

## Vector bundles with a fixed determinant on an irreducible nodal curve

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**Abstract.** Let  $M$  be the moduli space of generalized parabolic bundles (GPBs) of rank  $r$  and degree  $d$  on a smooth curve  $X$ . Let  $M_{\bar{L}}$  be the closure of its subset consisting of GPBs with fixed determinant  $\bar{L}$ . We define a moduli functor for which  $M_{\bar{L}}$  is the coarse moduli scheme. Using the correspondence between GPBs on  $X$  and torsion-free sheaves on a nodal curve  $Y$  of which  $X$  is a desingularization, we show that  $M_{\bar{L}}$  can be regarded as the compactified moduli scheme of vector bundles on  $Y$  with fixed determinant. We get a natural scheme structure on the closure of the subset consisting of torsion-free sheaves with a fixed determinant in the moduli space of torsion-free sheaves on  $Y$ . The relation to Seshadri–Nagaraj conjecture is studied.

**Keywords.** Nodal curves; torsion-free sheaves; fixed determinant.

### 1. Introduction

Generalized parabolic vector bundles (GPBs) on a smooth curve  $X$  are vector bundles on  $X$  together with parabolic structures on finitely many disjoint divisors  $D_j$ ,  $j = 1, \dots, m$  [1, 2]. There is an open subscheme  $M''$  of the moduli space  $M$  of GPBs on which one can define a determinant morphism into the moduli space of generalized parabolic line bundles  $\bar{L}$ , the map does not extend to  $M$ . Let  $M''_{\bar{L}}$  be its locally closed subset consisting of GPBs with a fixed determinant  $\bar{L}$ . In this note, we define a moduli functor and construct a coarse moduli scheme  $M_{\bar{L}}$  for it. The moduli scheme contains  $M''_{\bar{L}}$  as an open dense subscheme.

Let  $Y$  be an irreducible projective nodal curve with nodes  $y_j$ ,  $j = 1, \dots, m$  and  $p: X \rightarrow Y$  its desingularization with  $D_j$  the inverse image of  $y_j$ . Denote by  $U$  the moduli variety of torsion-free sheaves of rank  $r$ , degree  $d$  on  $Y$ . Let  $U'$  be the open subvariety of  $U$  corresponding to vector bundles on  $Y$ . There is a surjective morphism  $f$  from  $M$  onto  $U$  [1, 2]. The restriction of the morphism  $f$  to  $M' = f^{-1}U'$  is an isomorphism onto the open subvariety  $U'$  of  $U$ . A GPB  $\bar{L}$  gives a torsion-free sheaf  $\mathcal{L}$  on  $Y$ . If  $\mathcal{L}$  is locally free, let  $U'_{\mathcal{L}}$  be the closed subset of  $U'$  consisting of vector bundles with fixed determinant  $\mathcal{L}$ . Using  $f$ ,  $U'_{\mathcal{L}}$  may be identified with  $M'_{\mathcal{L}}$  and  $M_{\bar{L}}$  can be regarded as compactified moduli variety of vector bundles on  $Y$  with determinant  $\mathcal{L}$ .

We show that  $f(M_{\bar{L}}) = \overline{U'_{\mathcal{L}}}$ , the closure of  $U'_{\mathcal{L}}$  in  $U$ , thus giving  $\overline{U'_{\mathcal{L}}}$  the scheme structure of an image subscheme of  $U$ . Let  $I_j$  denote the ideal sheaf at the node  $y_j$ . For a torsion-free sheaf  $F$  of rank  $r$  on  $Y$ , let  $N = \Lambda^r F / (\text{torsion})$  where  $(\text{torsion})$  denotes the torsion subsheaf. Then we show that for any  $\bar{L}$ , the image  $f(M_{\bar{L}})$  can be described (as a set) by

$$f(M_{\bar{L}}) = \{F \in U: I_j^r \mathcal{L} \subset N \subset \mathcal{L}, \quad \forall j\}.$$

