

Boundary regularity of correspondences in \mathbb{C}^n

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Abstract. Let M, M' be smooth, real analytic hypersurfaces of finite type in \mathbb{C}^n and \hat{f} a holomorphic correspondence (not necessarily proper) that is defined on one side of M , extends continuously up to M and maps M to M' . It is shown that \hat{f} must extend across M as a locally proper holomorphic correspondence. This is a version for correspondences of the Diederich–Pinchuk extension result for CR maps.

Keywords. Correspondences; Segre varieties.

1. Introduction and statement of results

1.1 Boundary regularity

Let U, U' be domains in \mathbb{C}^n and let $M \subset U, M' \subset U'$ be relatively closed, connected, smooth, real analytic hypersurfaces of finite type (in the sense of D'Angelo). A recent result of Diederich and Pinchuk [DP3] shows that a continuous CR mapping $f: M \rightarrow M'$ is holomorphic in a neighbourhood of M . The purpose of this note is to show that their methods can be adapted to prove the following version of their result for correspondences. We assume additionally that M (resp. M') divides the domain U (resp. U') into two connected components U^+ and U^- (resp. U'^{\pm}).

Theorem 1.1. *Let $\hat{f}: U^- \rightarrow U'$ be a holomorphic correspondence that extends continuously up to M and maps M to M' , i.e., $\hat{f}(M) \subset M'$. Then \hat{f} extends as a locally proper holomorphic correspondence across M .*

We recall that if $\mathcal{D} \subset \mathbb{C}^p$ and $\mathcal{D}' \subset \mathbb{C}^m$ are bounded domains, a holomorphic correspondence $\hat{f}: \mathcal{D} \rightarrow \mathcal{D}'$ is a complex analytic set $A \subset \mathcal{D} \times \mathcal{D}'$ of pure dimension p such that $\overline{A} \cap (\mathcal{D} \times \partial\mathcal{D}') = \emptyset$, where $\partial\mathcal{D}'$ is the boundary of \mathcal{D}' . In this situation, the natural projection $\pi: A \rightarrow \mathcal{D}$ is proper, surjective and a finite-to-one branched covering. If in addition the other projection $\pi': A \rightarrow \mathcal{D}'$ is proper, the correspondence is called proper. The analytic set A can be regarded as the graph of the multiple valued mapping $\hat{f} := \pi' \circ \pi^{-1}: \mathcal{D} \rightarrow \mathcal{D}'$. We also use the notation $A = \text{Graph}(\hat{f})$.

The branching locus σ of the projection π is a codimension one analytic set in \mathcal{D} . Near each point in $\mathcal{D} \setminus \sigma$, there are finitely many well-defined holomorphic inverses of π . The symmetric functions of these inverses are globally well-defined holomorphic functions on \mathcal{D} . To say that \hat{f} is continuous up to $\partial\mathcal{D}$ simply means that the symmetric functions extend

continuously up to ∂D . Thus in Theorem 1.1 the various branches of \hat{f} are continuous up to M and each branch maps points on M to those on M' .

We say that \hat{f} in Theorem 1.1 extends as a *holomorphic correspondence* across M if there exist open neighbourhoods \tilde{U} of M and \tilde{U}' of M' , and an analytic set $\tilde{A} \subset \tilde{U} \times \tilde{U}'$ of pure dimension n such that (i) $\text{Graph}(\hat{f})$ intersected with $(\tilde{U} \cap U^-) \times (\tilde{U}' \cap U')$ is contained in \tilde{A} and (ii) the projection $\tilde{\pi}: \tilde{A} \rightarrow \tilde{U}$ is proper. Without condition (ii), \hat{f} is said to extend as an *analytic set*. Finally, the extension of \hat{f} is a proper holomorphic correspondence if in addition to (i) and (ii), $\tilde{\pi}': \tilde{A} \rightarrow \tilde{U}'$ is also proper.

COROLLARY 1.1

Let D and D' be bounded pseudoconvex domains in \mathbb{C}^n with smooth real-analytic boundary. Let $\hat{f}: D \rightarrow D'$ be a holomorphic correspondence. Then \hat{f} extends as a locally proper holomorphic correspondence to a neighbourhood of the closure of D .

The corollary follows immediately from Theorem 1.1 and [BS] where the continuity of \hat{f} is proved. This generalizes a well-known result of [BR] and [DF] where the extension past the boundary of D is proved for holomorphic mappings.

1.2 Preservation of strata

Let M_s^+ (resp. M_s^-) be the set of strongly pseudoconvex (resp. pseudoconcave) points on M . The set of points where the Levi form \mathcal{L}_ρ has eigenvalues of both signs on $T^{\mathbb{C}}(M)$ and no zero eigenvalue will be denoted by M^\pm and finally M^0 will denote those points where \mathcal{L}_ρ has at least one zero eigenvalue on $T^{\mathbb{C}}(M)$. M^0 is a closed real analytic subset of M of real dimension at most $2n - 2$. Then

$$M = M_s^+ \cup M_s^- \cup M^\pm \cup M^0.$$

Further, let M^+ (resp. M^-) be the pseudoconvex (resp. pseudoconcave) part of M , which equals the relative interior of M_s^+ (resp. M_s^-). For non-negative integers i, j such that $i + j = n - 1$, let $M_{i,j}$ denote those points at which \mathcal{L}_ρ has exactly i positive and j negative eigenvalues on $T^{\mathbb{C}}(M)$. Each (non-empty) $M_{i,j}$ is relatively open in M and semi-analytic whose relative boundary is contained in M^0 . With this notation, $M_{0,n-1} = M_s^-$ and $M_{n-1,0} = M_s^+$. Moreover, M^\pm is the union of all (non-empty) $M_{i,j}$ where both i, j are at least 1 and $i + j = n - 1$. Note that points in M_s^-, M^\pm are in the envelope of holomorphy of U^- . Following [B], there is a semi-analytic stratification for M^0 given by

$$M^0 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4, \quad (1.1)$$

where Γ_4 is a closed, real analytic set of dimension at most $2n - 4$ and $\Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ is also a closed, real analytic set of dimension at most $2n - 3$. Further, $\Gamma_1, \Gamma_2, \Gamma_3$ are either empty or smooth, real analytic manifolds; Γ_2, Γ_3 have dimension $2n - 3$, and Γ_1 has dimension $2n - 2$. Finally, Γ_2 and Γ_3 are CR manifolds of complex dimension $n - 2$ and $n - 3$ respectively. The set of points, denoted by Γ_h^1 in Γ_1 where the complex tangent space to Γ_1 has dimension $n - 1$ is semi-analytic and has real dimension at most $2n - 3$, as otherwise there would exist a germ of a complex manifold in M contradicting the finite type hypothesis. Then $\Gamma_1 \setminus \Gamma_h^1$ is a real analytic manifold of dimension $2n - 2$ and has CR dimension $n - 2$. Using the same letters to denote the various strata of M^0 , there exists a

refinement of (1.1), so that $\Gamma_1, \Gamma_2, \Gamma_3$ are all smooth, real analytic manifolds of dimensions $2n-2, 2n-3, 2n-3$ respectively, while the corresponding CR dimensions are $n-2, n-2,$ and $n-3$. Finally, Γ_4 is a closed, real analytic set of dimension at most $2n-4$.

