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A functional central limit theorem for a class of urn models

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Abstract. We construct an independent increments Gaussian process associated to a class of multicolor urn models. The construction uses random variables from the urn model which are different from the random variables for which central limit theorems are available in the two color case.

Keywords. Urn models; functional central limit theorem; Gaussian processes.

1. Introduction

Consider a four-color urn model in which the replacement matrix is actually a stochastic matrix $\mathbf{R}$ as in ref. [4]. That is, we start with one ball of any color, which is the 0-th trial. Let $\mathbf{W}_n$ denote the column vector of the number of balls of the four colors up to the n-th trial, where the components of $\mathbf{W}_n$ are nonnegative real numbers. Then a color is observed by random sampling from a multinomial distribution with probabilities $(1/(n+1))\mathbf{W}_n$. Depending on the color that is observed, the corresponding row of $\mathbf{R}$ is added to $\mathbf{W}_n'$ and this gives $\mathbf{W}_{n+1}'$. A special case of the main theorem of Gouet [4] is that if the stochastic matrix $\mathbf{R}$ is irreducible, then $(1/(n+1))\mathbf{W}_n'$ converges almost surely (a.s.) to the stationary distribution π of the irreducible stochastic matrix $\mathbf{R}$ (it should be carefully noted that the multicolor urn model is vastly different from the Markov chain evolving according to the transition matrix equal to the stochastic matrix $\mathbf{R}$, also notice that π is a row vector). Suppose the nonprincipal eigenvalues of $\mathbf{R}$ satisfy $\lambda_1 < 1/2$, $\lambda_2 = 1/2$, $\lambda_3 > 1/2$ respectively, which are assumed to be real (and hence lie in $(-1, 1)$), and ξ_1, ξ_2, ξ_3 be the corresponding eigenvectors. Using $\pi\xi_i = \pi\mathbf{R}\xi_i = \lambda_i\pi\xi_i$ it is seen that $(1/(n+1))\mathbf{W}_n'\xi_i \to 0$.

Central and functional central limit theorems for $\mathbf{W}_n'\xi_i$ have been the subject of several papers in the literature [2,3,7] especially for two-color models and also some multicolor models. The norming in the central limit theorems in the two color urn models depends on the nonprincipal eigenvalue as follows: for $\lambda < 1/2$ the rate is $\sqrt{n}$, for $\lambda = 1/2$ the rate is $\sqrt{n\log n}$ and the limits are normal in these two cases. However for $\lambda > 1/2$ the rate is $\Pi_0^{n-1}(1 + (\lambda/(j+1))$ and in this case the limit exists almost surely.

Functional central limit theorems (FCLT) for a class of two-color urn models have been considered by Gouet [3]. These FCLT's of Gouet [3] use the same norming, as stated in the previous paragraph, under which central limit theorems have been proved in [2] and [3]. Ref. [5] contains a survey of the literature on such FCLT's. In this article we prove a different FCLT that uses random variables with the norming $\Pi_0^{n-1}(1 + (\lambda/(j+1))$ irrespective of whether λ is less than 1/2, equal to 1/2 or greater than 1/2. This is the main result of the paper. For the sake of convenience we restrict ourselves to real eigenvalues only. We state the result for the above four-color model but it can be seen from the proof that it can be extended to urn models with any number of colors.

The article is organized as follows. In §2 we develop the notation, state the main result and give its proof. Some of the calculations have been done separately in §3.

2. Main result

We write

$$Z_{i,n} = \frac{\mathbf{W}_n' \xi_i}{\Pi_0^{n-1}\left(1 + \frac{\lambda_i}{j+1}\right)}, \tag{1}$$

where ξ_i is the eigenvector corresponding to the eigenvalue λ_i. From the description of the urn model we have $\mathbf{W}_{n+1}'\xi_i = \mathbf{W}_n'\xi_i + \chi_{n+1}'\mathbf{R}\xi_i = \mathbf{W}_n'\xi_i + \lambda_i\chi_{n+1}'\xi_i$, where χ_{n+1} is the column vector consisting of the indicator functions of balls of the four colors respectively. We also have