This gives a proof of a conjecture by Seshadri and Nagaraj (Conjecture (a), p. 136 of [3]). Proving Seshadri–Nagaraj conjecture was not the aim of this note. The conjecture was proved by Sun [6] by degeneration methods. However he does not get a scheme structure on  $\overline{U}'_{\mathcal{L}}$  or a moduli functor (except in some low rank cases). Our aim is to give a moduli functor and an explicit construction of a projective moduli space for it which contains an open subvariety isomorphic to  $U'_{\mathcal{L}}$  if  $\mathcal{L}$  is a line bundle. We also deal with the case when  $\mathcal{L}$  is torsion-free but not locally free. The construction is much simpler than that of Schmitt [4] and hence the moduli space is easier to study. For example, properties like reduced, irreducible, Cohen–Macaulay follow immediately for our moduli space. Normality is true in rank 2 and is expected to be true in general. These properties have been used in computation of Picard groups in the rank two case.

## 2. The moduli scheme of GPBs with fixed determinant

### 2.1.

Let  $X$  be a nonsingular projective curve over an algebraically closed base field  $k$ . Let  $D_j$ ,  $j = 1, \dots, m$  be disjoint divisors on  $X$  with  $D_j = x_j + x'_j$ , where  $x_j, x'_j$  are distinct closed points. We recall here some basics on generalized parabolic bundles (GPBs), details may be found in [1, 2].

#### DEFINITION 2.1

A generalized parabolic bundle (GPB, in short) of rank  $r$  and degree  $d$  on  $X$  is a vector bundle  $E$  of rank  $r$  and degree  $d$  on  $X$  together with  $r$ -dimensional vector subspaces  $F_j(E)$  of  $E_{x_j} \oplus E_{x'_j}$ . For a subbundle  $N$  of  $E$ , define  $F_j(N) = F_j(E) \cap (N_{x_j} \oplus N_{x'_j})$  and  $f_j(N) = \dim F_j(N)$ .

#### DEFINITION 2.2

Fix a rational number  $\alpha \in (0, 1]$ . A GPB  $(E, F_j(E))$  is  $\alpha$ -stable (resp.  $\alpha$ -semistable) if for every proper subbundle  $N$  of  $E$ , one has  $(d(N) + \alpha \sum_j f_j(N))/r(N) < (\text{resp. } \leq) (d(E) + \alpha rm)/r$ .

#### DEFINITION 2.3

Let  $p_j: F_j(E) \rightarrow E_{x_j}$ ,  $p'_j: F_j(E) \rightarrow E_{x'_j}$  be the projections. Assume that for each  $j$ , at least one of  $p_j, p'_j$  is an isomorphism. The subspace  $F_j(E)$  determines an element  $F_j(E)$  of  $\text{Gr}(r, E_{x_j} \oplus E_{x'_j}) \subset \mathbf{P}(\Lambda^r(E_{x_j} \oplus E_{x'_j}))$ . One has a (rational) morphism  $\delta: \mathbf{P}(\Lambda^r(E_{x_j} \oplus E_{x'_j})) \rightarrow \mathbf{P}(\Lambda^r E_{x_j} \oplus \Lambda^r E_{x'_j})$ . Let  $\det F_j(E)$  denote the one-dimensional subspace of  $\Lambda^r E_{x_j} \oplus \Lambda^r E_{x'_j}$  determined by  $\delta(F_j(E))$ . Define the determinant of  $(E, F_j(E))$  to be the generalized parabolic line bundle  $(\det E, \det F_j(E))$ .

#### DEFINITION 2.4

A family of GPBs of rank  $r$ , degree  $d$  parametrized by a scheme  $T$  is a tuple  $(\mathcal{E}, F_j(\mathcal{E})_j)$  where  $\mathcal{E} \rightarrow T \times X$  is a family of vector bundles of rank  $r$ , degree  $d$  on  $X$  which is flat over  $T$  and  $F_j(\mathcal{E})$  is a rank  $r$  subbundle of  $\mathcal{E}|_{T \times x_j} \oplus \mathcal{E}|_{T \times x'_j}$ . The notion of equivalence of families is the obvious one.

We fix a generalized parabolic line bundle  $\bar{L} := (L, F_j(L))$ . Fix isomorphisms  $h_j: L_{x_j} \rightarrow k, h'_j: L_{x'_j} \rightarrow k$ . Then  $F_j(L)$  can be identified to a point  $F_j(L)$  of  $\mathbf{P}^1$  of the form  $(1 : 0), (0 : 1)$  or  $(1 : \lambda_j), \lambda_j \in k^*$ .