Theorem 1.2. *With the hypothesis of Theorem 1.1, the extended correspondence $\hat{f}: M \rightarrow M'$ satisfies the additional properties: $\hat{f}(M^+) \subset M'^+, \hat{f}(M^+ \cap M^0) \subset M'^+ \cap M'^0$ and $\hat{f}(M^-) \subset M'^-, \hat{f}(M^- \cap M^0) \subset M'^- \cap M'^0$. Moreover, $\hat{f}(M^+ \cap \Gamma_j) \subset M'^+ \cap \Gamma'_j$ and $\hat{f}(M^- \cap \Gamma_j) \subset M'^- \cap \Gamma'_j$ for $j = 1, 3, 4$. Finally, \hat{f} maps the relative interior of \overline{M}^\pm to the relative interior of \overline{M}'^\pm .*

Preservation of Γ_2 is not always possible even for holomorphic mappings as the following example shows: the domain $\Omega = \{(z_1, z_2): |z_1|^2 + |z_2|^4 < 1\}$ is mapped to the unit ball in \mathbb{C}^2 by the proper holomorphic mapping $f(z_1, z_2) = (z_1, z_2^2)$. Points of the form $\{(e^{i\theta}, 0)\} \subset \partial\Omega$ are weakly pseudoconvex and in fact form $\Gamma_2 \subset \partial\Omega$, and f maps them to strongly pseudoconvex points.

2. Segre varieties

We will write $z = (z, z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$ for a point $z \in \mathbb{C}^n$. The word ‘analytic’ will always mean complex analytic unless stated otherwise. The techniques of Segre varieties will be used and here is a synopsis of the main properties that will be needed. The proofs of these can be found in [DF] and [DW]. As described above, let M be a smooth, real analytic hypersurface of finite type in \mathbb{C}^n that contains the origin. If U is small enough, the complexification $\rho(z, \bar{w})$ of ρ is well-defined by means of a convergent power series in $U \times U$. Note that $\rho(z, \bar{w})$ is holomorphic in z and anti-holomorphic in w . For any $w \in U$, the associated Segre variety is defined as

$$Q_w = \{z \in U: \rho(z, \bar{w}) = 0\}.$$

By the implicit function theorem, it is possible to choose neighbourhoods $U_1 \subset\subset U_2$ of the origin such that for any $w \in U_1$, Q_w is a closed, complex hypersurface in U_2 and

$$Q_w = \{z = (z, z_n) \in U_2: z_n = h(z, \bar{w})\},$$

where $h(z, \bar{w})$ is holomorphic in z and anti-holomorphic in w . Such neighbourhoods will be called a standard pair of neighbourhoods and they can be chosen to be polydiscs centered at the origin. It can be shown that Q_w is independent of the choice of ρ . For $\zeta \in Q_w$, the germ Q_w at ζ will be denoted by ${}_\zeta Q_w$. Let $\mathcal{S} := \{Q_w: w \in U_1\}$ be the set of all Segre varieties, and let $\lambda: w \mapsto Q_w$ be the so-called Segre map. Then \mathcal{S} admits the structure of a finite dimensional analytic set. It can be shown that the analytic set

$$I_w := \lambda^{-1}(\lambda(w)) = \{z: Q_z = Q_w\}$$

is contained in M if $w \in M$. Consequently, the finite type assumption on M forces I_w to be a discrete set of points. Thus λ is proper in a small neighbourhood of each point of M . For $w \in U_1^+$, the symmetric point ${}^s w$ is defined to be the unique point of intersection of the complex normal to M through w and Q_w . The component of $Q_w \cap U_2^-$ that contains the symmetric point is denoted by Q_w^c .

Finally, for all objects and notions considered above, we simply add a prime to define their corresponding analogs in the target space.

3. Localization and extension across an open dense subset of M

In the proof of Theorem 1.1 in order to show extension of \hat{f} as a holomorphic correspondence, it is enough to consider the problem in an arbitrarily small neighbourhood of any point $p \in M$. The reason is the following. Firstly, since the projection $\pi: \text{Graph}(\hat{f}) \rightarrow U^-$ is proper, the closure of $\text{Graph}(\hat{f})$ has empty intersection with $U^- \times \partial U'$. Therefore, by [C] § 20.1, to prove the continuation of \hat{f} across M as an analytic set, it is enough to do that in a neighbourhood of any point in M . Secondly, once the extension of \hat{f} as a holomorphic correspondence in a neighbourhood of any point $p \in M$ is established, then globally there exists a holomorphic correspondence defined in a neighbourhood \tilde{U} of M which extends \hat{f} . To see that simply observe that if $F \subset \tilde{U} \times \tilde{U}'$ is an analytic set extending \hat{f} , then by choosing smaller \tilde{U} we may ensure that the projection to the first component is proper, as otherwise there would exist a point z on M such that $\hat{F}(z)$ has positive dimension (here \hat{F} is the map associated with the set F). This however contradicts local extension of \hat{f} near z as a holomorphic correspondence.

Since the projection $\pi: \text{Graph}(\hat{f}) \rightarrow U^-$ is proper, $\text{Graph}(\hat{f})$ is contained in the analytic set $A \subset U^- \times U'$, defined by the zero locus of holomorphic functions $P_1(z, z'_1), P_2(z, z'_2), \dots, P_n(z, z'_n)$ given by

$$P_j(z, z'_j) = z_j^l + a_{j1}(z)z_j^{l-1} + \dots + a_{jl}(z), \quad (3.1)$$

where l is the generic number of images in $\hat{f}(z)$, and $1 \leq j \leq n$ (for details, see [C]). The coefficients $a_{\mu\nu}(z)$ are holomorphic in U^- and extend continuously up to M . This is the definition of continuity of the correspondence \hat{f} up to M which is equivalent to that given in §1.1.

The discriminant locus is $\{R_j(z) = 0\}$, $1 \leq j \leq n$, where $R_j(z)$ is a universal polynomial function of $a_{j\mu}(z)$ ($1 \leq \mu \leq l$) and hence by the uniqueness theorem, it follows that $\overline{\{R_j(z) = 0\}} \cap M$ is nowhere dense in M , for all j . The set of points S on M which do not belong to $\overline{\{R_j(z) = 0\}} \cap M$ for any j is therefore open and dense in M . Near each point p on S , \hat{f} splits into well-defined holomorphic maps $f_1(z), f_2(z), \dots, f_l(z)$ each of which is continuous up to M .

