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Basic topological and geometric properties of Cesàro–Orlicz spaces

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Abstract. Necessary and sufficient conditions under which the Cesàro–Orlicz sequence space ces_ϕ is nontrivial are presented. It is proved that for the Luxemburg norm, Cesàro–Orlicz spaces ces_ϕ have the Fatou property. Consequently, the spaces are complete. It is also proved that the subspace of order continuous elements in ces_ϕ can be defined in two ways. Finally, criteria for strict monotonicity, uniform monotonicity and rotundity (= strict convexity) of the spaces ces_ϕ are given.

Keywords. Cesàro–Orlicz sequence space; Luxemburg norm; Fatou property; order continuity; strict monotonicity; uniform monotonicity; rotundity.

1. Introduction

As usual, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ denote the sets of reals, nonnegative reals and natural numbers, respectively. The space of all real sequences $x = (x(i))_{i=1}^{\infty}$ is denoted by l^0.

A map $\phi\colon \mathbb{R} \to [0, +\infty]$ is said to be an Orlicz function if ϕ is even, convex, left continuous on $\mathbb{R}_+$, continuous at zero, $\phi(0) = 0$ and $\phi(u) \to \infty$ as $u \to \infty$. If ϕ takes value zero only at zero we will write $\phi > 0$ and if ϕ takes only finite values we will write $\phi < \infty$ [1, 13, 17–20].

The arithmetic mean map σ is defined on l^0 by the formula:

$$\sigma x = (\sigma x(i))_{i=1}^{\infty}, \quad \text{where } \sigma x(i) = \frac{1}{i}\sum_{j=1}^{i} |x(j)|.$$

Given any Orlicz function ϕ, we define on l^0 the following two convex modulars [18, 19]

$$I_\phi(x) = \sum_{i=1}^{\infty} \phi(x(i)), \quad \rho_\phi(x) = I_\phi(\sigma x).$$

The space

$$ces_\phi = \{x \in l^0 : \quad \rho_\phi(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

where ϕ is an Orlicz function which is called the Cesàro–Orlicz sequence space. We equip this space with the Luxemburg norm

$$\|x\|_\phi = \inf\left\{\lambda > 0 : \quad \rho_\phi\left(\frac{x}{\lambda}\right) \le 1\right\}.$$

In the case when $\phi(u) = |u|^p$, $1 \le p < \infty$, the space ces_ϕ is nothing but the Cesàro sequence space ces_p (see [5–7, 14, 16, 21]) and the Luxemburg norm generated by this power function is then expressed by the formula

$$\|x\|_{ces_p} = \left[\sum_{i=1}^{\infty}\left(\frac{1}{i}\sum_{j=1}^{i}|x(j)|\right)^p\right]^{\frac{1}{p}}.$$

A Banach space $(X, \|\cdot\|)$ which is a subspace of l^0 is said to be a Köthe sequence space, if:

(i) for any $x \in l^0$ and $y \in X$ such that $|x(i)| \le |y(i)|$ for all $i \in \mathbb{N}$, we have $x \in X$ and $\|x\| \le \|y\|$,
(ii) there is $x \in X$ with $x(i) \ne 0$ for all $i \in \mathbb{N}$.

Any nontrivial Cesàro–Orlicz sequence space belongs to the class of Köthe sequence spaces.

An element x from a Köthe sequence space $(X, \|\cdot\|)$ is called order continuous if for any sequence (x_n) in X_+ (the positive cone of X) such that $x_n \le |x|$ and $x_n \to 0$ coordinatewise, we have $\|x_n\| \to 0$.

A Köthe sequence space X is said to be order continuous if any $x \in X$ is order continuous. It is easy to see that X is order continuous if and only if $\|(0, \ldots, 0, x(n+1), x(n+2), \ldots)\| \to 0$ as $n \to \infty$ for any $x \in X$.

A Köthe sequence space X is called monotone complete if for any $x \in X_+$ and any sequence (x_n) in X_+ such that $x_n(i) \le x_{n+1}(i) \le \cdots \le x(i)$ for all $i \in \mathbb{N}$ and $x_n \to x$ coordinatewise, we have $\|x_n\| \to \|x\|$.

We say a Köthe sequence space X has the Fatou property if for any sequence (x_n) in X_+ and any $x \in l^0$ such that $x_n \to x$ coordinatewise and $\sup_n \|x_n\| < \infty$, we have that $x \in X$ and $\|x_n\| \to \|x\|$. For the above properties of Köthe sequence (and function) spaces we refer to [12] and [15].

We say an Orlicz function ϕ satisfies the Δ_2-condition at zero ($\phi \in \Delta_2(0)$ for short) if there are $K > 0$ and $a > 0$ such that $\phi(a) > 0$ and $\phi(2u) \le K\phi(u)$ for all $u \in [0, a]$.

A modular ρ (for its definition see [4, 18, 19]) is said to satisfy the Δ_2-condition if for any $\epsilon > 0$ there exist constants $k \ge 2$ and $a > 0$ such that $\rho(2x) \le k\rho(x) + \epsilon$ for all $x \in X$ with $\rho(x) \le a$.

If ρ satisfies the Δ_2-condition for any $a > 0$ and $\epsilon > 0$ with $k \ge 2$ dependent on a and ϵ, we say that ρ satisfies the strong Δ_2-condition ($\rho \in \Delta_2^S$ for short) (see [4]).

We say a Köthe sequence space X is strictly monotone, and then we write $X \in (SM)$, if $\|x\| < \|y\|$ for all $x, y \in X$ such that $0 \le x \le y$ and $x \ne y$.

We say a Köthe sequence space X is uniformly monotone, and then we write $X \in (UM)$, if for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for any $x, y \ge 0$ such that $\|x\| = 1$ and $\|y\| \ge \epsilon$, we have $\|x + y\| \ge 1 + \delta(\epsilon)$.

Let $B(X)$ (resp. $S(X)$) be the closed unit ball (resp. the unit sphere) of X. A point $x \in S(X)$ is called an *extreme point* of $B(X)$ if for every $y, z \in B(X)$ the equality $2x = y + z$ implies $y = z$. Let Ext $B(X)$ denote the set of all extreme points of $B(X)$. A Banach space X is said to be *rotund* (write **(R)** for short), if Ext $B(X) = S(X)$. For these and other geometric notions of rotundity type and their role in mathematics we refer to the monographs [1, 8, 19] and also to the papers [2, 3, 10, 11, 22].

We say that $u \in \mathbb{R}$ is a *point of strict convexity* of ϕ if $\phi\left(\frac{v+w}{2}\right) < \frac{\phi(v)+\phi(w)}{2}$, whenever $u = \frac{v+w}{2}$ and $v \neq w$. We denote by S_ϕ the set of all *points of strict convexity* of ϕ.

An interval $[a, b]$ is called a *structurally affine interval* for an Orlicz function ϕ, or simply, SAI of ϕ, provided that ϕ is affine on $[a, b]$ for any $\varepsilon > 0$ and it is not affine either on $[a - \varepsilon, b]$ or on $[a, b + \varepsilon]$. Let $\{[a_i, b_i]\}_i$ be all the SAIs of ϕ. It is obvious that

$$S_\phi = \mathbb{R} \setminus \bigcup_i (a_i, b_i).$$

2. Results

First we present necessary and sufficient conditions for nontriviality of ces_ϕ.

