x^*x + \ker \| \cdot \|_n\| = \|x^*x\|_n \leq \|x\|^2. \end{aligned}$$

Thus $\|\pi(x)\| \leq \|x\|_n$ for all x in A^∞ . Since A^∞ is dense in A , π can be uniquely extended as a $*$ -representation $\pi: A \rightarrow B(H)$ satisfying that $\|\pi(x)\| \leq \|x\|_n$ for all x in A . Then by the definition of the C^* -seminorm $| \cdot |_n$ on A , π extends as a continuous $*$ -homomorphism $\tilde{\pi}: E(A) \rightarrow B(H)$ such that $\|\tilde{\pi}(x)\| \leq |x|_n$ for all x in $E(A)$. This also implies that $E(A^\infty) = E(A)$ and $| \cdot |_{n,m} = | \cdot |_n$ for all n, m .

Lemma 3.4. Let B be a σ - C^* -algebra. Let $j: A \rightarrow E(A)$ be $j(x) = x + \text{srad}(A)$. Let $\pi: A \rightarrow B$ be a $*$ -homomorphism. Then there exists a unique $*$ -homomorphism $\tilde{\pi}: E(A) \rightarrow B$ such that $\pi = \tilde{\pi} \circ j$.

This follows immediately by taking $B = \varprojlim B_n$, where B_n 's are C^* -algebras, and by the universal property of $E(A)$.

4. Proof of Theorem 1

Step I. $\text{Rep}(S(\mathbb{R}, A^\infty, \alpha)) = \text{Rep}(S(\mathbb{R}, E(A), \alpha)) = \text{Rep}(L^1(\mathbb{R}, E(A), \alpha))$ up to one-to-one correspondence.

By Lemma 3.1, the Frechet algebras $S(\mathbb{R}, E(A), \alpha)$ and $L^1(\mathbb{R}, E(A), \alpha)$ are $*$ -algebras with the continuous involution $y \rightarrow y^*, y^*(t) = \alpha_t(y(-t))^*$. By Lemma 3.2, $\text{Rep}(S(\mathbb{R}, E(A), \alpha)) = \text{Rep}(L^1(\mathbb{R}, E(A), \alpha))$ each identified with the set of all covariant representations. Let $\rho: S(\mathbb{R}, A^\infty, \alpha) \rightarrow B(H)$ be in $\text{Rep}(S(\mathbb{R}, A^\infty, \alpha))$. There exists $c > 0$ and appropriate n, l, m such that for all y ,

$$\|\rho(y)\| \leq c\|y\|_{n,l,m} = c \sum_{i+j=n} \int_R (1 + |r|)^i \|y^{(j)}(r)\|_{l,m} dr. \tag{1}$$

By Lemma 3.2, there exists a covariant representation (π, U, H) of $(\mathbb{R}, A^\infty, \alpha)$ on H such that $\rho = \pi \times U$. Thus $\pi: A^\infty \rightarrow B(H)$ is a $*$ -homomorphism and $U: \mathbb{R} \rightarrow \mathcal{U}(H)$ is a strongly continuous unitary representation such that

(i) $\rho(f) = \int_R \pi(f(t))U_t dt$ for all f in $S(\mathbb{R}, A^\infty, \alpha)$, (2)

(ii) $\pi(\alpha_t(x)) = U_t \pi(x) U_t^*$ for all $x \in A^\infty, t \in \mathbb{R}$, (3)

(iii) there exists $K > 0$ such that $\|\pi(x)\| \leq k\|x\|_{l,m}$ for all $x \in A^\infty$.

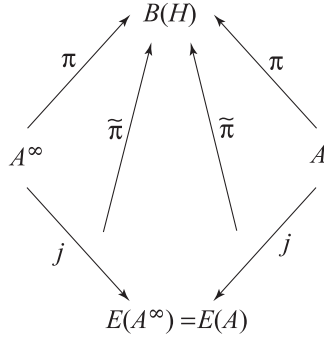
The l, m in (iii) are the same as in (1). Let $\{ | \cdot |_{l,m}: l \text{ in } \mathbb{Z}^+, m \text{ in } \mathbb{N} \}$ be the sequence of C^* -seminorms on A^∞ (and also on $E(A^\infty)$ via $\text{srad } A^\infty$) which are defined by the submultiplicative $*$ -seminorms $\{ \| \cdot \|_{l,m}: l \text{ in } \mathbb{Z}^+, m \text{ in } \mathbb{N} \}$. Then $| \cdot |_{l,m}$ is the greatest C^* -seminorm on A^∞ satisfying that there exists $M = M_{l,m} > 0$ such that $| \cdot |_{l,m} \leq M \| \cdot \|_{l,m}$. Hence by (iii) above, π can be uniquely extended as a continuous $*$ -homomorphism $\tilde{\pi}: E(A^\infty) \rightarrow B(H)$ such that $\tilde{\pi}(j(x)) = \pi(x)$ for all $x \in A^\infty$; and