### 2.2 The moduli functor

For simplicity, let us assume that there is only one divisor  $D = x_1 + x_2$ . Let  $(\mathcal{E}, F(\mathcal{E})) \rightarrow T \times X$  be a family of GPBs of rank  $r$ , degree  $d$  on  $X$  with  $\mathcal{E}_t, t \in T$ , of fixed determinant  $L$ . For  $i = 1, 2$  we have vector bundles

$$\mathcal{E}_{x_i} = \mathcal{E}|_{T \times x_i} \rightarrow T.$$

Let  $\mathcal{G}r \rightarrow T$  denote the Grassmannian bundle of rank  $r$  subbundles of  $\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2}$ . It is embedded as a closed subvariety in  $\mathbf{P}(\Lambda^r(\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2}))$  by Plücker embedding. Note that  $F(\mathcal{E})$  defines a section of  $\mathcal{G}r$ . Since  $\det \mathcal{E}|_{T \times X} = L$ , it follows that  $\det \mathcal{E} = p_T^* N \otimes p_X^* L$  for some line bundle  $N$  on  $T$ . Hence for  $i = 1, 2$  one has  $\det \mathcal{E}_{x_i} = N \otimes L_{x_i} = N$ , using the isomorphism  $h_i$ . Let  $q_i: \Lambda^r(\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2}) \rightarrow \det \mathcal{E}_{x_i} = N$  be the projections,  $i = 1, 2$ . Define a hyperplane subbundle  $\mathcal{H}$  of  $\mathbf{P}(\Lambda^r(\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2}))$  by  $q_2 = 0$  if  $F(L) = (1 : 0)$ , by  $q_1 = 0$  if  $F(L) = (0 : 1)$  and by  $q_2 - \lambda_j q_1 = 0$  if  $F(L) = (1 : \lambda)$ . Let  $H_T := \mathcal{G}r \cap \mathcal{H}$ . It is a closed reduced subscheme of  $\mathcal{G}r \subset \mathbf{P}(\Lambda^r(\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2}))$ . Note that  $H_T$  is independent of the choice of  $h_1, h_2$ .

More generally, if we consider parabolic structures over finitely many disjoint divisors  $D_j = x_j + x'_j$ , for each  $j$  one constructs the hyperplane bundle  $\mathcal{H}_j$  and Grassmannian bundle  $\mathcal{G}r_j$  over  $T$ . Let  $\mathcal{G}r$  be the fibre product of  $\mathcal{G}r_j$  over  $T$  and  $H_T$  the fibre product of  $\mathcal{G}r_j \cap \mathcal{H}_j$  over  $T$ .

#### DEFINITION 2.5

Let  $F_L^{ss}$  be the functor  $F_L^{ss}: \text{Schemes} \rightarrow \text{Sets}$  which associates to a scheme  $T$  the set of equivalence classes of families  $(\mathcal{E}, F(\mathcal{E})) \rightarrow T \times X$  of  $\alpha$ -semistable GPBs of rank  $r$  and degree  $d$  with  $\det \mathcal{E}_t \cong L$  for all  $t \in T$  such that the section of  $\mathcal{G}r$  defined by  $(F_j(\mathcal{E}))_j$  maps into  $H_T$ .

One similarly defines a full subfunctor  $F_L^s$  of  $F_L^{ss}$  with semistable replaced by stable.

### 2.3 Construction of the moduli space

Let  $S$  denote the set of semistable GPBs  $(E, F(E))$ , where  $E$  is a vector bundle of rank  $r$ , degree  $d$  with fixed determinant  $L$  and  $F(E)$  is a subspace of  $E_{x_j} \oplus E_{x'_j}$  of dimension  $r$  with fixed weights  $(0, \alpha), 0 < \alpha < 1$ . For  $m \gg 0$ , all GPBs in  $S$  satisfy the condition

$$(*) \quad H^1(E(m)) = 0, H^0(E(m)) \cong \mathbf{C}^n, H^0(E(m)) \rightarrow H^0(E(m) \otimes (\oplus_j \mathcal{O}_{D_j}))$$

is surjective.