Theorem 2.1. *The following conditions are equivalent*:

1) $ces_\phi \neq \{0\}$,
2) $\exists_{n_1} \sum_{n=n_1}^{\infty} \phi\left(\frac{1}{n}\right) < \infty$,
3) $\forall_k > 0 \quad \exists_{n_k} \sum_{n=n_k}^{\infty} \phi\left(\frac{k}{n}\right) < \infty$.

Proof.

(1) $\Rightarrow$ (2). Let $0 \neq z \in ces_\phi$. Since $z \neq 0$, there exists $l \in \mathbb{N}$ such that $z(l) \neq 0$. Hence $y = (0, \ldots, 0, z(l), 0, \ldots) \in ces_\phi$, and consequently, $x = (0, \ldots, 0, 1, 0 \ldots) \in ces_\phi$, which means that there exists $k > 0$ such that $\rho_\phi(kx) = \sum_{n=l}^{\infty} \phi\left(\frac{k}{n}\right) < \infty$. We will consider two cases:

1. $k > 1$. Then for all n we have $\frac{1}{n} < \frac{k}{n}$. From monotonicity of the function ϕ we have $\phi(\frac{1}{n}) < \phi(\frac{k}{n})$ for all n. Therefore
$$\sum_{n=l}^{\infty} \phi\left(\frac{1}{n}\right) < \sum_{n=l}^{\infty} \phi\left(\frac{k}{n}\right) < \infty.$$
So it is enough to take $n_1 = l$.
2. $0 < k < 1$. Then there exists $m \in \mathbb{N}$ such that $\frac{1}{m} \leq k$, whence $\frac{1}{mn} \leq \frac{k}{n}$ for all $n \in \mathbb{N}$ and so, $\sum_{n=l}^{\infty} \phi\left(\frac{1}{mn}\right) \leq \sum_{n=l}^{\infty} \phi(\frac{k}{n})$. Consequently,
$$\sum_{n=ml}^{\infty} \phi\left(\frac{1}{n}\right) = \phi\left(\frac{1}{ml}\right) + \phi\left(\frac{1}{ml+1}\right) + \cdots + \phi\left(\frac{1}{ml+(m-1)}\right)$$
$$+ \phi\left(\frac{1}{m(l+1)}\right) + \phi\left(\frac{1}{m(l+1)+1}\right)$$
$$+ \cdots + \phi\left(\frac{1}{m(l+1)+(m-1)}\right) + \cdots \leq \phi\left(\frac{1}{ml}\right) + \phi\left(\frac{1}{ml}\right)$$

$$+\cdots+\phi\left(\frac{1}{ml}\right)+\phi\left(\frac{1}{m(l+1)}\right)+\phi\left(\frac{1}{m(l+1)}\right)$$

$$+\cdots+\phi\left(\frac{1}{m(l+1)}\right)+\cdots=m\phi\left(\frac{1}{ml}\right)$$

$$+m\phi\left(\frac{1}{m(l+1)}\right)+\cdots=m\sum_{n=l}^{\infty}\phi\left(\frac{1}{mn}\right)\leq m\sum_{n=l}^{\infty}\phi\left(\frac{k}{n}\right)<\infty.$$

Taking $n_1 := ml$, we get the thesis of condition (2).

(2) $\Rightarrow$ (3). Assume that there exists n_1 such that $\sum_{n=n_1}^{\infty}\phi\left(\frac{1}{n}\right)<\infty$ and consider two cases.

1. $0<k<1$. Then $\frac{k}{n}<\frac{1}{n}$ and $\sum_{n=n_1}^{\infty}\phi\left(\frac{k}{n}\right)<\sum_{n=n_1}^{\infty}\phi\left(\frac{1}{n}\right)<\infty$. Taking $n_k := n_1$, we have $\sum_{n=n_k}^{\infty}\phi\left(\frac{k}{n}\right)<\infty$.
2. $k>1$. Then there exists $m\in\mathbb{N}$ such that $k\leq m$. Defining $n_k := n_1 m$, we have

$$\sum_{n=n_k}^{\infty}\phi\left(\frac{k}{n}\right)\leq\sum_{n=n_k}^{\infty}\phi\left(\frac{m}{n}\right)=\sum_{n=n_1 m}^{\infty}\phi\left(\frac{m}{n}\right)=\phi\left(\frac{m}{n_1 m}\right)$$

$$+\phi\left(\frac{m}{n_1 m+1}\right)+\cdots+\phi\left(\frac{m}{n_1 m+(m-1)}\right)+\phi\left(\frac{m}{(n_1+1)m}\right)$$

$$+\phi\left(\frac{m}{(n_1+1)m+1}\right)+\cdots+\phi\left(\frac{m}{(n_1+1)m+(m-1)}\right)+\cdots$$

$$\leq\phi\left(\frac{1}{n_1}\right)+\phi\left(\frac{1}{n_1}\right)+\cdots+\phi\left(\frac{1}{n_1}\right)$$

$$+\phi\left(\frac{1}{n_1+1}\right)+\phi\left(\frac{1}{n_1+1}\right)+\cdots+\phi\left(\frac{1}{n_1+1}\right)+\cdots$$

$$=m\phi\left(\frac{1}{n_1}\right)+m\phi\left(\frac{1}{n_1+1}\right)+\cdots=m\sum_{n=n_1}^{\infty}\phi\left(\frac{1}{n}\right)<\infty.$$

(3) $\Rightarrow$ (1). Take $k=1$. By the assumption that condition (3) holds, there exists $n_1\in\mathbb{N}$ such that $\sum_{n=n_1}^{\infty}\phi\left(\frac{1}{n}\right)<\infty$. Define $x=(\underbrace{0,\ldots,0,}_{n_1-1\text{ times}}1,0,\ldots)$. Clearly, $x\in l^0$ and

$$\rho_\phi(kx)=\rho_\phi(x)=\sum_{n=n_1}^{\infty}\phi\left(\frac{1}{n}\right)<\infty.$$

Hence $x\in ces_\phi$. □

We will assume in the following that ces_ϕ is nontrivial, that is, conditions (2) and (3) from Theorem 2.1 hold. Our next theorem gives some sufficient conditions for the nontriviality of ces_ϕ in terms of some lower index for the generating Orlicz function ϕ.

Theorem 2.2. *For the conditions*:

(a) $\liminf_{t\to 0} \frac{t\phi'(t)}{\phi(t)} > 1$,

(b) $\exists_{\epsilon>0}\exists_{A>0}\exists_{u_0>0}\,\forall_{0\le u\le u_0}\,\phi(u) \le Au^{1+\epsilon}$,

(c) $\exists n_1 \sum_{n=n_1}^{\infty} \phi\left(\frac{1}{n}\right) < \infty$,

we have the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

Proof.

(a) $\Rightarrow$ (b). Although this implication appeared for example in [9] we will present its proof for the sake of completeness.