$$\|\tilde{\pi}(x)\| \leq |x|_{l,m} \text{ for all } x \in E(A^\infty). \tag{4}$$

Here j is the map $j: A^\infty \rightarrow E(A^\infty)$, $j(x) = x + \text{srad } A^\infty$. Let l denote $\max(l, m)$. Then we have

$$\begin{aligned} \|\rho(y)\| &\leq c\|y\|_{n,l,l} \text{ for all } y \in S(\mathbb{R}, A^\infty, \alpha); \\ \|\pi(x)\| &\leq k\|x\|_{l,l} \text{ for all } x \in A^\infty; \\ \|\tilde{\pi}(z)\| &\leq |z|_{l,l} \text{ for all } z \in E(A^\infty). \end{aligned} \tag{5}$$

By Lemma 3.3, $\tilde{\pi}: E(A) \rightarrow B(H)$ is a $*$ -representation satisfying $\|\tilde{\pi}(x)\| \leq |x|_l$ for all x in $E(A)$. We have the following commutative diagram.



Now, let $\alpha: \mathbb{R} \rightarrow \text{Aut}^* E(A)$ be the action on $E(A)$ induced by α as in Lemma 3.1 satisfying

$$\alpha_t(j(x)) = j(\alpha_t(x)) \text{ for all } x \text{ in } A. \tag{6}$$

Then $(\tilde{\pi}, U, H)$ is a covariant representation of $(\mathbb{R}, E(A), \alpha)$. Indeed, let $x \in A^\infty$, $y = j(x)$. Then for all $t \in \mathbb{R}$,

$$\begin{aligned} \tilde{\pi}(\alpha_t(y)) &= \tilde{\pi}(\alpha_t(j(x))) = \tilde{\alpha}(j(\alpha_t(x))) = \pi(\alpha_t(x)) = U_t \pi(x) U_t^* \\ &= U_t \tilde{\pi}(j(x)) U_t^* = U_t \tilde{\pi}(y) U_t^*. \end{aligned}$$

By the continuity of $\tilde{\pi}$ and α_t , it follows that $\tilde{\pi}(\alpha_t(y)) = U_t \tilde{\pi}(y) U_t^*$ for all $y \in E(A)$ and all $t \in \mathbb{R}$. Hence by Lemma 3.2, $\tilde{\rho} = \tilde{\pi} \times U$ is a non-degenerate $*$ -representation of each of $S(\mathbb{R}, E(A), \alpha)$ and $L^1(\mathbb{R}, E(A), \alpha)$ satisfying, for some constants c and c' , the following (using (5)):

- (iv) For all f in $L^1(\mathbb{R}, E(A), \alpha)$, $\|\rho(f)\| \leq c\|f\|_l = c \int_{\mathbb{R}} |f(t)|_l dt$.
- (v) For all f in $S(\mathbb{R}, E(A), \alpha)$, $\|\tilde{\rho}(f)\| \leq c'\|f\|_{n,l,m}$. (7)

Thus given a $*$ -representation ρ of $S(\mathbb{R}, A^\infty, \alpha)$, there is canonically associated a $*$ -representation $\tilde{\rho}$ of each of $S(\mathbb{R}, E(A), \alpha)$ and $L^1(\mathbb{R}, E(A), \alpha)$.

Conversely, given ρ in $\text{Rep}(S(\mathbb{R}, E(A), \alpha))$, $\rho = \pi \times U$ for a covariant representation (π, U) of $(\mathbb{R}, E(A), \alpha)$, $\pi \circ j$ is a covariant representation of A , and then $(\pi \circ j) \times U$ is in $\text{Rep}(S(\mathbb{R}, A^\infty, \alpha))$.