If $p \in S \cap (M^- \cup M^\pm)$, the functions $a_{\mu\nu}(z)$ extend holomorphically to a neighbourhood of p and hence \hat{f} extends as a holomorphic correspondence across p . It is therefore sufficient to show that \hat{f} extends across an open dense subset of $S \cap M^+$. But this follows from Lemma 3.2 and Corollary 3.3 in [DP3]. We denote by $\Sigma \subset M$ the non-empty open dense subset of M across which \hat{f} extends as a holomorphic correspondence.

4. Extension as an analytic set

Fix $0 \in M$ and let $p'_1, p'_2, \dots, p'_k \in \hat{f}(0) \cap M'$. The continuity of \hat{f} allows us to choose neighbourhoods $0 \in U_1$ and $p'_i \in U'_i$ and local correspondences $\hat{f}_i: U_1^- \rightarrow U'_i$ that are irreducible and extend continuously up to M . Moreover, $\hat{f}_i(0) = p'_i$ for all $1 \leq i \leq k$. It will suffice to focus on one of the \hat{f}_i 's, say \hat{f}'_1 and to show that it extends holomorphically across the origin. Abusing notation, we will write $\hat{f}'_1 = \hat{f}$, $U_1^- = U'$ and $p'_1 = 0'$. Thus $\hat{f}: U_1^- \rightarrow U'$ is an irreducible holomorphic correspondence and $\hat{f}(0) = 0'$. Define

$$V^+ = \{(w, w') \in U_1^+ \times U': \hat{f}(Q_w^c) \subset Q_{w'}'\}.$$

Then V^+ is non-empty. Indeed, \hat{f} extends across an open dense set near the origin and [V] shows that the invariance property of Segre varieties then holds. Moreover, a similar argument as in [S2] shows that $V^+ \subset U_1^+ \times U'$ is an analytic set of dimension n and exactly the same arguments as in Lemmas 4.2–4.4 of [DP3] show that: first, the projection $\pi: V^+ \rightarrow U^+ := \pi(V^+) \subset U_1^+$ is proper (and hence that $U^+ \subset U_1^+$ is open) and second, the projection $\pi': V^+ \rightarrow U'$ is locally proper. Thus, to V^+ is associated a correspondence $F^+: U^+ \rightarrow U'$ whose branches are $\hat{F}^+ = \pi' \circ \pi^{-1}$.

Let $a \in M$ be a point close to the origin, across which \hat{f} extends as a holomorphic correspondence. If \hat{f} is well-defined in the ball $B(a, r)$, $r > 0$ and $w \in B(a, r)^-$, it follows from Theorem 4.1 in [V] that all points in $\hat{f}(w)$ have the same Segre variety. By analytic continuation, the same holds for all $w \in U_1^-$. Using this observation, it is possible to define another correspondence $F^-: U_1^- \rightarrow U'$ whose branches are $\hat{F}^-(w) = (\lambda')^{-1} \circ \lambda' \circ \hat{f}(w)$. Let $U := U_1^- \cup U^+ \cup (\Sigma \cap U_1)$. The invariance property of Segre varieties shows that the correspondences \hat{F}^+ , \hat{F}^- can be glued together near points on $\Sigma \cap U_1$. Hence, there is a well-defined correspondence $\hat{F}: U \rightarrow U'$ whose values over U^+ and U_1^- are \hat{F}^+ and \hat{F}^- respectively. Note that

$$F := \text{Graph}(\hat{F}) = \{(w, w') \in U \times U' : w' \in \hat{F}(w)\}$$

is an analytic set in $U \times U'$ of pure dimension n , with proper projection $\pi: F \rightarrow U$. Once again, the invariance property shows that all points in $\hat{F}(w)$, $w \in U$, have the same Segre variety.

Lemma 4.1. The correspondence \hat{F} satisfies the following properties:

- (i) For $w_0 \in \partial U \cap U_1^+$, $\text{cl}_{\hat{F}}(w_0) \subset \partial U'$.
- (ii) $\text{cl}_{\hat{F}}(0) \subset Q'_{0'}$.
- (iii) If $\text{cl}_{\hat{F}}(0) = \{0'\}$, then $0 \in \Sigma$.
- (iv) $F \subset (U_1 \setminus (M \setminus \Sigma)) \times U'$ is a closed analytic set.

Proof.

- (i) Choose $(w_j, w'_j) \in F$ converging to $(w_0, w'_0) \in (\partial U \cap U_1^+) \times \overline{U}'$. Then $\hat{f}(Q^c_{w_j}) \subset Q'_{w'_j}$ for all j . If $w'_0 \in U'$, then passing to the limit, we get $\hat{f}(Q^c_{w_0}) \subset Q'_{w'_0}$ which shows that $(w_0, w'_0) \in F$ and hence $w_0 \in U$, which is a contradiction. This also proves (iv).
- (ii) Choose $w_j \in U$ converging to 0. There are two cases to consider. First, if $w_j \in U_1^- \cup (\Sigma \cap U_1)$ for all j , it follows that $\hat{f}(w_j) \rightarrow 0'$. Moreover, for any $w'_j \in \hat{F}(w_j)$, $Q'_{w'_j} = Q'_{\hat{f}(w_j)}$. If U' is small enough, the equality $Q'_{w'_j} = Q'_{0'}$ implies that $w'_j = 0'$ and thus we conclude that $w'_j \rightarrow 0' \in Q'_{0'}$. Second, if $w_j \in U^+$ for all j , then $\hat{f}(Q^c_{w_j}) \subset Q'_{w'_j}$ for any $w'_j \in \hat{F}(w_j)$. Let $w'_j \rightarrow w'_0 \in U'$. If $\zeta \in Q^c_{w_j}$, then $\hat{f}(\zeta) \in Q'_{w'_j} \rightarrow Q'_{w'_0}$. But $w_j \rightarrow 0$ implies that $\text{dist}(Q^c_{w_j}, 0) \rightarrow 0$ and hence $\hat{f}(\zeta) \rightarrow 0'$. Thus $0' \in Q'_{w'_0}$ which shows that $w'_0 \in Q'_{0'}$.
- (iii) If $\text{cl}_{\hat{F}}(0) = \{0'\}$, then (i) shows that $0 \notin \partial U \cap U_1^+$. Let $B(0, r)$ be a small ball around the origin such that $B(0, r) \cap \partial U = \emptyset$. The correspondence \hat{F} over $B(0, r)^+$ is the union of some components of the zero locus of a system of monic pseudo-polynomials

whose coefficients are bounded holomorphic functions on $B(0, r)^+$. By Trepreau's theorem, all these coefficients extend holomorphically to $B(0, r)$, and the extended zero locus contains the graph of \hat{f} near the origin since Σ is dense. It follows that $0 \in \Sigma$. \square