$$E\{\chi_{n+1}'\xi_i|\mathcal{F}_n\} = \frac{1}{n+1}\mathbf{W}_n'\xi_i, \tag{2}$$

where $\mathcal{F}_n$ is the σ-field of observations up to the n-th trial. From this it follows that $Z_{i,n}$ is a martingale. From §3, it follows that $Z_{3,n}$ is L^2-bounded so that it converges almost surely. However in the two color case, for $\lambda < 1/2$, $\mathbf{W}_n'\xi/\sqrt{n}$ and for $\lambda = 1/2$, $\mathbf{W}_n'\xi/\sqrt{n\log n}$ converge to normal distributions and the FCLT's proved in [3] use such normalizations. Thus the question of using the same norming $\Pi_0^{n-1}(1 + (\lambda/(j+1)) \sim n^{\lambda}$, to get an FCLT irrespective of $\lambda < 1/2, \lambda = 1/2$ or $\lambda > 1/2$, is of interest. Our main result, Proposition 2.1, is a step in this direction using the tails of the sequence $(Z_{1,n}, Z_{2,n}, Z_{3,n})$ whereas the FCLT's in the literature are based on partial sums starting from the beginning.

PROPOSITION 2.1

The sequence of processes $\mathbf{G}_n(t) = (G_{1,n}(t), G_{2,n}(t), G_{3,n}(t))$ *where*

$$G_{i,n}(t) = \sum_{m=n}^{[ne^t]} m^{\lambda_i - 1/2}(Z_{i,m+1} - Z_{i,m}), \quad i = 1, 2, 3, \; t \geq 0,$$

converges to an independent increments Gaussian process $\mathbf{G}(t)$ *with covariance function* $c_{i,j}(t) = t\lambda_i\lambda_j\langle\xi_i\xi_j, \pi'\rangle, i, j = 1, 2, 3,$ *where the vector of the coordinate-wise product of the components of the two vectors* ξ_i *and* ξ_j *is denoted by* $\xi_i\xi_j$ *and the Euclidean inner product of the two vectors is denoted by* $\langle .,. \rangle$.

Note that the process $\mathbf{G}$ can be viewed as a multidimensional Wiener process with covariance function $c_{i,j}(\cdot)$.

Proof. From eq. (1) we have the following expansion:

$$Z_{i,m+1} - Z_{i,m} \sim -\frac{\lambda_i}{m} Z_{i,m} + \lambda_i \frac{\chi'_{m+1}\xi_i}{\Pi_{k=0}^{m}\left(1 + \frac{\lambda_i}{k+1}\right)}$$

$$\sim -\frac{\lambda_i}{m} Z_{i,m} + \lambda_i \frac{\chi'_{m+1}\xi_i}{m^{\lambda_i}}. \tag{3}$$

Since the components of $\mathbf{G}_n(t)$ are martingales, an independent increments Gaussian process as a limiting process is expected. In particular we follow Theorem 1.4, p. 339 of [1], by which it is enough to show that the joint characteristics of the martingales converge to a joint covariance function. Note that from (3) $m^{\lambda_i - \frac{1}{2}}(Z_{i,m+1} - Z_{i,m}) = O(1/\sqrt{m})$, as $\mathbf{W}'_m \xi_i / m$ and $\chi'_{m+1}\xi_i$ are bounded. This takes care of continuity of the paths and cross quadratic variations which is condition (b) of that theorem. Thus it remains to show that the cross quadratic variations converge to $c_{i,j}(t)$. We first do this for $i = 1$, $j = 2$. From (3) we have

$$m^{\lambda_1 - 1/2}(Z_{1,m+1} - Z_{1,m}) \sim -\lambda_1 m^{\lambda_1 - 1/2} \frac{\mathbf{W}'_m \xi_1}{m^{\lambda_1 + 1}} + \lambda_1 m^{\lambda_1 - 1/2} \frac{(\chi'_{m+1}\xi_1)}{m^{\lambda_1}},$$

$$m^{\lambda_2 - 1/2}(Z_{2,m+1} - Z_{2,m}) \sim -\lambda_2 m^{\lambda_2 - 1/2} \frac{\mathbf{W}'_m \xi_2}{m^{\lambda_2 + 1}} + \lambda_2 m^{\lambda_2 - 1/2} \frac{(\chi'_{m+1}\xi_2)}{m^{\lambda_2}}. \tag{4}$$