Step II. The σ - C^* -algebra $C^*(\mathbb{R}, E(A), \alpha)$ is universal for the $*$ -representations of the Frechet algebra $S(\mathbb{R}, A^\infty, \alpha)$.

Let $\tilde{j}: S(\mathbb{R}, A^\infty, \alpha) \rightarrow L^1(\mathbb{R}, E(A), \alpha)$ be the map

$$\begin{aligned} \tilde{j}(f) &= j \circ f = \tilde{f} \text{ (say), i.e.,} \\ \tilde{j}(f)(r) &= j(f(r)) = f(r) + \text{srad}(A)^\infty \text{ for all } r \in \mathbb{R}. \end{aligned} \tag{8}$$

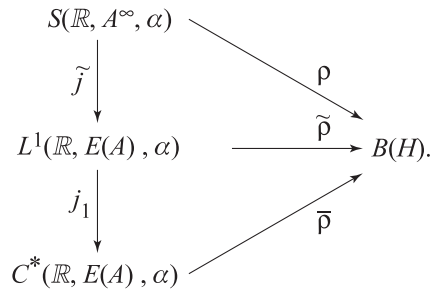
Notice that the map \tilde{j} is defined and is continuous; because $(S(\mathbb{R}, A^\infty, \alpha)) \subset L^1(\mathbb{R}, A^\infty, \alpha) \subset L^1(\mathbb{R}, A, \alpha)$, and for n in \mathbb{N} and m in \mathbb{Z}^+ , all f in $S(\mathbb{R}, A^\infty, \alpha)$,

$$|\tilde{f}(t)|_n \leq \|f(t)\|_n \leq M \|f(t)\|_{m,n}, \text{ and hence}$$

$$\int_{\mathbb{R}} |\tilde{f}(t)|_t dt \leq \int_{\mathbb{R}} \|f(t)\|_{m,n} dt < \infty$$

so that $f \in L^1(\mathbb{R}, E(A), \alpha)$. Let $j_1: L^1(\mathbb{R}, E(A), \alpha) \rightarrow C^*(\mathbb{R}, E(A), \alpha)$ be the natural map $j_1(f) = f + \text{srad}(L^1(\mathbb{R}, E(A), \alpha))$. This gives the continuous *-homomorphism

$$J: j_1 \circ \tilde{j}: S(\mathbb{R}, A^\infty, \alpha) \rightarrow C^*(\mathbb{R}, E(A), \alpha). \tag{9}$$



Let $\rho \in \text{Rep}(S(\mathbb{R}, A^\infty, \alpha))$, $\rho = \pi \times U$ in usual notations with $\pi: A^\infty \rightarrow B(H)$ in $\text{Rep}(E(A))$ such that $\pi = \tilde{\pi} \circ j$. Let $\tilde{\rho}: L^1(\mathbb{R}, E(A), \alpha) \rightarrow B(H)$ be $\tilde{\rho} = \tilde{\pi} \times U$. Then for all f in $S(\mathbb{R}, A^\infty, \alpha)$,

$$\begin{aligned} \tilde{\rho}(\tilde{j}(f)) &= (\tilde{\pi} \times U)(\tilde{j}(f)) = \int_{\mathbb{R}} \tilde{\pi}(\tilde{j}(f)(t))U_t dt = \int_{\mathbb{R}} \tilde{\pi}(j \circ f)(t)U_t dt \\ &= \int_{\mathbb{R}} \tilde{\pi}(j(f(t)))U_t dt = \int_{\mathbb{R}} \tilde{\pi}(f(t) + \text{srad}(A))U_t dt \\ &= \int_{\mathbb{R}} \pi(f(t))U_t dt = \rho(f). \end{aligned}$$

Thus $\tilde{j} \circ \tilde{\rho} = \rho$; and hence $J \circ \bar{\rho} = \rho$, where $J = j_1 \circ \tilde{j}$ and $\bar{\rho} \in \text{Rep}(C^*(\mathbb{R}, E(A), \alpha))$ is defined by $j_1 \circ \bar{\rho} = \tilde{\rho}$ in view of $C^*(\mathbb{R}, E(A), \alpha) = E(L^1(\mathbb{R}, E(A), \alpha))$.