Following [S1], for any $w_0 \in U$, it is possible to find a neighbourhood Ω of w_0 , relatively compact in U and a neighbourhood $V \subset U_1$ of $Q_{w_0} \cap U_1$ such that for $z \in V$, $Q_z \cap \Omega$ is non-empty and connected. Associated with the pair (Ω, V) is

$$\tilde{F} := \tilde{F}(w_0, \Omega, V) = \{(z, z') \in V \times U' : \hat{F}(Q_z \cap \Omega) \subset Q'_{z'}\} \quad (4.1)$$

which (see [DP4]) is an analytic set of dimension at most n . If $w_0 \in \Sigma$, then Corollary 5.3 of [DP3], shows that $F \cap (V \times U')$ is the union of irreducible components of \tilde{F} of dimension n . As in [DP3] we call $(w_0, z_0) \in U \times Q_{w_0}$ a pair of reflection if there exist neighbourhoods $\Omega(w_0) \ni w_0$ and $\Omega(z_0) \ni z_0$ such that for all $w \in \Omega(w_0)$, $\hat{F}(Q_w \cap \Omega(z_0)) \subset Q'_{\hat{F}(w)}$. It follows from the invariance property of Segre varieties that the definition of the pair of reflection is symmetric. As an example we note that if the set \tilde{F} defined in (4.1) contains $F \cap (V \times U')$, then (w_0, z) is a point of reflection for any point z in a connected component of $Q_{w_0} \cap U$ containing w_0 .

Let $w_0 \in U$, $z_0 \in Q_{w_0} \cap \Sigma$ be a pair of reflection. Fix $B(z_0, r)$, a small ball around z_0 where \hat{f} is well-defined and let $S(w_0, z_0) \subset \tilde{F} \cap ((Q_{w_0} \cap U_1) \times U')$ be the union of those irreducible components that contain $\text{Graph}(\hat{f})$ over $Q_{w_0} \cap B(z_0, r)$. Note that $S(w_0, z_0)$ is an analytic set of dimension $n - 1$ and is contained in $(Q_{w_0} \cap U_1) \times U'$ and moreover, the invariance property shows that

$$S(w_0, z_0) \subset ((Q_{w_0} \cap U_1) \times (Q'_{\hat{F}(w_0)} \cap U')).$$

Furthermore, from the above considerations it follows that for any $z \in \pi(S(w_0, z_0))$ the point (w_0, z) is a pair of reflection. Finally, let the cluster set of a sequence of closed sets $\{C_j\} \subset \mathcal{D}$, where \mathcal{D} is some domain, be the set of all possible accumulation points in \mathcal{D} of all possible sequences $\{c_j\}$ where $c_j \in C_j$.

PROPOSITION 4.1

Let $\{z_\nu\} \in \Sigma$ converge to 0. Suppose that the cluster set of the sequence $\{S(z_\nu, z_\nu)\}$ contains a point $(\zeta_0, \zeta'_0) \in U \times U'$. Then \hat{f} extends as an analytic set across the origin.

Proof. First, the pair (z_ν, z_ν) is an example of a pair of reflection and hence $S(z_\nu, z_\nu)$ is well-defined. Also, note that $(z_\nu, \hat{f}(z_\nu)) \rightarrow (0, 0')$. Choose $(\zeta_\nu, \zeta'_\nu) \in S(z_\nu, z_\nu)$ that converges to $(\zeta_0, \zeta'_0) \in U \times U'$. It follows that (ζ_ν, z_ν) is a pair of reflection. Let Ω, V be neighbourhoods of ζ_0 and Q_{ζ_0} as in the definition of $\tilde{F}(\zeta_0, \Omega, V)$. Since $\zeta_0 \in U$, it follows that $\tilde{F}(\zeta_0, \Omega, V)$ is a non-empty, analytic set in $V \times U'$. Shrinking U_1 if needed, $Q_{\zeta_\nu} \cap U_1 \subset V$ and $\zeta_\nu \in \Omega$ for all large ν . This shows that $\tilde{F}(\zeta_\nu, \Omega, V) = \tilde{F}(\zeta_0, \Omega, V)$ for all large ν . Lemma 5.2 of [DP3] shows that $\tilde{F}(\zeta_\nu, \Omega, V)$ contains the graph of all branches of \hat{f} near z_ν and hence $\tilde{F}(\zeta_0, \Omega, V)$ contains the graph of \hat{f} near $(0, 0')$. Therefore, $\tilde{F}(\zeta_0, \Omega, V)$ extends the graph of \hat{f} across the origin. \square

Remarks. First, as in [DP3] this proposition will be valid if the pair (z_ν, z_ν) were replaced by a pair of reflection $(w_\nu, z_\nu) \in U \times \Sigma$ that converges to $(0, 0')$ and $\hat{F}(w_\nu)$ clusters at some point in U' . Second, this proposition shows the relevance of studying the cluster

set of a sequence of analytic sets (see [SV] also). In general, the hypothesis that the cluster set of $\{S(z_\nu, z_\nu)\}$ (or $S(w_\nu, z_\nu)$ in case (w_ν, z_ν) is a pair of reflection) contains a point in $U \times U'$ cannot be guaranteed since the projection $\pi: S(z_\nu, z_\nu) \rightarrow U$ is not known to be proper. However, the following version of Lemma 5.9 in [DP3] holds.

Lemma 4.2. *There are sequences $(w_\nu, z_\nu) \in U \times \Sigma$, $w'_\nu \in \hat{F}(w_\nu)$ and analytic sets $\sigma_\nu \subset U$ of pure dimension $p \geq 1$ (p independent of ν) such that:*

- (i) $(w_\nu, z_\nu) \rightarrow (0, 0)$ and (w_ν, z_ν) is a pair of reflection for all ν .
- (ii) $w'_\nu \rightarrow w'_0 \in U'$ and $z_\nu \in \sigma_\nu \subset \pi(S(w_\nu, z_\nu))$.

Proof. Choose a sequence $z_\nu \in \Sigma$ that converges to the origin. If the projections $\pi: S(z_\nu, z_\nu) \rightarrow U$ were proper for all ν , then for some fixed $r > 0$ and ν large enough, let $\sigma_\nu := Q_{z_\nu} \cap B(z_\nu, r)$, $w_\nu = z_\nu$ and $w'_\nu \in \hat{f}(z_\nu)$. It can be seen that the lemma holds with these choices. On the other hand, if π is not known to be proper on $S(z_\nu, z_\nu)$, no fixed value of r , as described above, exists. Hence, for arbitrarily small values of $r' > 0$, there exist $(w_\nu, w'_\nu) \in S(z_\nu, z_\nu) \cap (U^+ \times U')$ such that $w_\nu \rightarrow 0$ and $w'_\nu \rightarrow w'_0$ with $|w'_0| = r'$. Since M' is of finite type, we may assume that $Q'_{w'_0} \neq Q'_{0'}$. Moreover, note that $w'_0 \in Q'_{0'} \cap U'$ (which shows that $0' \in Q'_{w'_0}$) and (w_ν, z_ν) is a pair of reflection for all ν . By making a small holomorphic perturbation of coordinates in the target space, if needed, it follows that $0' \in Q'_{w'_0} \cap \{z' \in U': z'_2 = \dots = z'_n = 0\}$ is an isolated point. Therefore, there exists an $\epsilon > 0$ such that after shrinking U' , if needed, $q'_0 := Q'_{w'_0} \cap \{z' \in U': z'_2 = \dots = z'_{n-1} = 0, |z'_n| < \epsilon\}$ (which is an analytic set of dimension 1 in $U' \cap \{|z'_n| < \epsilon\}$ containing the origin) has no limit points on $\partial U' \cap \{|z'_n| < \epsilon\}$. Let l be the multiplicity of $\hat{f}: U_1^- \rightarrow U'$. Let $\hat{f}(z_\nu) = \{\zeta_\nu^j\}$, $1 \leq j \leq l$ counted with multiplicity. For large ν , the l sets