By the assumption that $\liminf_{t\to 0} \frac{t\phi'(t)}{\phi(t)} > 1$ we know that there exists t_0 such that $\alpha := \inf_{0<t\le t_0} \frac{t\phi'(t)}{\phi(t)} > 1$. Then for all $0 \le t \le t_0$ we have that $\frac{t\phi'(t)}{\phi(t)} \ge \alpha$, that is, $\frac{\phi'(t)}{\phi(t)} \ge \frac{\alpha}{t}$. Take $0 < \lambda < 1$. Then $\lambda t < t$ and so for $0 < t \le t_0$:

$$\int_{\lambda t}^{t} \frac{\phi'(s)}{\phi(s)} \mathrm{d}s \ge \alpha \int_{\lambda t}^{t} \frac{\mathrm{d}s}{s},$$

whence

$$\ln \frac{\phi(t)}{\phi(\lambda t)} \ge \ln \frac{t^{\alpha}}{(\lambda t)^{\alpha}}$$

and consequently

$$\phi(\lambda t) \le \lambda^{\alpha} \phi(t).$$

Let us take $t = t_0$. Then, for all $0 < \lambda < 1$, we have $\phi(\lambda t_0) \le \phi(t_0)\lambda^{\alpha}$, so $\phi(\lambda t_0) \le \frac{\phi(t_0)}{t_0^{\alpha}} \cdot (\lambda t_0)^{\alpha}$. If we take $\epsilon = \alpha - 1$, $A = \frac{\phi(t_0)}{t_0^{\alpha}}$ and $u_0 = t_0$, we get (b).

(b) $\Rightarrow$ (c). Take $\epsilon > 0$, $A > 0$ and $u_0 > 0$ such that for all $0 \le u \le u_0$, we have $\phi(u) \le Au^{1+\epsilon}$. Since $\frac{1}{n} \to 0$ there exists $n_1 \in \mathbb{N}$ such that $\frac{1}{n} \le u_0$ for all $n \ge n_1$. Therefore,

$$\sum_{n=n_1}^{\infty} \phi\left(\frac{1}{n}\right) \le \sum_{n=n_1}^{\infty} A\left(\frac{1}{n}\right)^{1+\epsilon} \le A \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}} < \infty. \qquad \square$$

Lemma (*Fatou property*). *If* $x \in l^0$, $\{x_n\} \subset ces_{\phi}$, $\sup \|x_n\| < \infty$ *and* $0 \le x_n \uparrow x$ *coordinatewise*, *then* $x \in ces_{\phi}$ *and* $\|x_n\| \to \|x\|$.

Proof. Assume that $x_n \in ces_{\phi}$ for all $n \in \mathbb{N}$, $\sup \|x_n\| < \infty$ and $0 \le x_n(i) \uparrow x(i)$ for each $i \in \mathbb{N}$. Denote $A = \sup_n \|x_n\|$. We know that $\|x_n\| \le A < \infty$ for all $n \in \mathbb{N}$, so $0 \le \frac{x_n}{A} \le \frac{x_n}{\|x_n\|}$ for all $n \in \mathbb{N}$. Therefore $\rho_{\phi}\left(\frac{x_n}{A}\right) \le 1$ and since the modular ρ_{ϕ} is monotone, we get

$$\rho_{\phi}\left(\frac{x_n}{A}\right) \le \rho_{\phi}\left(\frac{x_n}{\|x_n\|}\right) \le 1.$$

Then, by the Beppo Levi theorem and the fact that $A^{-1}x_n(i) \to A^{-1}x(i)$ for each $i \in \mathbb{N}$, we get

$$\rho_{\phi}\left(\frac{x}{A}\right) = \lim_{n\to\infty} \rho_{\phi}\left(\frac{x_n}{A}\right) = \sup_{n} \rho_{\phi}\left(\frac{x_n}{A}\right) \le 1,$$

whence $x \in ces_\phi$ and $\|x\| \leq A$. By the assumption that $x_n \uparrow x$ coordinatewise and by monotonicity of the norm, we get $\sup_n \|x_n\| \leq \|x\|$. Therefore, we have $\|x\| = \sup_n \|x_n\| = \lim_{n\to\infty} \|x_n\|$. □

It is known that for any Köthe sequence (function) space the Fatou property implies its completeness (see [17]). Therefore, ces_ϕ is a Banach space.

Theorem 2.3. *Let* $A_\phi = \{x \in ces_\phi \colon \forall k > 0\, \exists n_k\ \sum_{n=n_k}^{\infty} \phi\left(\frac{k}{n}\sum_{i=1}^{n} |x(i)|\right) < \infty\}$. *Then the following assertions are true*:

(i) A_ϕ *is a closed separable subspace of* ces_ϕ,
(ii) $A_\phi = cl\{x \in ces_\phi \colon x(i) \neq 0$ *only for finite number of* $i \in \mathbb{N}\}$,
(iii) A_ϕ *is the subspace of all order continous elements of* ces_ϕ.

Proof. It is easy to see that A_ϕ is a subspace of ces_ϕ. Next we will prove that A_ϕ is closed in ces_ϕ. We must show that if $x_m \in A_\phi$ for each $m \in \mathbb{N}$ and $x_m \to x \in ces_\phi$, then $x \in A_\phi$. Take any $k > 0$. We will show that there exists $n_k \in \mathbb{N}$ such that $\sum_{n=n_k}^{\infty} \phi\left(\frac{k}{n}\sum_{i=1}^{n} |x(i)|\right) < \infty$. Since $\rho_\phi(k(x - x_m)) \to 0$ for all $k > 0$, there exists $M \in \mathbb{N}$ such that $\rho_\phi(2k(x - x_M)) < 1$. Since $x_M \in A_\phi$, there exists n_M such that $\sum_{n=n_M}^{\infty} \phi\left(\frac{2k}{n}\sum_{i=1}^{n} |x_M(i)|\right) < \infty$. As we will see, we can take $n_k = n_M$. Indeed,

$$\sum_{n=n_M}^{\infty} \phi\left(\frac{k}{n}\sum_{i=1}^{n} |x(i)|\right) = \sum_{n=n_M}^{\infty} \phi\left(\frac{k}{n}\sum_{i=1}^{n} \left|\frac{2(x(i) - x_M(i))}{2} + \frac{2x_M(i)}{2}\right|\right)$$

$$\leq \sum_{n=n_M}^{\infty} \phi\left(\frac{k}{n}\sum_{i=1}^{n} \left|\frac{2(x(i) - x_M(i))}{2}\right| + \left|\frac{2x_M(i)}{2}\right|\right)$$

$$= \sum_{n=n_M}^{\infty} \phi\left(\frac{1}{2}\frac{k}{n}\sum_{i=1}^{n} |2(x(i) - x_M(i))| + \frac{1}{2}\frac{k}{n}\sum_{i=1}^{n} |2x_M(i)|\right)$$

$$\leq \sum_{n=n_M}^{\infty} \left(\frac{1}{2}\phi\left(\frac{2k}{n}\sum_{i=1}^{n} |x(i) - x_M(i)|\right) + \frac{1}{2}\phi\left(\frac{2k}{n}\sum_{i=1}^{n} |x_M(i)|\right)\right)$$

$$= \frac{1}{2}\sum_{n=n_M}^{\infty} \phi\left(\frac{2k}{n}\sum_{i=1}^{n} |x(i) - x_M(i)|\right) + \frac{1}{2}\sum_{n=n_M}^{\infty} \phi\left(\frac{2k}{n}\sum_{i=1}^{n} |x_M(i)|\right)$$

$$\leq \frac{1}{2}\rho_\phi(2k(x - x_M) + \frac{1}{2}\sum_{n=n_M}^{\infty} \phi\left(\frac{2k}{n}\sum_{i=1}^{n} |x_M(i)|\right) < \infty.$$

By the arbitrariness of $k > 0$, we get that $x \in A_\phi$, which proves that A_ϕ is the closed subspace in the norm topology in ces_ϕ.