Step III. Given a *-homomorphism $\rho: S(\mathbb{R}, A^\infty, \alpha) \rightarrow B$ from $S(\mathbb{R}, A^\infty, \alpha)$ to a σ - C^* -algebra B , there exists *-homomorphisms $\tilde{\rho}: L^1(\mathbb{R}, E(A), \alpha) \rightarrow B$, $\bar{\rho}: C^*(\mathbb{R}, E(A), \alpha) \rightarrow B$ such that $\rho = \tilde{\rho} \circ \tilde{j} = \bar{\rho} \circ J$ and $\tilde{\rho} = \bar{\rho} \circ j_1$.

This follows by applying Step II to each of the factor C^* -algebra B_n in the inverse limit decomposition of B .

Step IV. $C^*(\mathbb{R}, E(A), \alpha) = E(S(\mathbb{R}, A^\infty, \alpha))$ up to homeomorphic *-isomorphism.

Let $k: S(\mathbb{R}, E(A), \alpha) \rightarrow E(S(\mathbb{R}, A^\infty, \alpha))$ be $k(f) = f + \text{srad } S(\mathbb{R}, A^\infty, \alpha)$. Then there exists a $*$ -homomorphism $\bar{k}: C^*(\mathbb{R}, E(A), \alpha) \rightarrow E(S(\mathbb{R}, A^\infty, \alpha))$ such that $\bar{k} \circ J = k$. We show that \bar{k} is the desired homeomorphic $*$ -isomorphism making the following diagram commutative.

$$\begin{array}{ccc}
 & S(\mathbb{R}, A^\infty, \alpha) & \\
 J \swarrow & & \searrow k \\
 C^*(\mathbb{R}, E(A), \alpha) & \xrightleftharpoons[\bar{J}]{\bar{k}} & E(S(\mathbb{R}, A^\infty, \alpha)).
 \end{array} \tag{10}$$

By the universal property of $E(S(\mathbb{R}, A^\infty, \alpha))$, there exists a $*$ -homomorphism $\bar{J}: E(S(\mathbb{R}, A^\infty, \alpha)) \rightarrow C^*(\mathbb{R}, E(A), \alpha)$ such that $\bar{J} \circ k = J$. We claim that $\bar{k}|_{\text{Im}(J)}$ is injective. Indeed, let $f \in S(\mathbb{R}, A^\infty, \alpha)$ be such that $\bar{k}(J(f)) = 0$. Hence $k(f) = 0$, so that $f \in \text{srad}(S(\mathbb{R}, A^\infty, \alpha))$. Thus, for all $\rho \in \text{Rep}(S(\mathbb{R}, A^\infty, \alpha))$, $\rho(f) = 0$. Therefore, by Step I, $\sigma(\tilde{f}) = 0$ for all $\sigma \in \text{Rep}(L^1(\mathbb{R}, E(A), \alpha))$. (Recall that $\tilde{f} = j \circ f = \tilde{j}(f)$.) Hence $\tilde{j}(f)$ is in $\text{srad}(L^1(\mathbb{R}, E(A), \alpha))$, and so $j_1(\tilde{j}(f)) = 0$. Therefore $J(f) = 0$. It follows that \bar{k} is injective on $\text{Im}(J)$.

Now by (10) and the injectivity of \bar{k} on $\text{Im}(J)$, $\bar{J} \circ k = J$. Hence $J = \bar{J} \circ \bar{k} \circ J$, and so $\bar{J} \circ \bar{k} = \text{id}$ on $\text{Im}(J)$. Similarly $\bar{k} \circ \bar{J}(k(f)) = \bar{k}(J(f)) = k(f)$, hence $\bar{k} \circ \bar{J} = \text{id}$ on $\text{Im}(k)$. Thus $\bar{k} = (\bar{J})^{-1}$ on $\text{Im}(J)$. Thus \bar{k} is a homeomorphic $*$ -isomorphism from the dense $*$ -subalgebra $J(S(\mathbb{R}, A^\infty, \alpha))$ of $C^*(\mathbb{R}, E(A), \alpha)$ on the dense $*$ -subalgebra $k(S(\mathbb{R}, A^\infty, \alpha))$ of $E(S(\mathbb{R}, A^\infty, \alpha))$. It follows that $C^*(\mathbb{R}, E(A), \alpha)$ is homeomorphically $*$ -isomorphic to $E(S(\mathbb{R}, A^\infty, \alpha))$.

Step V. $E(L^1_{|\cdot|}(\mathbb{R}, A^\infty, \alpha)) = C^*(\mathbb{R}, E(A), \alpha)$.