$$q'_{\nu,j} = Q'_{w'_\nu} \cap \{z' \in U': z'_k = (\zeta_\nu^j)_k, \quad 2 \leq k \leq n-1, \quad |z'_n| < \epsilon\}$$

are analytic, of dimension 1, in $U' \cap \{|z'_n| < \epsilon\}$ without limit points on $\partial U' \cap \{|z'_n| < \epsilon\}$ and clearly contain (z_ν, ζ_ν^j) . Since $\pi'(S(w_\nu, z_\nu)) \subset Q'_{w'_\nu}$,

$$s_{\nu,j} := S(w_\nu, z_\nu) \cap \{(z, z'): z'_k = (\zeta_\nu^j)_k, \quad 2 \leq k \leq n-1\}$$

are analytic sets of dimension at least 1 in $U_1 \times (U' \cap \{|z'_n| < \epsilon\})$ for all $1 \leq j \leq l$. By construction, the analytic sets $q'_{\nu,j}$ do not have limit points on $\partial U' \cap \{|z'_n| < \epsilon\}$ and hence $s_{\nu,j}$ do not have limit points on $U_1 \times (\partial U' \cap \{|z'_n| < \epsilon\})$. By Lemma 4.1, $\text{cl}_{\hat{F}}(0) \subset Q'_{0'} = \{z'_n = 0\}$ and by shrinking U_1 if needed, this shows that $s_{\nu,j}$ have no limit points on $U_1 \times (U' \cap \{|z'_n| = \epsilon\})$. Thus for large ν and all j , the projections $\pi: s_{\nu,j} \rightarrow U_1$ are proper and their images $\sigma_{\nu,j} := \pi(s_{\nu,j})$ are analytic sets in U_1 of dimension at least 1 and $z_\nu \in \sigma_{\nu,j}$ for all ν, j . It remains to pass to subsequences if necessary to choose $\sigma_{\nu,j}$ with constant dimension. \square

One conclusion that follows now is: if \hat{f} does not extend as an analytic set across the origin, then $\text{cl}(\sigma_\nu) \subset M \setminus \Sigma$. Indeed, if there exists $\zeta_0 \in \text{cl}(\sigma_\nu) \cap (U_1 \setminus (M \setminus \Sigma))$, let $(\zeta_\nu, \zeta'_\nu) \in S(w_\nu, z_\nu)$ converge to $(\zeta_0, \zeta'_0) \in U_1 \times U'$. Proposition 4.1 now shows that $\zeta_0 \in \partial U \cap U_1$. But since $\zeta_0 \notin M \setminus \Sigma$, it follows from Lemma 4.1 that $\zeta'_0 \in \partial U'$ which is a contradiction.

The goal will now be to show that \hat{f} extends as an analytic set across the origin. For this, choose $\{z_\nu\} \in \Sigma$ converging to the origin and consider the analytic sets $S(z_\nu, z_\nu)$. By Proposition 4.1, it suffices to show that $\pi(\text{cl}(S(z_\nu, z_\nu))) \cap U \neq \emptyset$. Let

$$S' := \pi'(\text{cl}(S(z_\nu, z_\nu)) \cap (\{0\} \times U')) \subset Q'_{0'}$$

and let m be the dimension of \hat{S}' – the smallest closed analytic set containing S' (the so-called Segre completion of [DP3]). If $m = 0$, then $0'$ is an isolated point in S' and after shrinking U_1, U' suitably, it follows that $\text{cl}(S(z_\nu, z_\nu))$ has no limit points on $U_1 \times \partial U'$. Thus $\pi: S(z_\nu, z_\nu) \rightarrow U_1$ are proper projections and therefore $\pi(S(z_\nu, z_\nu)) = Q_{z_\nu} \cap U_1$ for all large ν . Hence $\pi(\text{cl}(S(z_\nu, z_\nu))) = Q_0 \cap U_1$. If \hat{f} did not extend as an analytic set across the origin, the aforementioned remark shows that with $\sigma := Q_{z_\nu} \cap U_1, Q_0 \cap U_1 = \text{cl}(\sigma_\nu) \subset M \setminus \Sigma \subset M$. This cannot happen as M is of finite type. Hence \hat{f} extends as an analytic set across the origin in case $m = 0$. We may therefore suppose that $m > 0$. We recall the following lemma proved by Diederich and Pinchuk:

Lemma 4.3 ([DP3], Lemma 9.8). Let S' be a subset of $Q'_{0'}$, $0' \in S'$ and $m = \dim \hat{S}'$. Then after possibly shrinking U_1 , there are points $w^1, \dots, w^k \in S'$ ($k \leq n - 1$) such that one of the following holds:

- (1) $k = m$ and $\dim(\hat{S}' \cap Q'_{w^1} \cap \dots \cap Q'_{w^k}) = 0$;
- (2) $k \geq 2m - n + 1$ and $\dim(\hat{S}' \cap Q'_{w^1} \cap \dots \cap Q'_{w^k}) = m - k$.

Thus there are two cases to consider.