Let  $Q$  denote the quot scheme of coherent quotients of  $\mathcal{O}_X^n \otimes \mathcal{O}_X(-m)$  with fixed Hilbert polynomial determined by  $r, d$ . Let  $R$  be the nonsingular subvariety of  $Q$  corresponding to quotient vector bundles  $E$  satisfying condition  $(*)$ . Denote by  $R_0$  the nonsingular closed subvariety in  $R$  corresponding to  $E$  with  $\Lambda^r(E) = L$ . Let  $\mathcal{E} \rightarrow R \times X$  be the universal quotient bundle. Over  $R$ , we have the fibre bundle  $\mathcal{G}r$  with each fibre an  $m$ -fold product of  $\text{Gr}(r, 2r)$  as in §2.2. Over  $R_0$ , we have the fibre bundle  $\bar{R}_0 := H_{R_0}$  whose fibres are

$m$ -fold products of hyperplane sections of  $\text{Gr}(r, 2r)$ . Then  $\tilde{R}_0$  is a closed subvariety of  $\mathcal{G}r$ . Let  $\tilde{R}_0^s$  (resp.  $\tilde{R}_0^{ss}$ ) be the open set of stable (resp. semistable) points in  $\tilde{R}_0$ .

The GIT quotient  $M$  of  $\mathcal{G}r$  by  $\text{PGL}(n)$  for a suitable polarization (depending on  $\alpha$ ) is the coarse moduli space for GPBs [1, 2]. It is a normal projective variety. Since  $\tilde{R}_0$  is a  $\text{PGL}(n)$ -invariant closed subscheme (subvariety) of  $\mathcal{G}r$ , the GIT quotient  $M_{\bar{L}}$  of  $\tilde{R}_0^{ss}$  by  $\text{PGL}(n)$  is a closed subvariety of  $M$  (with a natural reduced subscheme structure). The GIT quotient  $M_{\bar{L}}^s = \tilde{R}_0^s // \text{PGL}(n)$  is an open subscheme of  $M_{\bar{L}}$ . It is easy to see that  $M_{\bar{L}}$  (resp.  $M_{\bar{L}}^s$ ) is the coarse moduli space for the functor  $F_{\bar{L}}^{ss}$  (resp.  $F_{\bar{L}}^s$ ).

**Theorem 1.** *Let  $\alpha \in (0, 1)$ . Then there is a coarse moduli space  $M_{\bar{L}}$  (resp.  $M_{\bar{L}}^s$ ) for the functor  $F_{\bar{L}}^{ss}$  (resp.  $F_{\bar{L}}^s$ ). The moduli space  $M_{\bar{L}}$  is a projective (irreducible) variety, containing  $M_{\bar{L}}^s$  as an open subvariety.*

Let  $M'$  (resp.  $M'_{\bar{L}}$ ) be the open subscheme of  $M$  (resp.  $M_{\bar{L}}$ ) consisting of GPBs  $(E, F_j(E))$  such that the projections  $p_j, p'_j$  are isomorphisms for all  $j$ . Then  $M'_{\bar{L}}$  corresponds to GPBs in  $M$  with fixed determinant  $\bar{L}$  with  $F_j(L) = (1 : \lambda_j), \lambda_j \in k^*$  and  $M_{\bar{L}}$  is the closure of  $M'_{\bar{L}}$ .

### 3. Application to nodal curves

#### 3.1.

Let  $Y$  be an irreducible projective nodal curve with nodes  $y_j, j = 1, \dots, m$  and  $X$  its desingularization with  $D_j$ , the inverse image of  $y_j$ . Then there is a correspondence from GPBs on  $X$  of rank  $r$ , degree  $d$  to torsion-free sheaves on  $Y$  of the same rank and degree [1, 2]. The correspondence induces a surjective morphism  $f$  from  $M$  onto  $U$ , where  $U$  is the moduli space of torsion-free sheaves of rank  $r$ , degree  $d$  on  $Y$ . The restriction of the morphism  $f$  to  $M'$  is an isomorphism onto the open subvariety  $U'$  of  $U$  corresponding to vector bundles on  $Y$ . One has  $f^{-1}U' = M'$ .