Let \mathbb{R} act on $L^1_{|\cdot|}(\mathbb{R}, A, \alpha)$ by $xf(y) = f(x - y)$. For this action, $(L^1_{|\cdot|}(\mathbb{R}, A, \alpha))^\infty = S(\mathbb{R}, A, \alpha)$ by Theorem 2.1.7 of [14]. Thus $S(\mathbb{R}, A^\infty, \alpha) = (L^1_{|\cdot|}(\mathbb{R}, A, \alpha))^\infty$. Hence by Lemma 3.4, $E(L^1_{|\cdot|}(\mathbb{R}, A^\infty, \alpha))^\infty = E(S(\mathbb{R}, A^\infty, \alpha)) = C^*(\mathbb{R}, E(A), \alpha)$. This completes the proof of Theorem 1. \square

5. Proof of Theorem 2

Let the Frechet algebra A be hermitian and a Q -algebra. Hence A is spectrally bounded, i.e., the spectral radius $r(x) = r_A(x) < \infty$ for all $x \in A$. Let $s_A(x) := r(x^*x)^{1/2}$ be the Ptak's spectral function on A . By Corollary 2.2 of [1], $E(A)$ is a C^* -algebra, the complete C^* -norm of $E(A)$ being defined by the greatest C^* -seminorm $p_\infty(\cdot)$ (automatically continuous) on A . Now for any $x \in A$,

$$\begin{aligned}
 p_\infty(x)^2 &= p_\infty(x^*x) = \|x^*x + \text{srad}(A)\| \\
 &= r_{E(A)}(x^*x + \text{srad}(A)) \leq r_A(x^*x) = s_A(x)^2.
 \end{aligned}$$

Hence $p_\infty(x) \leq s_A(x)$ for all $x \in A$. By the hermiticity and Q -property, $s_A(\cdot)$ is a C^* -seminorm (Theorem 8.17 of [8]), hence $p_\infty(\cdot) = s(\cdot) \geq r(\cdot)$. In this case, $\text{rad}(A) = \text{srad}(A)$. Let $A_q = A/\text{rad}(A)$ which is a dense $*$ -subalgebra of the C^* -algebra $E(A)$ and

is also a Frechet Q -algebra with the quotient topology t_q . Let $[x] = x + \text{rad}(A)$ for all $x \in A$. Since the spectrum

$$\text{sp}_A(x) = \text{sp}_{A_q}([x]), \quad r_A(x) = r_{A_q}([x]), \quad s_A(x) = s_{A_q}([x]),$$

and so $r_{A_q}([x]) \leq s_{A_q}([x]) = \|[x]\|_\infty$. Hence $\|\cdot\|_\infty$ is a spectral norm on A_q , i.e., $(A_q, \|\cdot\|_\infty)$ is a Q -algebra. Thus A_q is spectrally invariant in $E(A)$. Hence by Corollary 7.9 of [10], $K_*(A_q) = RK_*(A_q) = K_*(E(A))$.

Now consider the maps

$$A \xrightarrow{j} A_q \xrightarrow{\text{id}} E(A)$$

and, for each positive integer n , the induced maps

$$M_n(A) \xrightarrow{j_n = j \otimes \text{id}_n} M_n(A_q) = [M_n(A)]_q \xrightarrow{\text{id}} M_n(E(A)) = E(M_n(A)).$$

By the spectral invariance of A in A_q via the map j , $j(\text{inv}(A)) = \text{inv}(A_q)$, where $\text{inv}(K)$ denotes the group of invertible elements of K . Let $\text{inv}_0(\cdot)$ denote the principle component in $\text{inv}(\cdot)$. We use the following.

Lemma 5.1. Let B be a Frechet Q -algebra or a normed Q -algebra. Then $\text{inv}_0(B)$ is the subgroup generated by the range $\exp B$ of the exponential function.

The Frechet Q -algebra case follows by adapting the proof of the corresponding Banach algebra result in Theorem 1.4.10 of [13]. If $(B, \|\cdot\|)$ is a Q -normed algebra, then $(B, \|\cdot\|)$ is advertably complete in the sense that if a Cauchy sequence (x_n) converges to an element $x \in \text{inv}(B^\sim)$ (B^\sim being the completion of B), then $x \in B$. Hence the exponential function is defined on B ; and then the Banach algebra proof can be adapted.