Case 1. Choose $(w_{1\nu}, w'_{1\nu}), (w_{2\nu}, w'_{2\nu}), \dots, (w_{m\nu}, w'_{m\nu}) \in S(z_\nu, z_\nu)$ so that $w_{\mu\nu} \rightarrow 0$ and $w'_{\mu\nu} \rightarrow w'_\mu$ for all $1 \leq \mu \leq m$. A generic choice of $w_{\mu\nu}$ (see p. 136 in [DP3]) ensures that $q_{m\nu} := Q_{w_{1\nu}} \cap Q_{w_{2\nu}} \cap \dots \cap Q_{w_{m\nu}}$ has dimension $n - m$. Each $(w_{\mu\nu}, z_\nu)$ is a pair of reflection and hence the analytic set

$$S_\nu^m := \bigcap_{1 \leq \mu \leq m} S(w_{\mu\nu}, z_\nu) \subset (q^{m\nu} \times q'^{m\nu}) \cap (U_1 \times U')$$

is well-defined. If $m = n - 1$, then Lemma 9.7 of [DP3] shows that the germ of $q'^{(n-1)}$ at the origin has dimension 1. Moreover, $\hat{S}' = Q'_{0'}$ and Lemma 4.3 implies that $q'^{(n-1)} \cap Q'_{0'}$ contains $0'$ as an isolated point. Since $\text{cl}_{\hat{F}}(0) \subset Q'_{0'}$, it follows that $0'$ is an isolated point of

$$\pi'(\text{cl}(S_\nu^{n-1}) \cap (\{0\} \times U')) \subset q'^{(n-1)} \cap Q'_{0'} = \{0'\}.$$

Shrinking U_1 , the projection $\pi: S_\nu^{n-1} \rightarrow U_1$ becomes proper and $\pi(S_\nu^{n-1}) = q^{n-1, \nu} \cap U_1$. By Theorem 7.4 of [DP3], there is a subsequence of $q^{n-1, \nu} \cap U_1$ that converges to an analytic set $A \subset U_1$ of pure dimension 1 and contains the origin. A contains points ζ_0 that do not belong to M because of the finite type assumption and $\zeta_0 \in \pi(\text{cl}(S_\nu^{n-1})) \subset \pi(\text{cl}(S(w_{\mu\nu}, z_\nu)))$. By Proposition 4.1, \hat{f} extends as an analytic set across the origin.

If $m < n - 1$, the dimension of $S_\nu^m \cap S(z_\nu, z_\nu)$ is at least $n - m - 1 > 0$. Now

$$\pi'(\text{cl}(S_\nu^m \cap S(z_\nu, z_\nu)) \cap (\{0\} \times U')) \subset q^m \cap \hat{S}' = \{0'\},$$

the last equality following from Lemma 4.3. The projection $\pi: S_\nu^m \cap S(z_\nu, z_\nu) \rightarrow U_1$ is therefore proper for small U_1 and that $\pi(S_\nu^m \cap S(z_\nu, z_\nu)) = q^{m\nu} \cap Q_{z_\nu} \cap U_1$. Again, by

Theorem 7.4 of [DP3], there is a subsequence of $q^{mv} \cap Q_{z_v} \cap U_1$ that converges to an analytic set $A \subset U_1$ of positive dimension and as before this shows that \hat{f} extends as an analytic set across the origin.

Case 2. As before, choose $(w_{1v}, w'_{1v}), (w_{2v}, w'_{2v}), \dots, (w_{kv}, w'_{kv}) \in S(z_v, z_v)$ such that $w_{\mu v} \rightarrow 0$ and $w'_{\mu v} \rightarrow w'_\mu$ for all $1 \leq \mu \leq k$ and $q_{kv} = Q_{w_{1v}} \cap Q_{w_{2v}} \cap \dots \cap Q_{w_{kv}}$, $\tilde{q}^{kv} := Q_{z_v} \cap q^{kv}$ have dimension $n - k$ and $n - k - 1$ respectively. Now note that $\dim(S_v^k \cap S(z_v, z_v)) \geq n - k - 1 > 1$. Indeed, the inequalities $2m - n + 1 \leq k < m$ show that $m \leq n - 2$ and hence $k < n - 2$. Since the dimension of $\hat{S}' \cap q^{tk}$ is $m - k$, choose coordinates so that

$$\hat{S}' \cap q^{tk} \cap \{z' \in U': z'_1 = z'_2 = \dots = z'_{m-k} = 0\} = \{0'\}.$$

Let $\hat{f}(z_v) = \{\zeta_v^j\}$, $1 \leq j \leq l$, l being the multiplicity of \hat{f} . The l sets

$$\begin{aligned} T_{v,j} &= \{(z, z') \in S_v^k \cap S(z_v, z_v): z'_1 = (\zeta_v^j)_1, \\ & z'_2 = (\zeta_v^j)_2, \dots, z'_{m-k} = (\zeta_v^j)_{m-k}\}, \end{aligned}$$

where $1 \leq j \leq l$ are analytic sets in $U_1 \times U'$ and have dimension at least $n - k - 1 - (m - k) = n - m - 1 > 0$. By construction,

$$\begin{aligned} \pi'(\text{cl}(T_{v,j}) \cap (\{0\} \times U')) &\subset \hat{S}' \cap q^{tk} \\ \cap \{z' \in U': z'_1 = z'_2 = \dots = z'_{m-k} = 0\} &= \{0'\} \end{aligned}$$

and hence by shrinking U_1, U' , the projections $\pi: T_{v,j} \rightarrow U_1$ are proper and the images $\sigma_{v,j} := \pi(T_{v,j}) \subset U_1$ are analytic and have dimension $n - m - 1$. Moreover $\sigma_{v,j} \subset \tilde{q}^{kv}$, and since \tilde{q}^{kv} depend anti-holomorphically on the k -tuple defining it, Theorem 7.4 of [DP3] shows that \tilde{q}^{kv} converges to an analytic set $\tilde{A} \subset U_1$ of dimension $n - k - 1$, after passing to a subsequence. Working with this subsequence, we see that $\text{cl}(\sigma_{v,j}) \subset \tilde{A}$. On the other hand, since $2m - n + 1 \leq k$, it follows, as in [DP3], that

$$\dim \tilde{A} = n - k - 1 \leq 2(n - m - 1) = 2 \dim \sigma_{v,j}.$$

Proposition 8.3 of [DP3] shows that $\text{cl}(\sigma_{v,j}) \not\subset M$ and hence by Proposition 4.1, it follows that \hat{f} extends as an analytic set across the origin.

To complete the proof, it suffices to show that extension as an analytic set implies extension as a locally proper holomorphic correspondence. This is achieved in the next lemma.

Lemma 4.4. *There exist neighbourhoods U of 0 and U' of $0'$ such that $F \subset U \times U'$ is a proper holomorphic correspondence which extends \hat{f} .*

Proof. Extension as a holomorphic correspondence essentially follows from [DP4]. All nuances in the proof of Proposition 2.4 in [DP4] work in this situation as well provided the following two modifications are made. Let U, U' be neighbourhoods of 0, $0'$ respectively and suppose that $F \subset U \times U'$ extends \hat{f} as an analytic set in $U \times U'$. Then it needs to be checked that $F \cap (U^+ \times U') \neq \emptyset$ and that there exists a sequence $\{z_v\} \in M$ converging to 0 such that \hat{f} extends as a correspondence across each z_v .