We want to show that in computing the cross quadratic variation, which is the limit of

$$\sum_{n}^{[ne^t]} E\{m^{\lambda_1 - 1/2} m^{\lambda_2 - 1/2}(Z_{1,m+1} - Z_{1,m})(Z_{2,m+1} - Z_{2,m}) | \mathcal{F}_m\},$$

only the second term from the right-hand side of each of eqs (4) contributes. Since χ_{n+1} consists of indicator functions, which implies that

$$\left(\sum_k \xi_{1,k}\chi_{n+1,k}\right)\left(\sum_l \xi_{2,l}\chi_{n+1,l}\right) = \sum_k \xi_{1,k}\xi_{2,k}\chi_{n+1,k},$$

this contribution is the limit of

$$\lambda_1\lambda_2 \sum_{n}^{[ne^t]} \frac{1}{m}\left\langle \xi_1\xi_2, \frac{\mathbf{W}_m}{m+1}\right\rangle,$$

which is $t\lambda_1\lambda_2\langle\xi_1\xi_2, \pi'\rangle$, since from [4] we know $\mathbf{W}'_m/(m+1) \to \pi$ a.s. Also notice that this part of the argument does not depend on whether λ_1 or λ_2 are less than or equal to 1/2.

To see why the contribution to the cross quadratic variation from the first terms of (4) goes to 0, by Cauchy–Schwarz inequality it is enough to show that the sum of squares over n to $[ne^t]$ of the first terms in each line of (4) goes to 0. This part of the argument will depend on the value of λ_i. Note the following which have been proved in §3:

$$\text{For } \lambda_1 < 1/2, \quad \frac{\mathbf{W}'_m \xi_1}{\sqrt{m}} \text{ is } L^2\text{-bounded,} \tag{5}$$

$$\text{For } \lambda_2 = 1/2, \quad \frac{\mathbf{W}'_m \xi_2}{\sqrt{m \log m}} \text{ is } L^2\text{-bounded.} \tag{6}$$

Consider the case $\lambda_1 < 1/2$. We need to show

$$\sum_n^{[ne^t]} \frac{(\mathbf{W}'_m \xi_1)^2}{m^3} \to 0 \text{ a.s.}$$

We know that for $\lambda_1 < 1/2$, $\mathbf{W}'_m \xi_1/\sqrt{m}$ is L^2-bounded, so that

$$E \sum_n^{[ne^t]} \frac{(\mathbf{W}'_m \xi_1)^2}{m^3} \leq E \sum_n^{\infty} \frac{(\mathbf{W}'_m \xi_1)^2}{m^3} \leq \text{const.} \sum_n^{\infty} \frac{1}{m^2} \to 0. \tag{7}$$

Since the sum inside the expectation in the middle is decreasing in n, it converges to 0 a.s. For $\lambda_2 = 1/2$, $\mathbf{W}'_m \xi_2/\sqrt{m \log m}$ is L^2-bounded, and one can proceed similarly. Thus we have proved that $c_{1,2}(t) = t\lambda_1\lambda_2\langle \xi_1\xi_2, \pi'\rangle$. Similarly $c_{i,j}(t)$, $i, j = 1, 2$, can be computed as given in Proposition 2.1.