Now, we will prove assertion (ii). Let us define the set $B_\phi = cl\{x \in ces_\phi \colon x(i) = 0$ for a.e. $i \in \mathbb{N}\}$. We will prove that A_ϕ and B_ϕ are equal.

First we will show that $B_\phi \subset A_\phi$. If $B_\phi = \emptyset$, the inclusion $B_\phi \subset A_\phi$ is obvious. So, assume that $B_\phi \neq \emptyset$. Take $x = (\underbrace{0, \ldots, 0}_{l-1 \text{ times}}, 1, 0, 0, \ldots) \in B_\phi$ and $k > 0$. We have from Theorem 2.1 that there exists n_k such that

$$\sum_{n=n_k}^{\infty} \phi\left(\frac{k}{n}\right) < \infty.$$

We can assume that $n_k \geq l$. Hence $x \in A_\phi$, and so, by the fact that A_ϕ is a linear subspace of ces_ϕ, we get the inclusion $B_\phi \subset A_\phi$.

Now, we will show that $A_\phi \subset B_\phi$. Let $x = (x_1, x_2, \ldots, x_k, x_{k+1}, \ldots) \in A_\phi$ and define $x^k = (x_1, x_2, \ldots, x_k, 0, 0, \ldots)$ for any $k \in \mathbb{N}$. Obviously $x^k \in B_\phi$. We will show that $\rho_\phi(\alpha(x - x_k)) \to 0$ for each $\alpha > 0$. Take any $\alpha > 0$ and $\epsilon > 0$. Since $x \in A_\phi$, so there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{n=k_0+1}^{\infty} \phi\left(\frac{\alpha}{n}\sum_{i=1}^{n}|x(i)|\right) < \epsilon.$$

Then for any $k \geq k_0$,

$$\begin{aligned}\rho_\phi(\alpha(x - x^k)) &\leq \rho_\phi(\alpha(x - x^{k_0})) = \rho_\phi(\alpha(0, \ldots, 0, x_{k_0+1}, x_{k_0+2}, \ldots)) \\ &= \sum_{n=k_0+1}^{\infty} \phi\left(\frac{\alpha}{n}\sum_{i=k_0+1}^{n}|x(i)|\right) \\ &\leq \sum_{n=k_0+1}^{\infty} \phi\left(\frac{\alpha}{n}\sum_{i=1}^{n}|x(i)|\right) < \epsilon.\end{aligned}$$

Next we will prove assertion (iii). Let $x \in A_\phi$. We will show that x is order continuous. Take any $k > 0$ and $\epsilon > 0$. Then there exists $n_k \in \mathbb{N}$ such that $\sum_{n=n_k}^{\infty} \phi\left(\frac{k}{n}\sum_{i=1}^{n}|x(i)|\right) < \frac{\epsilon}{2}$. Assume that $x_m \downarrow 0$ coordinatewise and $x_m \leq |x|$ for all $m \in \mathbb{N}$. Denote

$$\phi\left(\frac{k}{n}\sum_{i=1}^{n}|x(i)|\right) = \alpha(n)$$

and

$$\phi\left(\frac{k}{n}\sum_{i=1}^{n}|x_m(i)|\right) = \alpha_m(n) \quad \text{for any } n \in \mathbb{N}.$$

Since $x_m \downarrow 0$ coordinatewise, we get $\alpha_m(n) \to 0$ as $m \to \infty$ for any $n \in \mathbb{N}$. Consequently, there is $m_\epsilon \in \mathbb{N}$ such that $\sum_{n=1}^{n_k-1} \alpha_m(n) < \frac{\epsilon}{2}$ for any $m \geq m_\epsilon$. Moreover, $\sum_{n=n_k}^{\infty} \alpha_m(n) < \sum_{n=n_k}^{\infty} \alpha(n) < \frac{\epsilon}{2}$ for all $n \geq n_k$ and $m \in \mathbb{N}$. Therefore $\rho_\phi(kx_m) < \epsilon$ for all $m \geq m_\epsilon$, which means that $\rho_\phi(kx_m) \to 0$. By the arbitrariness of $k > 0$, this means that $\|x_m\| \to 0$.

Let $x \in ces_\phi$ be an order continuous element. Since

$$\|(0, \ldots, 0, x(n+1), x(n+2), \ldots)\| \to 0 \quad \text{as } n \to \infty,$$

so it easy to see that $x \in cl\{x \in ces_\phi\colon x(i) = 0 \text{ for a.e. } i \in \mathbb{N}\}$.

Finally, we will show that A_ϕ is separable. Roughly speaking, this follows by the fact that the counting measure on $\mathbb{N}$ is separable and A_ϕ is order continuous.

Define the set $C_\phi = cl\{x \in ces_\phi\colon x(i) = 0$ for a.e. $i \in \mathbb{N}$ and $x(i) \in Q\}$ which is countable. It is obvious that $C_\phi \subset B_\phi$. Now, we will show that $B_\phi \subset C_\phi$. Let $x = (x(1), x(2), \dots, x(k), 0, 0, \dots) \in B_\phi$ and $x_m = (x_m(1), \dots, x_m(k), 0, \dots) \in C_\phi$ will be such that $x_m(i) \to x(i)$ as $m \to \infty$. We will show that $\|x_m - x\| \to 0$.