For  $r = 1$ , a GPB  $\bar{L}$  corresponds to a torsion-free sheaf  $\mathcal{L}$  on  $Y$ . Then  $\mathcal{L}$  is a line bundle if and only if  $F_j(L) = (1, \lambda_j), \lambda_j \in k^*, \forall j$ . Suppose that  $\mathcal{L}$  is a line bundle and let  $U'_{\mathcal{L}}$  be the closed subset of  $U'$  corresponding to vector bundles with fixed determinant  $\mathcal{L}$ . Then  $f^{-1}U'_{\mathcal{L}} = M'_{\bar{L}}$  and the morphism  $f$  maps  $M'_{\bar{L}}$  isomorphically onto  $U'_{\mathcal{L}}$ . Hence, if  $\mathcal{L}$  is a line bundle, then  $f(M_{\bar{L}})$  contains  $U'_{\mathcal{L}}$  as an open subset. Since  $M_{\bar{L}}$  is an irreducible, closed subscheme of  $M$ , the image  $f(M_{\bar{L}})$  is a closed, irreducible subscheme of  $U$  and  $U'_{\mathcal{L}}$ , being open, is dense in it. It follows that  $f(M_{\bar{L}})$  is the closure of  $U'_{\mathcal{L}}$ .

*Remark 3.1.* The projective scheme  $M_{\bar{L}}$  can be regarded as the compactified moduli space of vector bundles of rank  $r$  with determinant  $\mathcal{L}$  on the nodal curve  $Y$ .

*Remark 3.2.* In fact, for any torsion-free sheaf  $\mathcal{L}$  of rank 1, the image  $f(M_{\bar{L}})$  is a closed, irreducible subscheme of  $U$  containing the subset of  $U$  consisting of torsion-free sheaves with fixed determinant  $\mathcal{L}$  as an open dense set.

#### 3.2 Relation to Seshadri–Nagaraj conjecture

For a torsion-free sheaf  $F$  of rank  $r$  on  $Y$ , let  $N := (\Lambda^r F)/(\text{torsion})$ , where  $(\text{torsion})$  denotes the maximum subsheaf with proper support. Denote by  $I_j$  the ideal sheaf of the

node  $y_j$ . Define  $U_{\mathcal{L}}$  as the set

$$U_{\mathcal{L}} = \{F \in U: I_j^r \mathcal{L} \subset N \subset \mathcal{L}, \quad \forall j\}.$$

Seshadri and Nagaraj had defined this set for  $Y$  with one node and conjectured that if  $\mathcal{L}$  is a line bundle, then  $U_{\mathcal{L}}$  is the closure of  $U_{\mathcal{L}}^i$  (Conjecture (a), page 136 of [3]). We prove this conjecture.

Let  $\tilde{R}_0^{1-ss}$  denote the subset consisting of 1-semistable points, then  $\tilde{R}_0^{ss} \subset \tilde{R}_0^{1-ss}$ . Let  $P$  be the moduli space of 1-semistable GPBs [5]. One has morphisms  $f: \tilde{R}_0^{ss} \rightarrow U$  inducing  $f: M_{\bar{L}} \rightarrow U$  and  $f_1: \tilde{R}_0^{1-ss} \rightarrow U$  inducing  $f_1: P \rightarrow U$ .

**PROPOSITION 3.3**

Let  $\bar{L}$  be any GPB of rank 1 on  $X$  and  $\mathcal{L}$  the corresponding torsion-free sheaf of rank 1 on  $Y$ .

- (1) If  $(E, F_j(E)) \in \tilde{R}_0^{1-ss}$ , then  $F = f_1((E, F_j(E))) \in U_{\mathcal{L}}$  and hence  $f(M_{\bar{L}}) \subset U_{\mathcal{L}}$ .
- (2) The morphism  $\tilde{R}_0^{1-ss} \rightarrow U$  surjects onto  $U_{\mathcal{L}}$ .
- (3)  $f(M_{\bar{L}}) = U_{\mathcal{L}}$  for  $\alpha$  sufficiently close to 1.