Now consider as to what will happen if we were computing say $c_{1,3}(t)$. For $\lambda_3 > 1/2$, the expansion (4) is similar, and in the cross quadratic variation the contribution of the second term from the right-hand side of

$$m^{\lambda_3 - 1/2}(Z_{3,m+1} - Z_{3,m}) \sim -\lambda_3 m^{\lambda_3 - 1/2} \frac{\mathbf{W}'_m \xi_3}{m^{\lambda_3+1}} + \lambda_3 m^{\lambda_3 - 1/2} \frac{(\chi'_{m+1}\xi_3)}{m^{\lambda_3}}$$

is similar to what we had before. For $\lambda_3 > 1/2$, $\mathbf{W}'_m \xi_3/\Pi_0^{m-1}(1 + (\lambda_3/(j+1))$ is a martingale and from Appendix 3.3,

$$\frac{\mathbf{W}'_m \xi_3}{m^{\lambda_3}} \text{ is } L^2\text{-bounded.} \tag{8}$$

So $\mathbf{W}'_m \xi_3/m^{\lambda_3}$ converges almost surely. This implies that the contribution of the first term

$$\sum_n^{[ne^t]} \frac{(\mathbf{W}'_m \xi_3)^2}{m^{2\lambda_3}} \frac{1}{m^{3-2\lambda_3}} \to 0 \text{ a.s.}$$

since $2\lambda_3 < 2$. Thus $c_{i,j}(t)$, $i = 1, 2, 3$, $j = 3$, can be computed as given in the statement of Proposition 2.1. This completes the proof. □

3. Appendix

Suppose real eigenvalues satisfy $\lambda_1 < 1/2, \lambda_2 = 1/2, \lambda_3 > 1/2$ and ξ_1, ξ_2, ξ_3 be the corresponding eigenvectors. In this section we prove that X_n, Y_n and Z_n are L^2-bounded where

$$X_n = \frac{\mathbf{W}'_n \xi_1}{\sqrt{n}}, \quad Y_n = \frac{\mathbf{W}'_n \xi_2}{\sqrt{n \log n}}, \quad Z_n = \frac{\mathbf{W}'_n \xi_3}{\Pi_0^{n-1}\left(1 + \frac{\lambda_3}{j+1}\right)}, \tag{9}$$

a fact which has been used in the proof of Proposition 2.1. For X_n and Y_n verification of L^2-boundedness is through Lemma 2.1 of [6]. This is done on a case by case basis depending on λ_1 and λ_2 in the next two subsections. For the reader's convenience we state Kersting's lemma from [6] here:

Lemma 2.1 [6]. *Let* α_n, β_n $(n \geq 1)$ *be nonnegative numbers such that* $\alpha_n \to 0, \sum_{n=1}^{\infty} \alpha_n = \infty$, *and for large* n,

$$\beta_{n+1} \leq \beta_n(1 - c\alpha_n) + d\alpha_n$$

with $c, d > 0$. *Then* $\limsup_{n\to\infty} \beta_n \leq d/c$.

3.1 L^2-boundedness of X_n

Using $\mathbf{W}'_{n+1}\xi_1 = \mathbf{W}'_n\xi_1 + \lambda_1\chi'_{n+1}\xi_1$ and the definition of X_n, we get

$$X_{n+1} = X_n\sqrt{\frac{n}{n+1}} + \lambda_1\frac{\chi'_{n+1}\xi_1}{\sqrt{n+1}}.$$

Taking conditional expectation and using (2) we get

$$E\{X_{n+1}^2|\mathcal{F}_n\} = X_n^2\left(1-\frac{1}{n+1}\right)\left(1+\frac{2\lambda_1}{n+1}\right) + \frac{\lambda_1^2}{n+1}\left\langle\frac{\mathbf{W}_n}{n+1},\xi_1^2\right\rangle,$$

from which taking further expectation we get

$$EX_{n+1}^2 = EX_n^2\left(1-\frac{1}{n+1}\right)\left(1+\frac{2\lambda_1}{n+1}\right) + \frac{\lambda_1^2}{n+1}\left\langle E\frac{\mathbf{W}_n}{n+1},\xi_1^2\right\rangle.$$

The last vector $E(\mathbf{W}_n/(n+1))$ consists of bounded components. Thus if $\lambda_1 < 0$, then

$$EX_{n+1}^2 \le EX_n^2\left(1-\frac{1}{n+1}\right) + \frac{\text{const}}{n+1},$$

and Kersting's lemma applies. If $\lambda_1 > 0$ then we still have $\lambda_1 < 1/2$ i.e. $2\lambda_1 < 1$. In this case

$$\begin{aligned}\left(1-\frac{1}{n+1}\right)\left(1+\frac{2\lambda_1}{n+1}\right) &\le 1+\frac{2\lambda_1}{n+1}-\frac{1}{n+1}\\ &= 1-\frac{1-2\lambda_1}{n+1},\end{aligned}$$

i.e. Kersting's lemma applies.