Let us take any $\lambda > 0$. We have

$$\lambda(|x(1) - x_m(1)| + |x(2) - x_m(2)| + \cdots + |x(k) - x_m(k)|) \le 1$$

for m large enough. Then by convexity of ϕ,

$$\begin{aligned}\rho_\phi(\lambda(x - x_m)) &\le \sum_{n=1}^{\infty} \phi\left(\lambda \frac{|x(1) - x_m(1)| + |x(2) - x_m(2)| + \cdots + |x(k) - x_m(k)|}{n}\right)\\ &\le \lambda(|x(1) - x_m(1)| + |x(2) - x_m(2)| + \cdots + |x(k) - x_m(k)|)\\ &\quad \times \sum_{n=1}^{\infty} \phi\left(\frac{1}{n}\right) \to 0\end{aligned}$$

as $m \to \infty$. By the arbitrariness of λ, we have $\|x_m - x\| \to 0$ as $m \to \infty$. Consequently, $B_\phi = C_\phi$. Since $B_\phi = A_\phi$ and the space C_ϕ is separable, we get the separability of A_ϕ. □

Theorem 2.4. *If $\phi \in \Delta_2(0)$, then $A_\phi = ces_\phi$.*

Proof. We should only show that $ces_\phi \subset A_\phi$. Let $x \in ces_\phi$. Then there exists $\alpha > 0$ such that $\rho_\phi(\alpha x) < \infty$. We will show that for any $\lambda > 0$ there exists n_λ such that $\sum_{n=n_\lambda}^{\infty} \phi\left(\frac{\lambda}{n} \sum_{i=1}^{n} |x(i)|\right) < \infty$. We take only $\lambda > \alpha$, because for $\lambda < \alpha$ we have $\sum_{n=n_1}^{\infty} \phi\left(\frac{\lambda}{n} \sum_{i=1}^{n} |x(i)|\right) < \sum_{n=n_1}^{\infty} \phi\left(\frac{\alpha}{n} \sum_{i=1}^{n} |x(i)|\right) < \infty$ from monotonicity of the function ϕ. Let $\lambda > \alpha$. By $\phi \in \Delta_2(0)$, we have that $\phi \in \Delta_l(0)$ for any $l > 1$, whence for $l := \frac{\lambda}{\alpha}$ there exists $k, u_0 > 0$ such that $\phi(lu) \le k\phi(u)$ for all $u \le u_0$. By $\rho_\phi(\alpha x) < \infty$, there exists n_λ such that $\frac{\alpha}{n} \sum_{i=1}^{n} |x(i)| < u_0$ for all $n \ge n_\lambda$. Therefore,

$$\begin{aligned}\sum_{n=n_\lambda}^{\infty} \phi\left(\frac{\lambda}{n} \sum_{i=1}^{n} |x(i)|\right) &= \sum_{n=n_\lambda}^{\infty} \phi\left(\frac{\lambda\alpha}{\alpha n} \sum_{i=1}^{n} |x(i)|\right)\\ &\le k \sum_{n=n_\lambda}^{\infty} \phi\left(\frac{\alpha}{n} \sum_{i=1}^{n} |x(i)|\right) < \infty,\end{aligned}$$

and the proof is finished. □

COROLLARY 2.1

If $\phi \in \Delta_2(0)$, then

(i) *the space ces_ϕ is a separable,*
(ii) *the space ces_ϕ is order continuous.*

We will assume in the following that the function ϕ is finite. We will prove some useful lemmas.

Lemma 2.1. *For any $x \in A_\phi$,*

$$\|x\| = 1 \quad \textit{if and only if } \rho_\phi(x) = 1.$$

Proof. We need only to show that $\|x\| = 1$ implies $\rho_\phi(x) = 1$ because the opposite implication holds in any modular space. Assume that $\phi < \infty$ and take $x \in A_\phi$ with $\|x\| = 1$. Note that $\rho_\phi(x) \leq 1$. Assume that $\rho_\phi(x) < 1$. Since $x \in A_\phi$, we have that $\rho_\phi(kx) < \infty$ for all $k > 0$. Let us define the function $f(\lambda) = \rho_\phi(\lambda x)$, which is convex and has finite values. Hence f is continous on $\mathbb{R}_+$ and $f(1) < 1$ by the assumption that $\rho_\phi(x) < 1$. Then, by the continuity of f there exists $r > 1$ such that $f(r) \leq 1$, that is, $\rho_\phi(rx) \leq 1$. Then $\|rx\| \leq 1$, whence $\|x\| \leq \frac{1}{r} < 1$, a contradiction, which shows that $\rho_\phi(x) = f(1) = 1$. □

Lemma 2.2. *If $\phi \in \Delta_2(0)$, then $\rho_\phi \in \Delta_2^S$.*

Proof. Take arbitrary $\epsilon > 0$, $a > 0$ and $\rho_\phi(x) \leq a$. Then $\rho_\phi(x) = \sum_{n=1}^{\infty} \phi(\sigma x(n)) \leq a$, whence $\phi(\sigma x(n)) \leq a$ for any $n \in \mathbb{N}$. If $b > 0$ is the number satisfying $\phi(b) = a$, then $\sigma x(n) \leq b$ for any $n \in \mathbb{N}$. Since $\phi \in \Delta_2(0)$ and $\phi < \infty$, so $\phi \in \Delta_2([0, b])$, i.e. there exists $K > 0$ such that $\phi(2u) \leq K\phi(u)$ for all $u \in [0, b]$. We have

$$\rho_\phi(2x) = \sum_{n=1}^{\infty} \phi(\sigma 2x(n)) = \sum_{n=1}^{\infty} \phi(2\sigma x(n))$$

$$\leq k \sum_{n=1}^{\infty} \phi(\sigma x(n)) = k\rho_\phi(x).$$

□

Lemma 2.3. *Assume that $\phi \in \Delta_2(0)$. Then for any $L > 0$ and $\epsilon > 0$ there exists $\delta = \delta(L, \epsilon) > 0$ such that*

$$|\rho_\phi(x + y) - \rho_\phi(x)| < \epsilon$$

for all $x, y \in ces_\phi$ with $\rho_\phi(x) \leq L$ and $\rho_\phi(y) \leq \delta(L, \epsilon)$.

Proof. In virtue of Lemma 2.2 it suffices to apply Lemma 2.1 in [4]. □

Lemma 2.4. *If $\phi \in \Delta_2(0)$, then for any sequence $(x_n) \in ces_\phi$ the condition $\|x_n\| \to 0$ holds if and only if $\rho_\phi(x_n) \to 0$.*

Proof. It suffices to apply Lemmas 2.2 and 2.3 in [4]. □

Lemma 2.5. *If $\phi \in \Delta_2(0)$, then for any $x \in ces_\phi$,*

$$\|x\| = 1 \text{ if and only if } \rho_\phi(x) = 1.$$

Proof. The result follows from Lemma 2.2 and Corollary 2.2 in [4]. □

Lemma 2.6. *If $\phi \in \Delta_2(0)$, then for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $\|x\| \geq 1 + \delta$ whenever $x \in ces_\phi$ and $\rho_\phi(x) \geq 1 + \epsilon$.*

Proof. The result follows by applying Lemmas 2.2 and 2.4 in [4]. □

Lemma 2.7. *Let $\phi \in \Delta_2(0)$. Then for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$ such that $\rho_\phi(x) > \delta$ whenever $\|x\| \geq \epsilon$.*

Proof. Suppose for the contrary there exists $\epsilon > 0$ such that for any $\delta > 0$, there exists x such that $\rho_\phi(x) \le \delta$ and $\|x\| \ge \epsilon$. Take $\delta_n = \frac{1}{n}$ and the sequence $(x_n)_{n\in\mathbb{N}}$ in ces_ϕ satisfying $\rho_\phi(x_n) \le \frac{1}{n}$ and $\|x_n\| \ge \epsilon$. Consequently $\rho_\phi(x_n) \to 0$ as $n \to \infty$. From Lemma 2.4 it follows that $\|x_n\| \to 0$, a contradiction finishing the proof. □

Lemma 2.8. *If* $\phi \in \Delta_2(0)$, *then* $\|x_n\| \to \infty$ *whenever* $\rho_\phi(x_n) \to \infty$.