We use the above lemma to verify the following:

Claim. $j_n(\text{inv}_0(M_n(A))) = \text{inv}_0(M_n(A_q))$.

Take $n = 1$. It is clear that $j(\text{inv}_0(A)) \subseteq \text{inv}_0(A_q)$. Let $y \in \text{inv}_0(A_q)$. Hence $y = \Pi \exp(z_i)$ for finitely many $z_i = [x_i] = x_i + \text{rad}(A)$ for some x_i in A . Then $y = [\Pi \exp(x_i)]$. Hence $y \in j(\text{inv}_0(A))$. Thus $j(\text{inv}_0(A)) = \text{inv}_0(A_q)$. Now take $n > 1$. As A is spectrally invariant in A_q , it follows from Theorem 2.1 of [16] that the Frechet Q -algebra $M_n(A)$ is spectrally invariant in $M_n(A_q)$ via j_n . Also, $M_n(A_q) = (M_n(A))_q$ is a Q -algebra in both the quotient topology as well as the C^* -norm induced from $M_n(E(A)) = E(M_n A)$. Applying arguments analogous to above, it follows that $j_n(\text{inv}_0(M_n(A))) = \text{inv}_0(M_n(A_q))$.

Now consider the surjective group homomorphisms

$$\text{inv}(M_n(A)) \xrightarrow{j_n} \text{inv}(M_n(A_q)) \xrightarrow{J} \text{inv}(M_n(A_q))/\text{inv}_0(M_n(A_q)).$$

It follows that $\ker(J \circ j_n) = \text{inv}_0(M_n(A))$, with the result, the group $\text{inv}(M_n(A))/\text{inv}_0(M_n(A))$ is isomorphic to the group $\text{inv}(M_n(A_q))/\text{inv}_0(M_n(A_q))$. Hence by the definition of the K -theory group K_1 ,

$$\begin{aligned} K_1(A) &= \lim_{\rightarrow} (\text{inv}(M_n(A))/\text{inv}_0(M_n(A))) \\ &= \lim_{\rightarrow} (\text{inv}(M_n(A_q))/\text{inv}_0(M_n(A_q))) = K_1(A_q). \end{aligned}$$

For B to be A or A_q , let the suspension of B be

$$SB = \{f \in C([0, 1], B) : f(0) = f(1) = 0\} \cong C_0(\mathbb{R}, B).$$

We use the Bott periodicity theorem $K_0(B) = K_1(SB)$ to show that $K_0(A) = K_0(A_q)$. It is standard that $\text{rad}(SA) = \text{rad}(C_0(\mathbb{R}, A)) \cong C_0(\mathbb{R}, \text{rad}(A))$. Hence

$$\begin{aligned} SA_q &= C_0(\mathbb{R}, A_q) = C_0(\mathbb{R}, A/\text{rad}(A)) \cong C_0(\mathbb{R}, A)/C_0(\mathbb{R}, \text{rad}(A)) \\ &= C_0(\mathbb{R}, A)/\text{rad}(C_0(\mathbb{R}, A)) = SA/\text{rad}(SA). \end{aligned}$$

Hence

$$K_0(A_q) = K_1(SA_q) = K_1(SA/\text{rad}(SA)) = K_0(A).$$

Thus we have

$$K_*(A) = K_*(A_q) = K_*(E(A)) = RK_*(A) = RK_*(A_q).$$

Now A^∞ is spectrally invariant in A (Theorem 2.2 of [15]); and the action α on A^∞ is smooth (Theorem A.2 of [14]). Then applying the Phillips–Schweitzer analogue of Thom isomorphism for smooth Frechet algebra crossed product (Theorem 1.2 of [11]) and Connes analogue of Thom isomorphism for C^* -algebra crossed product [7], it follows that

$$\begin{aligned} RK_*(S(\mathbb{R}, A^\infty, \alpha)) &= RK_{*+1}(A^\infty) = RK_{*+1}(A) = RK_{*+1}(E(A)) \\ &= RK_*(C^*(\mathbb{R}, E(A), \alpha)) = K_*(C^*(\mathbb{R}, E(A), \alpha)). \end{aligned}$$

When α is isometric, Theorem 1.3.4 of [11] implies that $RK_*(S(\mathbb{R}, A^\infty, \alpha)) = RK_*(L^1(\mathbb{R}, A, \alpha))$. This completes the proof. \square