*Proof.* We may assume that  $Y$  has only one node  $y$ . It is easy to see (from the proof) that the general case follows exactly on same lines. Consider a GPB  $(E, F(E))$ . Let  $p_i: F(E) \rightarrow E_{x_i}$ ,  $i = 1, 2$  be the projections and  $a_i = \dim \ker p_i$ . Let  $E_0 = p^*(F)/(\text{torsion})$ , then  $E_0 \subset E$ . Since  $F|_{Y-y} = p_*E|_{Y-y}$ , one has  $N|_{Y-y} = (p_*L)|_{Y-y} = \mathcal{L}|_{Y-y}$ . Hence to check that  $I^r \mathcal{L} \subset N \subset \mathcal{L}$ , we have only to check it locally at the node  $y$ .

Let  $(A, m)$  be the local ring at  $y$ . Its normalization  $\bar{A}$  is a semilocal ring with two maximal ideals  $m_1, m_2$ . The inclusion

$$F_y \subset (p_*E)_y$$

may be identified with the inclusion

$$(r - a_1 - a_2)A \oplus a_1m_1 \oplus a_2m_2 \subset r\bar{A}$$

(Proposition 4.2 of [2]).

- (1) We consider the following cases separately.

*Case (i).* Suppose that  $p_1, p_2$  are both isomorphisms. Then  $\det(E, F(E)) = \bar{L}$  corresponds to a torsion-free sheaf  $\mathcal{L}$  which is locally free at  $y$ . In this case,  $F$  is locally free at  $y$  with  $(\det F)_y = \mathcal{L}_y$  so that  $N = \mathcal{L} \supset I^r \mathcal{L}$ .

*Case (ii).* Assume that  $p_1$  is an isomorphism,  $p_2$  is not an isomorphism (the opposite case can be dealt with similarly). Then  $\det(E, F(E)) = (L, L_{x_1}) = \bar{L}$  corresponds to  $\mathcal{L}$  which is not locally free at  $y$ . One always has  $N \subset p_*L$  and  $N_y \subset \mathcal{L}_y$  if and only if  $N_y \otimes k(y) \subset F(L)$ . Locally,  $a_1 = 0$  so that  $F_y = (r - a_2)A \oplus a_2m_2$ . Then  $N_y = (m_2)^{a_2}$  so that  $N_y \otimes k(y) \subset L_{x_1} = F(L)$ . Hence  $N \subset \mathcal{L}$ . Since  $m^r \subset m_2^{a_2}$ , it follows that  $I^r \mathcal{L} \subset N \subset \mathcal{L}$ .

*Case (iii).* If both  $p_1$  and  $p_2$  are not isomorphisms, one has  $a_1 \geq 1, a_2 \geq 1$ . Then locally,  $N_y = m_1^{a_1}m_2^{a_2} \subset m_1m_2 = m$ . It follows that  $N_y$  maps to zero in  $p_*L \otimes k(y)$  so that  $N \subset I\mathcal{L} \subset \mathcal{L}$ . Since  $m^r \subset m_1^{a_1}m_2^{a_2}$ , one has  $I^r \mathcal{L} \subset N \subset \mathcal{L}$ .

Note that any  $(E, F(E)) \in M_{\bar{L}}$  with  $F(L) = (1 : \lambda), \lambda \in k^*$  (i.e.  $\mathcal{L}$  locally free at  $y$ ) occurs only in cases (i) or (iii). For  $(E, F(E)) \in M_{\bar{L}}$  with  $F(L) = (1 : 0)$  (or  $F(L) = (0 : 1)$ ) only cases (ii) and (iii) occur. Part (1) now follows. Note that  $(E, F_j(E))$  is 1-semistable if and only if  $F$  is semistable [1, 2].