Proof. Suppose $(\|x_n\|)$ is a bounded sequence, that is, there exists $M > 0$ such that $\|x_n\| \le M$ for all $n \in \mathbb{N}$. Take $s \in \mathbb{N}$ such that $M \le 2^s$. Then $\|x_n\| \le 2^s$, whence $\|\frac{x_n}{2^s}\| \le 1$ and $\rho_\phi\left(\frac{x_n}{2^s}\right) \le 1$. Consequently, $\phi((\sigma\frac{x_n}{2^s})(i)) \le 1$ for all $i \in \mathbb{N}$, and then, there exists some $L > 0$ such that $\left(\sigma\frac{x_n}{2^s}\right)(i) \le L$ for all $i \in \mathbb{N}$. Since $\phi \in \Delta_2(0)$ and $\phi < \infty$, $\phi \in \Delta_2([0, 2^{s-1}L])$. We have for all $n \in \mathbb{N}$,

$$\rho_\phi(x_n) = \rho_\phi\left(2^s\frac{x_n}{2^s}\right) \le k^s\rho_\phi\left(\frac{x_n}{2^s}\right) \le k^s,$$

whence $\rho_\phi(x_n) \not\to \infty$. □

Lemma 2.9. *If* $\phi \in \Delta_2(0)$, *then for any sequence* (x_n) *in* ces_ϕ, *we have*

$$\|x_n\| \to 1 \quad \text{if and only if } \rho_\phi(x_n) \to 1.$$

Proof. The implication $\rho_\phi(x_n) \to 1 \Rightarrow \|x_n\| \to 1$ is almost obvious. Namely, we have $\rho_\phi(x) \le \|x\|$ if $\rho_\phi(x) \le 1$ and $\|x\| \le \rho_\phi(x)$ if $\rho_\phi(x) > 1$. Therefore $|\|x_n\| - 1| \le |\rho_\phi(x_n) - 1|$ and the result follows. Now, assuming that $\|x_n\| \to 1$, we consider two cases:

1. $\|x_n\| \uparrow 1$. From Lemma 2.8 we know that the sequence $(\rho_\phi(2x_n))$ is bounded, that is, there exists $A > 0$ such that $\rho_\phi(2x_n) \le A$ for all $n \in \mathbb{N}$. Assume for the contrary that $\rho_\phi(x_n) \not\to 1$. We can assume that $\|x_n\| > \frac{1}{2}$ for all $n \in \mathbb{N}$ and there exists $\epsilon > 0$ such that $\rho_\phi(x_n) < 1 - \epsilon$ for all $n \in \mathbb{N}$. Take $a_n := \frac{1}{\|x_n\|} - 1$. Then $a_n \to 0$ and $a_n \le 1$. By Lemma 2.5, we have
$$\begin{aligned} 1 = \rho_\phi\left(\frac{x_n}{\|x_n\|}\right) &= \rho_\phi((a_n+1)x_n) \\ &= \rho_\phi(2a_nx_n + (1-a_n)x_n) \le a_n\rho_\phi(2x_n) + (1-a_n)\rho_\phi(x_n) \\ &\le a_n \cdot A + (1-a_n)(1-\epsilon) \to 1-\epsilon \end{aligned}$$
as $n \to \infty$, a contradiction.
2. $\|x_n\| \downarrow 1$. Assume that $\|x_n\| \le 2$ for $n \in \mathbb{N}$ and there exists $\epsilon > 0$ such that $\rho_\phi(x_n) > 1 + \epsilon$ for all $n \in \mathbb{N}$. From Lemma 2.8 we know that there exists $B > 0$ such that $\rho_\phi(2x_n) \le B$ for all $n \in \mathbb{N}$. By the assumption we have $0 \le 1 - \frac{1}{\|x_n\|} \le 1$, $0 \le 2 - \|x_n\| \le 1$. The inequality $\frac{1}{a} + a \ge 2$ for any $a > 0$ yields $0 \le \left(1 - \frac{1}{\|x_n\|}\right) + (2 - \|x_n\|) = 3 - \left(\frac{1}{\|x_n\|} + \|x_n\|\right) \le 3 - 2 = 1$ for any $n \in \mathbb{N}$. Therefore, we have
$$\begin{aligned} 1+\epsilon \le \rho_\phi(x_n) &= \rho_\phi\left(\left(1-\frac{1}{\|x_n\|}\right)\cdot 2x_n + (2-\|x_n\|)\frac{x_n}{\|x_n\|}\right) \\ &\le \left(1-\frac{1}{\|x_n\|}\right)\rho_\phi(2x_n) + (2-\|x_n\|)\rho_\phi\left(\frac{x_n}{\|x_n\|}\right) \\ &\le \left(1-\frac{1}{\|x_n\|}\right)B + \rho_\phi\left(\frac{x_n}{\|x_n\|}\right) \to 1, \end{aligned}$$

because $\rho_\phi\big(\frac{x_n}{\|x_n\|}\big) = 1$ for any $n \in \mathbb{N}$ and $1 - \frac{1}{\|x_n\|} \to 0$, a contradiction which finishes the proof. □

Now we will consider monotonicity properties of A_ϕ and ces_ϕ.

Theorem 2.5. *The space A_ϕ is strictly monotone if and only if $\phi > 0$.*

Proof. Denote $a_\phi = \sup\{t \geq 0\colon \phi(t) = 0\}$ and assume that $a_\phi > 0$. We will show that under this assumption there exists $x, y \in ces_\phi$ such that $x \leq y$, $x \neq y$ and $\|x\| = \|y\|$. We define the function $f(t) = \sum_{n=1}^{\infty} \phi\left(\frac{t}{n}\right)$ for $t \geq 0$. Since $a_\phi > 0$, $\frac{t}{n} \to 0$ as $n \to \infty$ and $a_\phi > 0$, so $\sum_{n=1}^{\infty} \phi\left(\frac{t}{n}\right)$ is convergent for all $t \in \mathbb{R}_+$. Since ϕ is a convex function, so f is convex, too. Then f is continuous on $\mathbb{R}_+$ and $f(t) \to \infty$ as $t \to \infty$, whence $f(\mathbb{R}_+) = \mathbb{R}_+$ and by the Darboux property of f we know that there exists $c \in \mathbb{R}$ such that $f(c) = \sum_{n=1}^{\infty} \phi\left(\frac{c}{n}\right) = 1$. Since $\frac{c+1}{n} \to 0$ as $n \to \infty$, there exists n_0 such that $\frac{c+1}{n_0} \leq a_\phi$. Consider two sequences $x = (c, 0, 0, \ldots)$ and $y = (\underbrace{c, 0, \ldots, 0}_{n_0 - 1 \text{ times}}, 1, 0, \ldots)$. It is obvious that $x \neq y$ and $x < y$. Moreover,

$$\rho_\phi(x) = \phi(c) + \phi\left(\frac{c}{2}\right) + \phi\left(\frac{c}{3}\right) + \cdots = f(c) = 1,$$

$$\rho_\phi(y) = \phi(c) + \phi\left(\frac{c}{2}\right) + \cdots + \phi\left(\frac{c}{n_0 - 1}\right) + \phi\left(\frac{c+1}{n_0}\right) + \phi\left(\frac{c+1}{n_0 + 1}\right) + \cdots = 1.$$

Since $\rho_\phi(x) = \rho_\phi(y) = 1$, we have $\|x\| = \|y\| = 1$, which means that $A_\phi \notin (SM)$.