6. An application to the differential structure in C^* -algebras

Let \mathcal{U} be a unital $*$ -algebra. Let $\|\cdot\|$ be a C^* -norm on \mathcal{U} . Let $(A, \|\cdot\|)$ be the completion of $(\mathcal{U}, \|\cdot\|)$. Following [5], a map $T : \mathcal{U} \rightarrow l^1(\mathbb{N})$ is a *differential seminorm* if $T(x) = (T_k(x))_0^\infty \in l^1(\mathbb{N})$ satisfies the following:

- (i) $T_k(x) \geq 0$ for all k and for all x .
- (ii) For all x, y in \mathcal{U} and scalars λ , $T(x + y) \leq T(x) + T(y)$, $T(\lambda x) = |\lambda|T(x)$.
- (iii) For all x, y in \mathcal{U} , for all k ,

$$T_k(xy) \leq \sum_{i+j=k} T_i(x)T_j(y).$$

- (iv) There exists a constant $c > 0$ such that $T_0(x) \leq c\|x\| \forall x \in \mathcal{U}$.

By (ii), each T_k is a seminorm. We say that T is a *differential $*$ -seminorm* if additionally;

- (v) $T_k(x^*) = T_k(x)$ for all x and for all k .

Further T is a *differential norm* if $T(x) = 0$ implies $x = 0$. Throughout we assume that $T_0(x) = \|x\|$, $x \in \mathcal{U}$. The *total norm* of T is $T_{\text{tot}}(x) = \sum_{k=0}^\infty T_k(x)$, $x \in \mathcal{U}$. Given T ,

the differential Frechet *-algebra defined by T is constructed as follows. For each k , let $p_k(x) = \sum_{i=0}^k T_i(x)$, $x \in \mathcal{U}$. Then each p_k is a submultiplicative *-norm; and on \mathcal{U} , we have

$$p_0 \leq p_1 \leq p_2 \leq \dots \leq p_k \leq p_{k+1} \leq \dots$$

and $(p_k)_0^\infty$ is a separating family of submultiplicative *-norms on \mathcal{U} . Let τ be the locally convex *-algebra topology on \mathcal{U} defined by $(p_k)_0^\infty$. Let $\mathcal{U}_\tau = (\mathcal{U}, \tau)^\sim$ the completion of \mathcal{U} in τ and let $\mathcal{U}_{(k)} = (\mathcal{U}, p_k)^\sim$ the completion of \mathcal{U} in p_k . Then \mathcal{U}_τ is a Frechet locally m -convex *-algebra, $\mathcal{U}_{(k)}$ is a Banach *-algebra. Let \mathcal{U}_T be the completion of $(\mathcal{U}, T_{\text{tot}})$. Then the Banach *-algebra $\mathcal{U}_T = \{x \in \mathcal{U}_\tau : \sup_n p_n(x) < \infty\}$, the bounded part of \mathcal{U}_τ . By the definitions, there exists continuous surjective *-homomorphisms $\phi_k: \mathcal{U}_{(k)} \rightarrow A$, $\phi: \mathcal{U}_\tau \rightarrow A$. The identity map $\mathcal{U} \rightarrow \mathcal{U}$ extends uniquely as continuous surjective *-homomorphisms $\phi_k: \mathcal{U}_{(k+1)} \rightarrow \mathcal{U}_{(k)}$ such that

$$\mathcal{U}_{(0)} \xleftarrow{\varphi_0} \mathcal{U}_{(1)} \xleftarrow{\varphi_1} \mathcal{U}_{(2)} \xleftarrow{\varphi_2} \mathcal{U}_{(3)} \xleftarrow{\dots} \dots$$

is a dense inverse limit sequence of Banach *-algebras and $\mathcal{U}_\tau = \varprojlim \mathcal{U}_{(k)}$.

Lemma 6.1 [4]. *Let $(\mathcal{U}, \|\cdot\|)$ be a C^* -normed algebra. Let A be the completion of \mathcal{U} . Let B denote $\mathcal{U}_{(k)}$ or \mathcal{U}_τ with respective topologies. Then the following hold:*

- (i) B is a hermitian Q -algebra.
- (ii) $E(B) = A$.
- (iii) $K_*(B) = K_*(A) = RK_*(B)$.

The K -theory result follows from the following.