Suppose that $F \cap (U^+ \times U') = \emptyset$. In this case, the proof of Proposition 3.1 (or even Proposition 4.1 in [SV]) shows that $(0, 0')$ is in the envelope of holomorphy of $\overline{U^- \times U'}$. The coefficients $a_{\mu\nu}(z)$ in (3.1) can be regarded as holomorphic functions on $U^- \times U'$ (i.e., independent of the z' variables) and thus each $a_{\mu\nu}(z)$ extends holomorphically across $(0, 0')$. This extension must be independent of the z' variables by the uniqueness theorem and hence $a_{\mu\nu}(z)$ extends holomorphically across the origin. This shows that \hat{f} extends as a holomorphic correspondence across the origin. To show the existence of the sequence $\{z_\nu\}$ claimed above, let $\pi: F \rightarrow U$ be the natural projection and define

$$A = \{(z, z') \in F: \dim(\pi^{-1}(z))_{(z, z')} \geq 1\},$$

where $(\pi^{-1}(z))_{(z, z')}$ denotes the germ of the fiber over z at (z, z') . Then A is an analytic subset of F , and since F contains the graph of \hat{f} over U^- , it follows that the dimension of A is at most $n - 1$. Since Lipschitz maps do not increase Hausdorff dimension, it follows that the Hausdorff dimension of $\pi(A)$ is at most $2n - 2$. Pick $p \in M \setminus \pi(A)$. The fiber $F \cap \pi^{-1}(p)$ is discrete and this shows that \hat{f} extends as a holomorphic correspondence across p .

Finally, we show that U' can be chosen so small that the projection $\pi': F \rightarrow U'$ is also proper. Indeed, for $z' \in M'$, $\pi'^{-1}(z')$ is an analytic subset of F . Since π is proper, it follows by Remmert's theorem that $\hat{F}^{-1}(z') = \pi \circ \pi'^{-1}(z')$ is an analytic set. The invariance property of Segre varieties yields $\hat{F}(Q_z \cap U) \subset Q_{z'}$ for any $z \in \hat{F}^{-1}(z')$. Since M is of finite type, the set $\bigcup_{z \in \hat{F}^{-1}(z')} Q_z$ has Hausdorff dimension n , and therefore cannot be mapped by \hat{F} into $Q_{z'}$, which has dimension $n - 1$. This shows that projection π' has discrete fibers on M' . It follows from the Cartan–Remmert theorem that there exists a neighbourhood U' of M' such that π' has only discrete fibers, and therefore the projection π' from F to U' will be proper.

This completes the proof of Theorem 1.1. \square

5. Preservation of strata

Fix $p \in M$ and let $p'_1, p'_2, \dots, p'_k \in \hat{f}(p) \subset M'$. Choose neighbourhoods U, U' of p, p'_1 respectively and let $\hat{f}_1: U^- \rightarrow U'$ be a component of \hat{f} such that $\hat{f}_1(p) = p'_1$. Then \hat{f}_1 extends as a holomorphic correspondence $F \subset U \times U'$ and to prove Theorem 1.2, it suffices to focus on \hat{f}_1 , which will henceforth be denoted by \hat{f} . The following two general observations can be made in this situation. First, the branching locus $\hat{\sigma}$ of \hat{F} is an analytic set in U and the finite-type assumption on M shows that the real dimension of $\hat{\sigma} \cap M$ is at most $2n - 3$. The branching locus of \hat{f} denoted by σ , is contained in $\hat{\sigma} \cap U^-$. Second, the invariance property of Segre varieties in [DP1], [V] shows that \hat{F} , the extended correspondence, preserves the two components U^\pm . That is, after possibly re-labelling U'^\pm , it follows that $\hat{F}(U^\pm) \subset U'^\pm$ and $\hat{F}(M) \subset M'$. The same holds for $\hat{G} := \hat{F}^{-1}: U' \rightarrow U$.

Proof of Theorem 1.2. Let $p \in M^+$ and suppose that $\{\zeta'_j\} \in M'$ is a sequence converging to p'_1 with the property that the Levi form \mathcal{L}_ρ restricted to the complex tangent space to M at ζ'_j has at least one negative eigenvalue. Fix $\zeta'_{j_0} \in U'$ for some large j_0 . By shifting ζ'_{j_0} slightly, we may assume that $\zeta'_{j_0} \notin \hat{\sigma}' \cup \hat{F}(M^0 \cap U)$, where $\hat{\sigma}'$ is the branching locus of \hat{G} , and at the same time retain the property of having at least one negative eigenvalue. Let g_1 be a locally biholomorphic branch of \hat{G} near ζ'_{j_0} . Then $g_1(\zeta'_{j_0})$ is clearly a pseudoconvex

point and this contradicts the invariance of the Levi form. This shows that $\hat{f}(M^+) \subset M'^+$. The same arguments show that $\hat{f}(M^-) \subset M'^-$.

Let $p \in M^+ \cap M^0$ and suppose that $p'_1 \in M'^+$. The extending correspondence $\hat{F}: U \rightarrow U'$ satisfies the invariance property, namely $\hat{F}(Q_w) \subset Q'_{w'}$ for all $(w, w') \in (U \times U') \cap \text{Graph}(\hat{F})$. But near p'_1 , the Segre map λ is injective and this shows that \hat{F} , and hence \hat{f} , is a single valued, proper holomorphic mapping, say $f: U \rightarrow U'$ with $f(p) = p'_1$. Two observations can be made at this stage: first, f cannot be locally biholomorphic near p due to the invariance of the Levi form. Second, if $V_f \subset U$ is the branching locus of f defined by the vanishing of the Jacobian determinant of f , then V_f intersects both U^\pm . Indeed, suppose that $V_f \cap U^- = \emptyset$. Choose a branch of f^{-1} near some fixed point $a' \in U'^-$ and analytically continue it along all paths in U'^- to get a well-defined mapping, say $g: U'^- \rightarrow U^-$. The analytic set $F \subset U \times U'$ extends g as a correspondence and hence [DP2] g is a well-defined holomorphic mapping in U' and this must be the single valued inverse of f . Thus f is locally biholomorphic near p and this is a contradiction. The same argument works to show that V_f must intersect U^+ as well. Note that $V_f \cap M$ has real dimension at most $2n - 3$. If $p \in \Gamma_1$, choose U so small that $M^0 \cap U \subset \Gamma_1$. Then there exists $q \in \Gamma_1 \setminus (V_f \cap M)$ near p , where f is locally biholomorphic. Thus q is mapped locally biholomorphically to $f(q)$ which is a strongly pseudoconvex point and this is a contradiction. If $p \in \Gamma_3$, then again we shrink U so that $M^0 \cap U \subset \Gamma_3$ and $(M \cap U) \setminus \Gamma_3 \subset M'^+$. Then f is locally biholomorphic near all points in $(M \cap U) \setminus \Gamma_3$ and therefore $V_f \cap U^-$ must cluster only along Γ_3 . Since the CR dimension of $\Gamma_3 = n - 3 < (n - 1) - 1$ which is one less than the dimension of V_f , it follows (Theorem 18.5 in [C]) that $\overline{V_f \cap U^-}$ is a closed, analytic set in U . Thus $V_f \cap U^-$ has two analytic continuations, namely V_f and $\overline{V_f \cap U^-}$ and therefore they must be the same. This shows that V_f cannot intersect U^+ which is a contradiction. The same argument works if $p \in \Gamma_4$, the only difference being that $\overline{V_f} \subset U^-$ is analytic because of Shiffman's theorem. Thus if $p \in M^+ \cap M^0$, then $p'_1 \in M'^+ \cap M'^0$.