Assume now that $a_\phi = 0$, $y \geq x \geq 0$, $x \neq y$ and $x, y \in A_\phi$. We can assume that $\|x\| = 1$. From Lemma 2.1 we know that $\rho_\phi(x) = 1$. In order to show that $\|y\| > 1$ we need to show that $\rho_\phi(y) > 1$. Note that $\rho_\phi(x + y) \geq \rho_\phi(x) + \rho_\phi(y)$ for all nonnegative $x, y \in A_\phi$. Therefore

$$\rho_\phi(y) = \rho_\phi(x + (y - x)) \geq \rho_\phi(x) + \rho_\phi(y - x) = 1 + \rho_\phi(y - x) > 1,$$

because of $y - x > 0$ and $\phi > 0$, whence $\rho_\phi(y - x) > 0$. This finishes the proof. □

From the last theorem, we get the following.

COROLLARY 2.2

If the space ces_ϕ is strictly monotone, then $\phi > 0$.

Before formulating the next theorem note that $\phi > 0$ whenever $\phi \in \Delta_2(0)$.

Theorem 2.6. *If $\phi \in \Delta_2(0)$, then ces_ϕ is uniformly monotone.*

Proof. Let $\epsilon > 0$ and $x, y \geq 0$ be such that $\|x\| = 1$ and $\|y\| \geq \epsilon$. From Lemma 2.5 we have $\rho_\phi(x) = 1$ and from Lemma 2.7 we have that $\rho_\phi(y) > \eta$ where $\eta > 0$ is independent of y. Then

$$\rho_\phi(x + y) \geq \rho_\phi(x) + \rho_\phi(y) \geq 1 + \eta.$$

By Lemma 2.6, there exists $\delta > 0$ independent of x and y such that $\|x + y\| \geq 1 + \delta$. □

Next we consider rotundity of ces_ϕ. In order to be able to prove criteria for rotundity of ces_ϕ, we need first to prove the following.

Lemma 2.10. *Let $\phi \in \Delta_2(0)$ and $y, z \in S(ces_\phi)$ satisfy $\frac{y+z}{2} \in S(ces_\phi)$. If $y \neq z$, then there exists $i_0 \in \mathbb{N}$ such that $|y(i_0)| \neq |z(i_0)|$.*

Proof. Assume for the contrary that the assumptions are satisfied, $y \neq z$ and $|y| = |z|$. Then there is $i_0 \in \mathbb{N}$ such that $y(i_0) \neq z(i_0)$, but $|y(i_0)| = |z(i_0)|$, whence $y(i_0) + z(i_0) = 0$. Consequently,

$$
\begin{aligned}
1 = \rho_\phi\left(\frac{y+z}{2}\right) &= \sum_{n=1}^{\infty} \phi\left(\frac{1}{n}\sum_{i=1}^{n} \frac{|y(i)+z(i)|}{2}\right) \\
&= \sum_{n=1}^{\infty} \phi\left(\frac{1}{2}\sum_{i=1}^{n} \frac{|y(i)+z(i)|}{n}\right) = \sum_{n=1}^{\infty} \phi\left(\frac{1}{2}\sum_{i\in\mathbb{N}\setminus\{i_0\}} \frac{|y(i)+z(i)|}{n}\right) \\
&\leq \sum_{n=1}^{\infty} \phi\left(\frac{1}{2}\left(\frac{1}{n}\sum_{i\in N\setminus\{i_0\}} |y(i)| + \frac{1}{n}\sum_{i\in N\setminus\{i_0\}} |z(i)|\right)\right) \\
&\leq \sum_{n=1}^{\infty} \left(\frac{1}{2}\phi\left(\frac{1}{n}\sum_{i\in N\setminus\{i_0\}} |y(i)|\right) + \frac{1}{2}\phi\left(\frac{1}{n}\sum_{i\in N\setminus\{i_0\}} |z(i)|\right)\right) \\
&< \frac{1}{2}\rho_\phi(y) + \frac{1}{2}\rho_\phi(z) = 1,
\end{aligned}
$$

a contradiction which finishes the proof. □

Given any Orlicz function ϕ with values in $\mathbb{R}_+$ such that $\sum_{i=1}^{\infty} \phi\left(\frac{1}{i}\right) < \infty$, define the function

$$
f(a) = 2\phi(a) + \sum_{i=3}^{\infty} \phi\left(\frac{2}{i}a\right). \tag{2.1}
$$

Since the function ϕ is convex, so f is convex as well. By Theorem 2.1 it has finite values. Therefore f is continuous and $f(a) \to \infty$ as $a \to \infty$, whence we deduce that there exists $\alpha \in \mathbb{R}$ such that $f(\alpha) = 1$.

Theorem 2.7. *If $\phi \in \Delta_2(0)$ then ces_ϕ is rotund if and only if ϕ is strictly convex on the interval $[0, \alpha]$, where $f(\alpha) = 1$ and f is defined by formula (2.1).*

Proof. Suppose ϕ is not strictly convex on $[0, \alpha]$. Then there exists an interval $[b, c] \subset (0, \alpha)$ on which ϕ is affine.

Since $c < \alpha$, we have

$$
2\phi(c) + \sum_{i=3}^{\infty} \phi\left(\frac{2c}{i}\right) < 1.
$$

Take $d > 0$ such that

$$
2\phi(c) + \sum_{i=3}^{\infty} \phi\left(\frac{2c+d}{i}\right) < 1.
$$

Choose b_1, c_1 such that $b < b_1 < c_1 < c$ and

$$\phi(b) + \phi\left(\frac{b+c}{2}\right) = \phi(b_1) + \phi\left(\frac{b_1+c_1}{2}\right),$$

$$b_1 - b < \frac{d}{2} \quad \text{and} \quad c - c_1 < \frac{d}{2}.$$

By $|b + c - b_1 - c_1| < d$, there is $k > 0$ for which either $b + c = b_1 + c_1 + k$ or $b + c + k = b_1 + c_1$.