3.2 L^2-boundedness of Y_n

Using $\mathbf{W}'_{n+1}\xi_2 = \mathbf{W}'_n\xi_2 + \lambda_2\chi'_{n+1}\xi_2$ and the definition of Y_n, we get

$$Y_{n+1} = Y_n\sqrt{\frac{n\log n}{(n+1)\log(n+1)}} + \lambda_2\frac{\chi'_{n+1}\xi_2}{\sqrt{(n+1)\log(n+1)}}.$$

Taking conditional expectation we get (recall $\lambda_2 = 1/2$)

$$\begin{aligned}E\{Y_{n+1}^2|\mathcal{F}_n\} = Y_n^2&\frac{n\log n}{(n+1)\log(n+1)}\left(1+\frac{1}{n+1}\right)\\ &+\frac{\lambda_2^2}{(n+1)\log(n+1)}\left\langle\frac{\mathbf{W}_n}{n+1},\xi_2^2\right\rangle,\end{aligned}$$

from which taking further expectation we get

$$\begin{aligned}EY_{n+1}^2 = EY_n^2&\left(1-\frac{(n+1)\log(n+1)-n\log n}{(n+1)\log(n+1)}\right)\left(1+\frac{1}{n+1}\right)\\ &+\frac{\lambda_2^2}{(n+1)\log(n+1)}\left\langle E\frac{\mathbf{W}_n}{n+1},\xi_2^2\right\rangle.\end{aligned}$$

The vector $E(\mathbf{W}_n/(n+1))$ consists of bounded components. Now we apply the second trick of the previous subsection to apply Kersting's lemma. The following calculation does the rest of the work.

We show that

$$(n+1)\log(n+1)\left\{\frac{(n+1)\log(n+1) - n\log n}{(n+1)\log(n+1)} - \frac{1}{n+1}\right\} \to c > 0.$$

We approximate $\log(n+1)$ by $\log n + \frac{1}{n}$. This gives

$$(n+1)\log(n+1) - n\log n \sim (n+1)\log n + (n+1)\frac{1}{n} - n\log n$$

$$= \log n + 1 + \frac{1}{n}.$$

Hence

$$\left(\log n + 1 + \frac{1}{n}\right) - \log(n+1) \sim \left(\log n + 1 + \frac{1}{n}\right) - \left(\log n + \frac{1}{n}\right) \to 1.$$

3.3 L^2-*boundedness of* Z_n

The proof follows Lemma 3.1 of [2]. We have earlier proved the approximation

$$Z_{n+1} - Z_n \sim -\frac{\lambda_3}{n} Z_n + \lambda_3 \frac{\chi'_{n+1}\xi_3}{n^{\lambda_3}}.$$

Now with the martingale property of Z_n, $Z_{n+1} = Z_n + (Z_{n+1} - Z_n)$, and by the above approximation we have

$$E(Z_{n+1}^2 | \mathcal{F}_n) \sim Z_n^2 + \left(\frac{\lambda_3^2}{n^2} Z_n^2 - 2\frac{\lambda_3^2}{n^2} Z_n^2 + \lambda_3^2 \frac{\left\langle \frac{\mathbf{W}_n}{n+1}, \xi_3^2 \right\rangle}{n^{2\lambda_3}}\right),$$

implying

$$E Z_{n+1}^2 \leq E Z_n^2 \left(1 - \frac{\lambda_3^2}{n^2}\right) + \lambda_3^2 \frac{\text{const}}{n^{2\lambda_3}}. \tag{10}$$

Since $2\lambda_3 > 1$, by iteration of (10) it follows that Z_n is L^2-bounded.

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