To study this further, suppose that $p \in M^+ \cap \Gamma_1$ and $p'_1 \in M'^+ \cap \Gamma'_2$. Choose U, U' small enough so that $M^0 \cap U \subset \Gamma_1$ and $M'^0 \cap U' \subset \Gamma'_2$. Pick $q \in \Gamma_1 \setminus (\hat{\sigma} \cap M)$. Then \hat{f} splits near q into finitely many well-defined holomorphic mappings each of which extends across q . Moving q slightly, if needed, on $\Gamma_1 \setminus (\hat{\sigma} \cap M)$, each of these holomorphic mappings are even locally biholomorphic near q . Working with one of these mappings, say f_1 , it follows that $f_1(q) \notin M'^+$ due to the invariance of the Levi form. This means that $f_1(q) \in \Gamma'_2$. In the same way, all points in Γ_1 that are sufficiently near q are mapped locally biholomorphically by f_1 to Γ'_2 . This cannot happen as Γ'_2 has strictly smaller dimension than Γ_1 . The same argument shows that $p'_1 \notin \Gamma'_3 \cup \Gamma'_4$. Hence $p'_1 \in M'^+ \cap \Gamma'_1$.

Suppose that $p \in M^+ \cap \Gamma_2$ and $p'_1 \in M'^+ \cap \Gamma'_1$. Considering $\hat{f}^{-1}: U' \rightarrow U$, the arguments used in the preceding lines show that this cannot happen. The case when $p'_1 \in \Gamma'_4$ can be dealt with similarly. Now suppose that $p'_1 \in \Gamma'_3$. As always, U, U' will be small enough so that $M^0 \cap U \subset \Gamma_2$ and $M'^0 \cap U' \subset \Gamma'_3$. The arguments used above show that the cluster set of points in $M^+ \cap U$ is contained in $M'^+ \cap U'$ and hence \hat{f} splits into finitely well-defined mappings each of which is locally biholomorphic near points in $M^+ \cap U$. This shows that the branching locus $\sigma \subset U^-$ of \hat{f} clusters only along Γ_2 . Then $\hat{F}(\sigma)$ is an analytic set of dimension $n - 1$ in U'^- . There are two cases to consider: first, if $\hat{F}(\sigma)$ clusters only along Γ'_3 , then arguing as above, $\overline{\hat{F}(\sigma)} \subset U'^-$ is a closed, analytic set in U' . The strong disk theorem shows that p'_1 is in the envelope of holomorphy of U'^- and this is a contradiction. Second, if there are points in $\overline{\hat{F}(\sigma)} \cap M'^+$, this means that $(\overline{\hat{F}(\sigma)} \cap M') \cap \Gamma'_3$

has real dimension at most $2n - 4$. Pick $q' \in \Gamma'_3 \setminus \overline{\hat{F}(\hat{\sigma})} \cap M'$ and note that the continuity of \hat{f} implies that $\hat{f}^{-1}(q') \in M_s^+$. As seen above, this cannot happen. Thus $p'_1 \in \Gamma'_2$ or $M_s^{'+}$. Similar arguments show that if $p \in M^+ \cap \Gamma_3$ or $M^+ \cap \Gamma_4$, then $p'_1 \in M'^+ \cap \Gamma'_3$ or $M'^+ \cap \Gamma'_4$ respectively.

By reversing the roles of U^\pm , the same arguments used in the preceding paragraphs can be applied to show that $\hat{f}(M^- \cap M^0) \subset M'^- \cap M'^0$ with the preservation of $M^- \cap \Gamma_j$ for $j = 1, 3, 4$.

Finally, fix integers i, j both at least 1 such that $i + j = n - 1$ and suppose that $p \in M_{i,j}$. Then there exists a point p_0 , in U (chosen so small that $M \cap U \subset M_{i,j}$) and arbitrarily close to p , where all branches of \hat{f} are well-defined and locally biholomorphic. The invariance of the Levi form shows that the images of p_0 under any of the branches of \hat{f} should all be in $M_{i,j}$. Note that each of these images is close to p'_1 . This cannot happen if p'_1 is in M'^+, M'^- or in $M'_{i',j'}$ for $i \neq i'$ and $j \neq j'$. The only possibility is that p'_1 is in the relative interior of $\overline{M'_{i,j}}$. The same argument works if p is in the relative interior of $\overline{M_{i,j}}$. \square

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References

- [BR] Baouendi M and Rothschild L, Germs of CR maps between real analytic hypersurfaces, *Invent. Math.* **93** (1988) 481–500
- [B] Bedford E, Proper holomorphic mappings from domains with real analytic boundary, *Am. J. Math.* **106** (1984) 745–760
- [BS] Berteloot F and Sukhov A, On the continuous extension of holomorphic correspondences, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **24** (1997) 747–766
- [C] Chirka E M, Complex analytic sets (Dordrecht: Kluwer) (1990)
- [DF] Diederich K and Fornaess J E, Proper holomorphic mappings between real analytic pseudoconvex domains in \mathbb{C}^n , *Math. Ann.* **282** (1988) 681–700
- [DP1] Diederich K and Pinchuk S, Proper holomorphic maps in dimension 2 extend, *Indiana Univ. Math. J.* **44** (1995) 1089–1126
- [DP2] Diederich K and Pinchuk S, Reflection principle in higher dimensions, *Doc. Math. J. Extra Volume ICM* (1998) Part II, pp. 703–712
- [DP3] Diederich K and Pinchuk S, Regularity of continuous CR maps in arbitrary dimension, *Mich. Math. J.* **51(1)** (2003) 111–140
- [DP4] Diederich K and Pinchuk S, Analytic sets extending the graphs of holomorphic mappings, *J. Geom. Anal.* **14(2)** (2004) 231–239
- [DW] Diederich K and Webster S, A reflection principle for degenerate real hypersurfaces, *Duke Math. J.* **47** (1980) 835–845
- [S1] Shafikov R, Analytic continuation of germs of holomorphic mappings between real hypersurfaces in \mathbb{C}^n , *Mich. Math. J.* **47(1)** (2001) 133–149
- [S2] Shafikov R, On boundary regularity of proper holomorphic mappings, *Math. Z.* **242(3)** (2002) 517–528
- [SV] Shafikov R and Verma K, A local extension theorem for proper holomorphic mappings in \mathbb{C}^2 , *J. Geom. Anal.* **13(4)** (2003) 697–714
- [V] Verma K, Boundary regularity of correspondences in \mathbb{C}^2 , *Math. Z.* **231(2)** (1999) 253–299