Without loss of generality, we may assume that $b + c + k = b_1 + c_1$, whence

$$\phi(b) + \phi\left(\frac{b+c}{2}\right) + \sum_{i=3}^{\infty} \phi\left(\frac{b+c+k}{i}\right)$$

$$= \phi(b_1) + \phi\left(\frac{b_1+c_1}{2}\right) + \sum_{i=3}^{\infty} \phi\left(\frac{b_1+c_1}{i}\right).$$

Take $k_1 > 0$ such that

$$\phi(b) + \phi\left(\frac{b+c}{2}\right) + \phi\left(\frac{b+c+k}{3}\right) + \sum_{i=4}^{\infty} \phi\left(\frac{b+c+k+k_1}{i}\right) = 1. \tag{2.2}$$

Since $b + c + k = b_1 + c_1$, we have

$$\phi(b_1) + \phi\left(\frac{b_1+c_1}{2}\right) + \phi\left(\frac{b_1+c_1}{3}\right) + \sum_{i=4}^{\infty} \phi\left(\frac{b_1+c_1+k_1}{i}\right) = 1. \tag{2.3}$$

Put

$$x = (b, c, k, k_1, 0, 0, \dots)$$

and

$$y = (b_1, c_1, 0, k_1, 0, 0, \dots).$$

By (2.2) and (2.3), we have $\rho_\phi(x) = 1 = \rho_\phi(y)$. So, Lemma 2.5 yields $x, y \in S(ces_\phi)$. Again, by (2.2) and (2.3) and the fact that ϕ is affine on $[b, c]$, we have

$$\begin{aligned}
\rho_\phi\left(\frac{x+y}{2}\right) &= \phi\left(\frac{b+b_1}{2}\right) + \phi\left(\frac{\frac{b+c}{2} + \frac{b_1+c_1}{2}}{2}\right) + \phi\left(\frac{b+c+k}{3}\right) \\
&\quad + \sum_{i=4}^{\infty} \phi\left(\frac{b+c+k+k_1}{i}\right) \\
&= \frac{1}{2}\left(\phi(b) + \phi(b_1)\right) + \frac{1}{2}\left(\phi\left(\frac{b+c}{2}\right) + \phi\left(\frac{b_1+c_1}{2}\right)\right) \\
&\quad + \phi\left(\frac{b+c+k}{3}\right) + \sum_{i=4}^{\infty} \phi\left(\frac{b+c+k+k_1}{i}\right) = 1.
\end{aligned}$$

Therefore Lemma 2.5 yields $\left\|\frac{x+y}{2}\right\| = 1$, which means that ces_ϕ is not rotund.

Conversely, let $x \in S(ces_\phi)$. We need to prove that x is an extreme point. If x is not an extreme point, then there exists $y, z \in S(ces_\phi)$ such that $2x = y + z$ and $y \neq z$. We will prove that $|y| = |z|$ and by Lemma 2.10, we will get a contradiction, finishing the proof.

Since $\phi \in \Delta_2(0)$, Lemma 2.5 yields that $\rho_\phi(x) = \rho_\phi(y) = \rho_\phi(z) = 1$ and

$$\begin{aligned}
1 = \rho_\phi(x) = \rho_\phi\left(\frac{y+z}{2}\right) &= \sum_{n=1}^{\infty} \phi\left(\frac{1}{n}\sum_{i=1}^{n}\frac{|y(i)+z(i)|}{2}\right) \\
&\leq \sum_{n=1}^{\infty} \phi\left(\frac{1}{n}\sum_{i=1}^{n}\frac{|y(i)|+|z(i)|}{2}\right) \\
&\leq \frac{1}{2}\left[\sum_{n=1}^{\infty} \phi\left(\frac{1}{n}\sum_{i=1}^{n}|y(i)|\right)\right. \\
&\qquad \left. + \sum_{n=1}^{\infty} \phi\left(\frac{1}{n}\sum_{i=1}^{n}|z(i)|\right)\right] \\
&= \frac{1}{2}[\rho_\phi(y) + \rho_\phi(z)] \\
&= 1.
\end{aligned}$$

Thus for each $n \in \mathbb{N}$ we have

$$\phi\left(\frac{1}{n}\sum_{i=1}^{n}\frac{|y(i)|+|z(i)|}{2}\right) = \frac{1}{2}\left[\phi\left(\frac{1}{n}\sum_{i=1}^{n}|y(i)|\right) + \phi\left(\frac{1}{n}\sum_{i=1}^{n}|z(i)|\right)\right]. \tag{2.4}$$

Case I. $\frac{1}{n}\sum_{i=1}^{n}|x(i)| \leq \alpha$ for each $n \in \mathbb{N}$. By condition (2.4) and the fact that ϕ is strictly convex on the interval $[0, \alpha]$, we have $\frac{1}{n}\sum_{i=1}^{n}|y(i)| = \frac{1}{n}\sum_{i=1}^{n}|z(i)|$ for each $n \in \mathbb{N}$. Consequently, $|y| = |z|$.

Case II. There exists n such that $\frac{1}{n}\sum_{i=1}^{n}|x(i)| > \alpha$. We claim that there exists only one such n. Assume for the contrary that there exists $n_0 < n_1$ such that $\frac{1}{n_0}\sum_{i=1}^{n_0}|x(i)| > \alpha$ and $\frac{1}{n_1}\sum_{i=1}^{n_1}|x(i)| > \alpha$. Then $n_1 \geq 2$ and we have

$$\begin{aligned}
1 = \rho_\phi(x) > 2\phi(\alpha) + \sum_{i=n_1+1}^{\infty}\phi\left(\frac{n_1\alpha}{i}\right) = 2\phi(\alpha) + \sum_{i=1}^{\infty}\phi\left(\frac{n_1\alpha}{n_1+i}\right) \\
\geq 2\phi(\alpha) + \sum_{i=1}^{\infty}\phi\left(\frac{2\alpha}{2+i}\right) = 2\phi(\alpha) + \sum_{i=3}^{\infty}\phi\left(\frac{2\alpha}{i}\right) = 1,
\end{aligned}$$

a contradiction, which proves the Claim. Let n_0 be the only natural number for which $\frac{1}{n_0}\sum_{i=1}^{n_0}|x(i)| > \alpha$. As in Case I, we can prove that $\frac{1}{n}\sum_{i=1}^{n}|y(i)| = \frac{1}{n}\sum_{i=1}^{n}|z(i)|$ for each $n \neq n_0$. Since $\rho_\phi(y) = \rho_\phi(z) = 1$, we get

$$\phi\left(\frac{1}{n_0}\sum_{i=1}^{n_0}|y(i)|\right) = 1 - \sum_{n\in\mathbb{N}\setminus\{n_0\}} \phi\left(\frac{1}{n}\sum_{i=1}^{n}|y(i)|\right)$$

$$= 1 - \sum_{n\in\mathbb{N}\setminus\{n_0\}} \phi\left(\frac{1}{n}\sum_{i=1}^{n}|z(i)|\right) = \phi\left(\frac{1}{n_0}\sum_{i=1}^{n_0}|z(i)|\right).$$

Consequently, $|y| = |z|$. This finishes the proof. □

Remark 2.1. Note that criteria for rotundity of Cesàro–Orlicz sequence spaces ces_ϕ are weaker than criteria for rotundity of Orlicz sequence spaces l_ϕ. Namely, we can easily conclude from [11] that an Orlicz sequence space l_ϕ is rotund if and only if ϕ attains value 1, $\phi \in \Delta_2(0)$ and ϕ is strictly convex on the interval $[0, a]$ where $\phi(a) = \frac{1}{2}$, which is smaller from the interval $[0, \alpha]$, where α is defined by (2.1).

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