- (2) Let  $F \in U_{\mathcal{L}}$ . Since  $I^r \mathcal{L} \subset N \subset \mathcal{L}$  it follows that  $L|_{X-D} = p^*N|_{X-D}$  where  $D = \sum_j (x_j + x'_j)$ . Since  $\det E_0 = p^*N$  outside  $D$ , one has  $L = \det E_0$  outside  $D$ . It follows that  $L = \det E_0 \otimes \mathcal{O}_X(\sum_j (a_j x_j + a'_j x'_j))$ ,  $a_j + a'_j \leq r$ . Let  $E$  be given by an extension

$$0 \rightarrow E_0 \rightarrow E \rightarrow \oplus_j (k(x_j)^{a_j} \oplus k(x'_j)^{a'_j}) \rightarrow 0.$$

The composite  $F \hookrightarrow p_*E_0 \hookrightarrow p_*E$  induces a linear map  $F \otimes k(y_j) \rightarrow p_*(E) \otimes k(y_j)$ . Let  $F_j(E)$  be the image of this linear map. Then  $(E, F_j(E))$  maps to  $F$  and it is 1-semistable as  $F$  is semistable [2]. By construction,  $\det E = \det E_0 \otimes \mathcal{O}_X(\sum_j (a_j x_j + a'_j x'_j)) = L$ . It follows that  $(E, F_j(E)) \in \tilde{R}_0^{1-ss}$ .

- (3) Let  $P_{\bar{L}}$  denote the closure of  $\tilde{R}_0^{1-ss}/\text{PGL}(n)$  in  $P$ . For  $\alpha$  close to 1, there is a surjective birational morphism  $\phi: M \rightarrow P$  with  $f = f_1 \circ \phi$ . It maps  $M_{\bar{L}}$  birationally into  $P_{\bar{L}}$ . Since both these spaces are irreducible and of the same dimension, it follows that  $\phi(M_{\bar{L}}) = P_{\bar{L}}$ . Since  $f(\tilde{R}_0^{1-ss})$  surjects onto  $U_{\mathcal{L}}$ , it follows that  $f_1(P_{\bar{L}}) \supset U_{\mathcal{L}}$  and hence  $f(M_{\bar{L}}) \supset U_{\mathcal{L}}$ . From (1), it follows that  $U_{\mathcal{L}} = f(M_{\bar{L}}) = f_1(P_{\bar{L}})$ .  $\square$

**COROLLARY 3.4**

*If  $\mathcal{L}$  is a line bundle, then  $U_{\mathcal{L}}$  is the closure of  $U'_{\mathcal{L}}$ .*

*Proof.* From Proposition 3.3, one has (as sets)  $f(M_{\bar{L}}) = U_{\mathcal{L}}$ . Since  $f(M_{\bar{L}})$  is the closure of  $U'_{\mathcal{L}}$  if  $\mathcal{L}$  is a line bundle, the result follows.  $\square$

*Remark 3.5.* The proof of Proposition 3.3(1) easily generalizes to  $(E, F_j(E))$  replaced by a family  $(\mathcal{E}, F_j(\mathcal{E})) \rightarrow T \times X$ .

*Remark 3.6.* Sun [6] had proved the conjecture (a) of Seshadri–Nagaraj by considering a smooth curve  $X$  degenerating to an irreducible nodal curve  $Y$ . However he does not get a moduli functor or a scheme structure on  $U_{\mathcal{L}}$  except in some cases (e.g. one node, rank 2, degree 1).

*Remark 3.7.* Schmitt [4] has constructed a moduli space  $\mathcal{M}$  of  $\alpha$ -semistable descending singular  $\text{SL}(r)$ -bundles  $(A, q, \tau)$  where  $(A, q)$  is a GPB on  $X$  and  $\tau: \text{Sym}^*(A \otimes \mathbf{C}^r)^{\text{SL}(r, \mathbf{C})} \rightarrow \mathcal{O}_X$  a nontrivial homomorphism. It is shown that  $\det A = \mathcal{O}_X$  and for  $\alpha \in (0, 1) \cap \mathbf{Q}$ , there is a forgetful morphism  $h: \mathcal{M} \rightarrow M$  (§5.1 of [4]). For  $\alpha$  close to 1, one has a forgetful morphism  $\mathcal{M} \rightarrow U = U(r, 0)$  whose set theoretic image is  $U_{\mathcal{L}}$  (Proposition 5.1.1 of [4]). Then, since  $\det A = \mathcal{O}_X$ , it follows that  $h(\mathcal{M}) = M_{\bar{L}}$  (as sets).

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