Lemma 6.2 [4]. *Let A be a Frechet algebra in which each element is bounded. Let A be spectrally invariant in $E(A)$. Then $K_*(A) = K_*(E(A))$.*

Now let α be an action of \mathbb{R} on A leaving \mathcal{U} invariant. Let T be α -invariant, i.e., $T_k(\alpha(x)) = T_k(x)$ for all k and for all x . Then α induces isometric actions of \mathbb{R} on each of $\mathcal{U}_{(k)}$, \mathcal{U}_τ and \mathcal{U}_T . Let B be as above. Hence the crossed product Frechet *-algebras $L^1(\mathbb{R}, B^\infty, \alpha)$, $L^1(\mathbb{R}, B, \alpha)$, $S(\mathbb{R}, B, \alpha)$ and $S(\mathbb{R}, B^\infty, \alpha)$ are defined. Theorem 2 and Lemma 6.1 give the following, which is Theorem 3(a).

COROLLARY 6.3

$$RK_*(S(\mathbb{R}, B^\infty, \alpha)) = RK_*(S(\mathbb{R}, B, \alpha)) = RK_*(C^*(\mathbb{R}, A, \alpha)) = K_{*+1}(A).$$

Now let $\tilde{\mathcal{U}}$ be the completion of \mathcal{U} in the family \mathcal{F} of all α -invariant differential *-norms on \mathcal{U} . Then $\tilde{\mathcal{U}}$ is a complete locally m -convex *-algebra admitting a continuous surjective *-homomorphism $\Psi: \tilde{\mathcal{U}} \rightarrow A$. This α -invariant smooth envelope $\tilde{\mathcal{U}}$ is different from the smooth envelope defined in [5], and it need not be a subalgebra of A .

Lemma 6.4. *Assume that $\tilde{\mathcal{U}}$ is metrizable. Then $\tilde{\mathcal{U}}$ is a hermitian Q -algebra, $E(\tilde{\mathcal{U}}) = A$, and $K_*(\tilde{\mathcal{U}}) = K_*(A)$.*

This supplements a comment in p. 279 of [5] that $K_*(A) = \dot{K}_*(\mathcal{U}_1)$ where \mathcal{U}_1 is the completion of \mathcal{U} in all, not necessarily α -invariant nor closable, differential semi-norms.

Proof. Since $\tilde{\mathcal{U}} = \lim_{\leftarrow} \mathcal{U}_\tau$, we have $E(\tilde{\mathcal{U}}) = \lim_{\leftarrow} E(\mathcal{U}_\tau) = A$; and $\tilde{\mathcal{U}}$ admits greatest continuous C^* -seminorm, say $p_\infty(\cdot)$ [1]. It is easily seen that for any $x \in \tilde{\mathcal{U}}$, the spectral radius in $\tilde{\mathcal{U}}r(x) \leq p_\infty(x)$; and $\tilde{\mathcal{U}}$ is a hermitian Q -algebra. This implies, in view of $\tilde{E}(\tilde{\mathcal{U}}) = A$, that the spectrum in $\tilde{\mathcal{U}}\text{sp}(x) = \text{sp}_A(j(x))$ for all x in $\tilde{\mathcal{U}}$, where $j(x) = x + \text{srad } \tilde{\mathcal{U}}$.

It follows from Lemma 6.2 that $K_*(A) = K_*(E(A))$. Hence Lemma 6.4 follows. \square

Now the action α induces an isometric action of \mathbb{R} on $\tilde{\mathcal{U}}$, with the result that the crossed product algebras $S(\mathbb{R}, \tilde{\mathcal{U}}, \alpha)$ and $L^1(\mathbb{R}, \tilde{\mathcal{U}}, \alpha)$ are defined and are complete locally m -convex $*$ -algebras with a C^* -enveloping algebras satisfying

$$\begin{aligned} E(S(\mathbb{R}, \tilde{\mathcal{U}}, \alpha)) &= E(L^1(\mathbb{R}, \tilde{\mathcal{U}}, \alpha)) \\ &= C^*(\mathbb{R}, E(\tilde{\mathcal{U}}), \alpha) \\ &= C^*(\mathbb{R}, A, \alpha). \end{aligned}$$

Theorem 2 quickly gives the following which is Theorem 3(b).

COROLLARY 6.5

Assume that $\tilde{\mathcal{U}}$ is metrizable. Then $RK_*(S(\mathbb{R}, \tilde{\mathcal{U}}, \alpha)) = K_{*+1}(A)$.

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